Control and Dynamics

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Analysis and Geometry in Control Theory
and its Applications
Let $M$ be a smooth compact manifold of dimension $n \geq 2$ be fixed. Let $H : T^* M \to \mathbb{R}$ be a Hamiltonian of class $C^k$, with $k \geq 2$, recall that the Hamiltonian vector field reads (in local coordinates)

$$X_H(x, p) = \left( \frac{\partial H}{\partial p}(x, p), -\frac{\partial H}{\partial x}(x, p) \right).$$

Examples:

- $H(x, p) = \|p\|_2^2 / 2$ (Riemannian)
- $H(x, p) = \|p\|_2^2 / 2 + V(x)$ (mechanical)
- $H(x, p) = \|p\|_2^2 / 2 + p \cdot X(x)$ (Mañé)
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- Tonelli Hamiltonians
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1. Change the behavior of an orbit: e.g. close a recurrent orbit or an orbit through a non-wandering point of the Hamiltonian flow into a periodic orbit
   \( \Rightarrow \) Closing Lemma

2. Change the behavior of \( \phi_t^H \) along a given orbit
   \( \Rightarrow \) Franks’ Lemma
The Pugh closing lemma

Let $X$ be a $C^1$ vector field on a compact manifold $M$ and $x \in M$ be a non-wandering point w.r.t to the flow of $X$.

**Proposition**

For every $\epsilon > 0$, there is a $C^1$ vector field $Y$ having $x$ as a periodic point such that $\|Y - X\|_{C^0} < \epsilon$. 

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The Franks Lemma for vector fields

Let $\bar{x}$ be a periodic point for the flow of $X$ of period $T > 0$. Fix a local section $\Sigma$ transverse to the flow at $\bar{x}$ and consider the Poincaré first return map

$$P : \Sigma \rightarrow \Sigma \quad \bar{x} \mapsto \phi_{\tau(x)}^X(x).$$

It is a local $C^1$ diffeomorphism fixing $\bar{x}$. 

Lemma (Franks, 1971)

For every $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that for every isomorphism $Q : T_{\bar{x}}\Sigma \rightarrow T_{\bar{x}}\Sigma$ satisfying $\|Q - d\bar{x}P\| < \delta$, there exists a $C^1$ vector field $Y$ which preserves the orbit of $\bar{x}$ such that $\|Y - X\|_{C^1} < \epsilon$ and $d\bar{x}P = Q$. 

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**Theorem (Pugh)**

Let $M$ be a smooth compact manifolds, the set of $C^1$ vector fields $X$ on $M$ such that

$$\overline{\text{Per}(X)} = \Omega(X),$$

is residual in $\mathcal{X}^1(M)$ (the set of $C^1$ vector fields on $M$).
Theorem (Pugh-Robinson, 1983)

Let \((N, \omega)\) be a symplectic manifold of dimension \(2n \geq 2\) and \(H : N \rightarrow \mathbb{R}\) be a given Hamiltonian of class \(C^2\). Let \(X\) be the Hamiltonian vector field associated with \(H\) and \(\phi^H\) the Hamiltonian flow. Suppose that \(x \in N\) is a non-wandering point of the flow of \(X\) and that \(U\) is a neighborhood of \(X\) in the \(C^1\) topology. Then there exists \(Y \in U\) such that \(Y\) is a Hamiltonian vector field and \(Y\) has a closed orbit through \(x\).
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- If \((N, \omega) = (T^* M, w_{can})\), can we close a recurrent orbit by adding a small potential \((H \rightsquigarrow H + V)\)?
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Questions:

- If \((N, \omega) = (T^* M, w_{can})\), can we close a recurrent orbit by adding a small potential \((H \leadsto H + V)\) ?
- If \(H = (1/2)\|p\|_x^2\), can we close a recurrent orbit by a small perturbation of the Riemannian metric?
## Theorem (Rifford, 2012)

Let $M$ be a smooth compact manifold and $g$ be a Riemannian metric on $M$ of class $C^k$ with $k \geq 3$ and let $(x, v) \in U^g M$ be a non-wandering point for the geodesic flow. Then for every $\epsilon > 0$, there exists a metric $\tilde{g}$ of class $C^{k-1}$ with 
\[ \| \tilde{g} - g \|_{C^1} < \epsilon \]
such that the geodesic starting from $x$ with initial velocity $v$ is periodic.
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Remark

By Poincaré’s recurrence theorem, the set of recurrent pairs has full measure in $U^g M$. Hence all points in $U^g M$ are non-wandering.

The closing lemma for the geodesic flow in the $C^2$ topology on the metric is open.
Let $M$ be a smooth compact manifold and $H : T^* M \to \mathbb{R}$ an Hamiltonian of class $C^k$, with $k \geq 2$. Let $\bar{\theta} = (\bar{x}, \bar{p})$ be a periodic point for the Hamiltonian flow of positive period $T > 0$. Fix a local section transversal to the flow at $\bar{\theta}$ and contained in the energy level of $\bar{\theta}$. Then consider the Poincaré first return map $P : \Sigma \rightarrow \Sigma \theta \mapsto \phi_H^{\tau}(\theta)$, which is a local diffeomorphism and for which $\bar{\theta}$ is a fixed point. The Poincaré map is symplectic, i.e. it preserves the restriction of the symplectic form to $T_{\bar{\theta}} \Sigma$. 
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Then consider the **Poincaré first return map**

$$P : \Sigma \quad \mapsto \quad \Sigma$$

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**Back to Franks’ Lemma**
The symplectic group

Let $\text{Sp}(m)$ be the symplectic group in $M_{2m}(\mathbb{R})$ ($m = n - 1$), that is the smooth submanifold of matrices $X \in M_{2m}(\mathbb{R})$ satisfying

$$X^* \mathbb{J} X = \mathbb{J} \quad \text{where} \quad \mathbb{J} := \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}.$$ 

Choosing a convenient set of coordinates, the differential of the Poincaré map is the symplectic matrix $X(T)$ where $X : [0, T] \to \text{Sp}(m)$ is solution to the Cauchy problem

$$\begin{cases} \dot{X}(t) = A(t)X(t) & \forall t \in [0, T], \\ X(0) = I_{2m}, \end{cases}$$

where $A(t)$ has the form

$$A(t) = \begin{pmatrix} 0 & I_m \\ -K(t) & 0 \end{pmatrix} \quad \forall t \in [0, T].$$
Problem:

Given $\epsilon > 0$, does the set of differentials of Poincaré maps (restricted to $T_\theta \Sigma$) associated with potentials $V : M \to \mathbb{R}$ of class $C^k$ such that

$$\|V\|_{C^k} < \epsilon,$$

the periodic orbit through $\bar{\theta}$ is preserved by the Hamiltonian flow associated with the perturbed Hamiltonian $H + V$? What's the radius of that ball in terms of $\epsilon$?
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answer: 

[Answer here]
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fill a ball around $d_{\tilde{\theta}}P$?
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What’s the radius of that ball in term of $\epsilon$?
Perturbation of the Poincaré map

Let $\gamma$ be the projection of the periodic orbit passing through $\bar{\theta}$, we are looking for a potential

$$V: M \longrightarrow \mathbb{R}$$

satisfying the following properties

$$V(\gamma(t)) = 0, \quad dV(\gamma(t)) = 0,$$

with

$$d^2V(\gamma(t)) \quad \text{free.}$$
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$$\iff \quad d^2V(\gamma(t)) \quad \text{is the control}.$$
A controllability problem on $\text{Sp}(m)$

The Poincaré map at time $T$ associated with the new Hamiltonian

$$H + V$$

is given by $X_u(T)$ where $X_u : [0, T] \rightarrow \text{Sp}(m)$ is solution to the control problem

$$\begin{aligned}
\dot{X}_u(t) &= A(t)X_u(t) + \sum_{i \leq j = 1}^{m} u_{ij}(t)\mathcal{E}(ij)X_u(t), \quad \forall t \in [0, T], \\
X(0) &= I_{2m},
\end{aligned}$$

where the $2m \times 2m$ matrices $\mathcal{E}(ij)$ are defined by

$$\mathcal{E}(ij) := \begin{pmatrix} 0 & 0 \\ E(ij) & 0 \end{pmatrix},$$

with

$$\begin{aligned}
(E(ii))_{k,l} &= \delta_{ik}\delta_{il} \quad \forall i = 1, \ldots, m, \\
(E(ij))_{k,l} &= \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \quad \forall i < j = 1, \ldots, m.
\end{aligned}$$
References:

- "Generic properties of closed orbits of Hamiltonian flows from Mañé’s viewpoint”
  L.R., Rafael Ruggiero, IMRN, 2012.

- "Franks’ Lemma for $C^2$-Mañé perturbations of Riemannian metrics and applications to persistence”
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Thank you for your attention !!