Optimal Transport on Surfaces

Ludovic Rifford

Université de Nice - Sophia Antipolis
Let \( (M, g) \) be a smooth compact connected Riemannian surface.
Let \((M, g)\) be a smooth compact connected Riemannian surface.
Denote by \(d_g\) the geodesic distance on \(M\) and define the quadratic cost \(c : M \times M \to [0, \infty)\) by

\[
c(x, y) := \frac{1}{2} d_g(x, y)^2 \quad \forall x, y \in M.
\]
Let \((M, g)\) be a smooth compact connected Riemannian surface. Denote by \(d_g\) the geodesic distance on \(M\) and define the quadratic cost \(c : M \times M \to [0, \infty)\) by

\[
c(x, y) := \frac{1}{2}d_g(x, y)^2 \quad \forall x, y \in M.
\]

Given two Borelian probability measures \(\mu_0, \mu_1\) on \(M\), find a measurable map \(T : M \to M\) satisfying

\[
T_\#\mu_0 = \mu_1 \quad \text{(i.e. } \mu_1(B) = \mu_0(T^{-1}(B)), \forall B \text{ borelian } \subset M),
\]

and minimizing

\[
\int_M c(x, T(x))d\mu_0(x).
\]
Theorem (McCann ’01)

Let $\mu_0, \mu_1$ be two probability measures on $M$. If $\mu_0$ is absolutely continuous w.r.t. the Lebesgue measure, then there is a unique optimal transport map $T : M \to M$ satisfying $T_{\#}\mu_0 = \mu_1$ and minimizing

$$\int_M c(x, T(x))d\mu_0(x).$$

It is characterized by the existence of a semiconvex function $\psi : M \to \mathbb{R}$ such that

$$T(x) = \exp_x (\nabla \psi(x)) \quad \text{for } \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$
We say that \((M, g)\) satisfies the **Transport Continuity Property (TCP)** if the following property is satisfied:

For any pair of probability measures \(\mu_0, \mu_1\) associated locally with **continuous positive densities** \(\rho_0, \rho_1\), that is

\[
\mu_0 = \rho_0 L^n, \quad \mu_1 = \rho_1 L^n,
\]

the optimal transport map between \(\mu_0\) and \(\mu_1\) is **continuous**.
Characterization of surfaces satisfying TCP

The surface \((M, g)\) satisfies \(\text{(TCP)}\) if and only if the two following properties hold:

- All the injectivity domains are convex,
- The cost \(c\) is regular.
Let $x \in M$ be fixed. We call exponential mapping from $x$, the mapping defined as

$$\exp_x : T_x M \longrightarrow M$$

$$v \longmapsto \exp_x(v) := \gamma_v(1),$$

where $\gamma_v : [0, 1] \rightarrow M$ is the unique geodesic starting at $x$ with speed $\dot{\gamma}_v(0) = v$. 

We call the injectivity domain of $x$ the set

$I(x) := \{v \in T_x M \mid \exists t > 1 \text{ s.t. } \gamma_{tv} \text{ is the unique minimizing geodesic between } x \text{ and } \exp_x(tv)\}$. 

It is a star-shaped (w.r.t. $0 \in T_x M$) domain with Lipschitz boundary.
Let $x \in M$ be fixed. We call exponential mapping from $x$, the mapping defined as

$$\exp_x : T_x M \longrightarrow M$$

$$\nu \longmapsto \exp_x(\nu) := \gamma_\nu(1),$$

where $\gamma_\nu : [0,1] \rightarrow M$ is the unique geodesic starting at $x$ with speed $\dot{\gamma}_\nu(0) = \nu$. We call injectivity domain of $x$ the set

$$\mathcal{I}(x) := \left\{ \nu \in T_x M \mid \exists t > 1 \text{ s.t. } \gamma_{tv} \text{ is the unique minimizing geodesic between } x \text{ and } \exp_x(tv) \right\}.$$
Let $x \in M$ be fixed. We call exponential mapping from $x$, the mapping defined as

$$\exp_x : T_x M \longrightarrow M$$

$$v \longmapsto \exp_x(v) := \gamma_v(1),$$

where $\gamma_v : [0, 1] \rightarrow M$ is the unique geodesic starting at $x$ with speed $\dot{\gamma}_v(0) = v$. We call **injectivity domain** of $x$ the set

$$\mathcal{I}(x) := \left\{ v \in T_x M \mid \exists t > 1 \text{ s.t. } \gamma_{tv} \text{ is the unique minimizing geodesic between } x \text{ and } \exp_x(tv) \right\}.$$

It is a star-shaped (w.r.t. $0 \in T_x M$) domain with Lipschitz boundary.
Injectivity domains: Examples...

Flat tori: all injectivity domains are convex.
Tori of revolution: injectivity domains are not necessarily convex.
Injectivity domains: Examples...

Round spheres: all injectivity domains are convex.
Injectivity domains: Examples...

$C^4$ small perturbations of round spheres:

**Theorem (Figalli-LR ’09)**

*Any small deformation of the round sphere $(\mathbb{S}^2, g^0)$ in $C^4$ topology has all its injectivity domains convex.*
Injectivity domains: examples...

Oblate ellipsoids of revolution:

\[ E_\mu : \quad x^2 + y^2 + \left( \frac{z}{\mu} \right)^2 = 1 \quad \mu \in (0, 1]. \]

**Theorem (Bonnard-Caillau-LR ’10)**

*The injectivity domain on an oblate ellipsoid of revolution is convex for any point if and only if the ratio between the minor and the major axis is greater or equal to \(1/\sqrt{3}\).*
The cost \( c = d^2/2 : M \times M \rightarrow \mathbb{R} \) is called \textbf{regular}, if for every \( x \in M \) and every \( v_0, v_1 \in \mathcal{I}(x) \), there holds

\[
\nu_t := (1 - t)\nu_0 + tv_1 \in \mathcal{I}(x) \quad \forall t \in [0, 1],
\]

and

\[
c(x, y_t) - c(x', y_t) \leq \max \left( c(x, y_0) - c(x', y_0), c(x, y_1) - c(x', y_1) \right),
\]

for any \( x' \in M \), where \( y_t := \exp_x \nu_t \).
An obvious remark

**Remark**

Assume that all the injectivity domains of $(M, g)$ are convex. Then the cost $c$ is regular if and only if for every $x, x' \in M$, the mapping

$$F_{x,x'} : v \in I(x) \mapsto c(x, \exp_x(v)) - c(x', \exp_x(v))$$

is quasiconvex (its sublevels sets are always convex).
An easy lemma

Lemma

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class $C^2$. Assume that for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$, the following property holds

$$\langle \nabla_v F, w \rangle = 0 \implies \langle \nabla^2_v F \, w, w \rangle > 0.$$

Then $F$ is quasiconvex.
Proof of the easy lemma

Proof.
Let $\nu_0, \nu_1 \in U$ be fixed.
Proof.
Let $v_0, v_1 \in U$ be fixed. Set $v_t := (1 - t)v_0 + tv_1$, for every $t \in [0, 1]$,
Proof.

Let $\nu_0, \nu_1 \in U$ be fixed. Set $\nu_t := (1 - t)\nu_0 + t\nu_1$, for every $t \in [0, 1]$, and define $h : [0, 1] \to \mathbb{R}$ by

$$h(t) := F(\nu_t) \quad \forall t \in [0, 1].$$
Proof.

Let \( \nu_0, \nu_1 \in U \) be fixed. Set \( \nu_t := (1 - t)\nu_0 + t\nu_1 \), for every \( t \in [0, 1] \), and define \( h : [0, 1] \to \mathbb{R} \) by

\[
h(t) := F(\nu_t) \quad \forall t \in [0, 1].
\]

If \( h \not\leq \max\{h(0), h(1)\} \), there is \( \tau \in (0, 1) \) such that

\[
h(\tau) = \max_{t \in [0, 1]} h(t) > \max\{h(0), h(1)\}.
\]
Proof.

Let $v_0, v_1 \in U$ be fixed. Set $v_t := (1 - t)v_0 + tv_1$, for every $t \in [0, 1]$, and define $h : [0, 1] \to \mathbb{R}$ by

$$h(t) := F(v_t) \quad \forall t \in [0, 1].$$

If $h \not\geq \max\{h(0), h(1)\}$, there is $\tau \in (0, 1)$ such that

$$h(\tau) = \max_{t \in [0, 1]} h(t) > \max\{h(0), h(1)\}.$$ 

There holds

$$\dot{h}(\tau) = \langle \nabla_{v_\tau} F, \dot{v}_\tau \rangle \quad \text{and} \quad \ddot{h}(\tau) = \langle \nabla^2_{v_\tau} F \dot{v}_\tau, \dot{v}_\tau \rangle.$$
Proof.

Let $v_0, v_1 \in U$ be fixed. Set $v_t := (1 - t)v_0 + tv_1$, for every $t \in [0, 1]$, and define $h : [0, 1] \to \mathbb{R}$ by

$$h(t) := F(v_t) \quad \forall t \in [0, 1].$$

If $h \not\leq \max\{h(0), h(1)\}$, there is $\tau \in (0, 1)$ such that

$$h(\tau) = \max_{t \in [0, 1]} h(t) > \max\{h(0), h(1)\}.$$

There holds

$$\dot{h}(\tau) = \langle \nabla_{v_\tau} F, \dot{v}_\tau \rangle \quad \text{and} \quad \ddot{h}(\tau) = \langle \nabla^2_{v_\tau} F \dot{v}, \dot{v}\rangle.$$

Since $\tau$ is a local maximum, we get a contradiction.
The following lemma is false !!

**FALSE Lemma**

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class $C^2$. Assume that for every $v \in U$ and every $w \in \mathbb{R}^n$, the following property holds

$$\langle \nabla_v F, w \rangle = 0 \implies \langle \nabla^2_v F w, w \rangle \geq 0.$$  

Then $F$ is quasiconvex.
However, the following result holds true.

**Lemma**

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class $C^2$. Assume that there is a constant $C > 0$ such that

$$\langle \nabla^2_v F, w \rangle \geq - C |\langle \nabla_v F, w \rangle| |w| \quad \forall v \in U, \forall w \in \mathbb{R}^n.$$

Then $F$ is quasiconvex.
The \textbf{MTW} tensor $\mathcal{S}$ is defined as

$$
\mathcal{S}(x,v)(\xi,\eta) = -\frac{3}{2} \left. \frac{d^2}{ds^2} \right|_{s=0} \left. \frac{d^2}{dt^2} \right|_{t=0} c(\exp_x(t\xi), \exp_x(v + s\eta)) ,
$$

for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$. 

The **MTW** tensor $\mathcal{S}$ is defined as

$$
\mathcal{S}_{(x,v)}(\xi, \eta) = -\frac{3}{2} \left. \frac{d^2}{ds^2} \right|_{s=0} \left. \frac{d^2}{dt^2} \right|_{t=0} c(\exp_x(t\xi), \exp_x(v + s\eta)),
$$

for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_x M$.

**Proposition (Villani ’09, Figalli-LR-Villani ’10)**

Assume that all the injectivity domains are convex. Then, the two following properties are equivalent:

- the cost $c$ is regular,
- the **MTW** tensor $\mathcal{S}$ is $\succeq 0$, that is, for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_x M$,

$$
\langle \xi, \eta \rangle_x = 0 \implies \mathcal{S}_{(x,v)}(\xi, \eta) \geq 0.
$$
Another characterization of (TCP) surfaces

Theorem (Figalli-LR-Villani ’10)

Let \((M,g)\) be a compact surface. Then, it satisfies (TCP) if and only if the following properties hold:

- all the injectivity domains are convex,
- the MTW tensor \(\mathcal{S}\) is \(\succeq 0\).
Another characterization of (TCP) surfaces

**Theorem (Figalli-LR-Villani ’10)**

Let \((M, g)\) be a compact surface. Then, it satisfies (TCP) if and only if the following properties hold:

- all the injectivity domains are convex,
- the MTW tensor \(\mathcal{G}\) is \(\succeq 0\).

Loeper noticed that for every \(x \in M\) and for any pair of unit orthogonal tangent vectors \(\xi, \eta \in T_x M\), there holds

\[
\mathcal{G}_{(x,0)}(\xi, \eta) = \kappa_x,
\]

where \(\kappa_x\) denotes the gaussian curvature of \(M\) at \(x\). Consequently, any \((M, g)\) satisfying TCP must have nonnegative gaussian curvatures.
The flat torus

The **MTW** tensor of the flat torus \((\mathbb{T}^n, g^0)\) satisfies

\[ \mathcal{S}_{(x, v)} \equiv 0 \quad \forall x \in \mathbb{T}^n, \forall v \in \mathcal{I}(x) \]

**Theorem (Cordero-Erausquin ’99)**

*The flat torus \((\mathbb{T}^n, g^0)\) satisfies TCP.*
Loeper checked that the **MTW** tensor of the round sphere \((S^n, g^0)\) satisfies for any \(x \in S^n, v \in \mathcal{I}(x)\) and \(\xi, \eta \in T_xS^n\),
\[\langle \xi, \eta \rangle_x = 0 \implies \mathcal{G}_{(x,v)}(\xi, \eta) \geq \|\xi\|_x^2 \|\eta\|_x^2.\]

**Theorem (Loeper '06)**

*The round sphere \((S^n, g^0)\) satisfies TCP.*
Let $G$ be a discrete group of isometries of $(M, g)$ acting freely and properly. Then there exists on the quotient manifold $N = M/G$ a unique Riemannian metric $h$ such that the canonical projection $p : M \to N$ is a Riemannian covering map.

**Theorem (Delanoë-Ge ’08)**

If $(M, g)$ satisfies TCP, then $(N = M/G, h)$ satisfies TCP.

Examples: $(\mathbb{RP}^n, g^0)$, the flat Klein bottle.
On $(\mathbb{S}^2, g^0)$, the **MTW** tensor is given by

$$\mathcal{S}_{(x,v)}(\xi, \xi^\perp) = 3 \left[ \frac{1}{r^2} - \frac{\cos(r)}{r \sin(r)} \right] \xi_1^4 + 3 \left[ \frac{1}{\sin^2(r)} - \frac{r \cos(r)}{\sin^3(r)} \right] \xi_2^4$$

$$+ \frac{3}{2} \left[ - \frac{6}{r^2} + \frac{\cos(r)}{r \sin(r)} + \frac{5}{\sin^2(r)} \right] \xi_1^2 \xi_2^2,$$

with $x \in \mathbb{S}^2$, $v \in \mathcal{I}(x)$, $r := \|v\|_x$, $\xi = (\xi_1, \xi_2)$, $\xi^\perp = (-\xi_2, \xi_1)$. 

Theorem (Figalli-LR '09) Any small deformation of the round sphere $(\mathbb{S}^2, g^0)$ in $C^4$ topology satisfies TCP.
Small deformations of \((S^2, g^0)\)

On \((S^2, g^0)\), the **MTW** tensor is given by

\[
\mathcal{G}_{(x,v)}(\xi, \xi_\perp) = 3 \left[ \frac{1}{r^2} - \frac{\cos(r)}{r \sin(r)} \right] \xi_1^4 + 3 \left[ \frac{1}{\sin^2(r)} - \frac{r \cos(r)}{\sin^3(r)} \right] \xi_2^4 + \frac{3}{2} \left[ -6 \frac{r^2}{r^2} + \frac{\cos(r)}{r \sin(r)} + \frac{5}{\sin^2(r)} \right] \xi_1^2 \xi_2^2,
\]

with \(x \in S^2, v \in I(x), r := \|v\|_x, \xi = (\xi_1, \xi_2), \xi_\perp = (-\xi_2, \xi_1)\).

**Theorem (Figalli-LR ’09)**

*Any small deformation of the round sphere \((S^2, g^0)\) in \(C^4\) topology satisfies **TCP**.*
Oblate ellipsoids

Any oblate ellipsoids of revolution

\[ E_\mu : \quad x^2 + y^2 + \left( \frac{z}{\mu} \right)^2 = 1 \]

with \( \mu < \frac{1}{\sqrt{3}} \)

does not satisfy **TCP**.
Jump of curvature

The surface made with two half-balls joined by a cylinder has not a regular cost.

Then, it does not satisfy TCP.
Thank you for your attention!
Assume that \((M^n, g)\) satisfies \((TCP)\) then the following properties hold:

- all the injectivity domains are convex,
- \(\mathcal{G} \succeq 0\).
Theorem (Necessary conditions)

Assume that \((M^n, g)\) satisfies \((TCP)\) then the following properties hold:

- all the injectivity domains are convex,
- \(\mathcal{S} \succeq 0\).

Theorem (Sufficient conditions)

Assume that \((M^n, g)\) satisfies the following properties:

- all the injectivity domains are strictly convex,
- \(\mathcal{S} \succ 0\).

Then \((M, g)\) satisfies \(TCP\).
**Theorem (Necessary conditions)**

Assume that \((M^n, g)\) satisfies \((TCP)\) then the following properties hold:

- all the injectivity domains are convex,
- \(\mathcal{G} \succeq 0\).

**Theorem (Sufficient conditions)**

Assume that \((M^n, g)\) satisfies the following properties:

- all the injectivity domains are strictly convex,
- \(\mathcal{G} \succ 0\).

Then \((M, g)\) satisfies \(TCP\).

There is a gap!