Recent progress in sub-Riemannian geometry

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Biosketch

Ludovic Rifford obtained his doctorate, which he prepared under the supervision of Francis Clarke, at the Université Lyon 1 in 2000. After holding assistant professor positions at the Lyon 1 and Paris-Sud (Orsay) universities, Ludovic Rifford took up a professorship at the Université Nice Sophia Antipolis in 2006. During his career, he has been interested in various problems at the interface between analysis, dynamics and geometry. He has largely contributed, *inter alia*, to major advances in two important conjectures of sub-Riemannian geometry (the so-called Sard conjecture) and Hamiltonian dynamics (the Mañé conjecture). Ludovic Rifford is the author of fifty or so articles and a monograph, and counts two Fields medallists (Cédric Villani and Alessio Figalli) among his regular co-authors.

Outline

I. Introduction to sub-Riemannian geometry

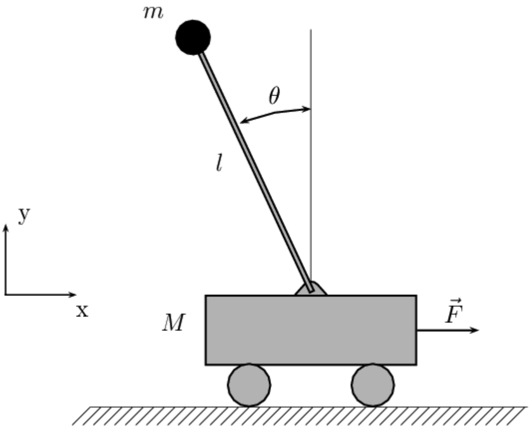
II. A few open problems

III. A few partial results

Part I

I. Introduction to sub-Riemannian geometry

Control of an inverted pendulum



Geometric Control Theory

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Proposition

Under classical assumptions on the datas, for every $x \in M$ and every measurable control $u : [0, T] \to U$ the Cauchy problem

$$\begin{cases} \dot{x}(t) = f\left(x(t), u(t)\right) & a.e. \;\; t \in [0, T], \\ x(0) = x \end{cases}$$

admits a unique solution

$$x(\cdot) = x(\cdot; x, u) : [0, T] \longmapsto M.$$

Controllability and Optimality issues

Controllability issue: Given two points x_1, x_2 in the state space M and $T > 0$, can we find a control u such that the solution of

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big) & \text{a.e. } t \in [0, T] \\ x(0) = x_1 \end{cases}$$

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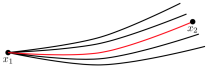


Optimality issue: Among all trajectories joining x_1 to x_2 which one is optimal?

Sub-Riemannian structures

Let M be a smooth connected manifold of dimension n.

Definition

A sub-Riemannian structure of rank m in M is given by a pair (Δ, g) where:

- Δ is a **totally nonholonomic distribution** of rank $m \leq n$ on M which is defined locally generated by a family of m linearly independent smooth vector fields satisfying the **Hörmander condition**.
- g_x is a **metric** over $\Delta(x)$.

The Hörmander condition

We say that a family of smooth vector fields $X^1, \ldots, X^m$, satisfies the **Hörmander condition** if

$$\mathrm{Lie}\left\{X^1, \ldots, X^m\right\}(x) = T_x M \qquad \forall x,$$

where $\mathrm{Lie}\{X^1, \ldots, X^m\}$ denotes the Lie algebra generated by $X^1, \ldots, X^m$, *i.e.* the smallest subspace of smooth vector fields that contains all the $X^1, \ldots, X^m$ and which is stable under Lie brackets.

Reminder

Given smooth vector fields X, Y in $\mathbb{R}^n$, the Lie bracket $[X, Y]$ at $x \in \mathbb{R}^n$ is defined by

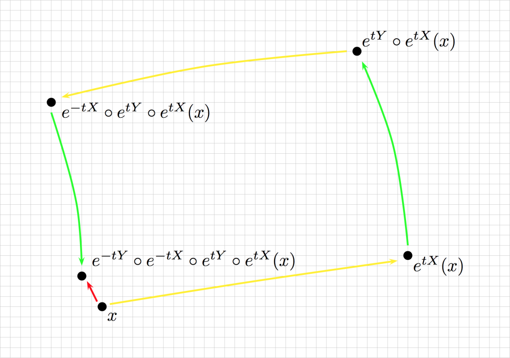
$$[X, Y](x) = DY(x)X(x) - DX(x)Y(x).$$

Lie Bracket: Dynamic Viewpoint

Exercise

There holds

$$[X,Y](x) = \lim_{t\downarrow 0} \frac{\left(e^{-tY}\circ e^{-tX}\circ e^{tY}\circ e^{tX}\right)(x) - x}{t^2}.$$



The Chow-Rashevsky Theorem

Definition

We call **horizontal path** any $\gamma \in W^{1,2}([0,1]; M)$ such that

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The following result is the cornerstone of the sub-Riemannian geometry. (Recall that M is assumed to be connected.)

Theorem (Chow-Rashevsky, 1938)

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Since the distribution is equipped with a metric, we can measure the lengths of horizontal paths and consequently we can associate a metric with the sub-Riemannian structure.

Examples of sub-Riemannian structures

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Every Riemannian manifold (M, g) *gives rise to a sub-Riemannian structure with* $\Delta = TM$.

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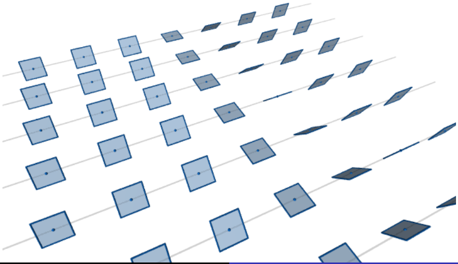
Example (Heisenberg)

In $\mathbb{R}^3$, $\Delta = Span\{X^1, X^2\}$ with k a positive integer and

$$X^1 = \partial_x, \quad X^2 = \partial_y + x^k \partial_z \quad et \quad g = dx^2 + dy^2.$$



Sub-Riemannian distance and minimizing geodesics

The **length** of an horizontal path γ is defined by

$$\text{length}^g(\gamma) := \int_0^T |\dot{\gamma}(t)|^g_{\gamma(t)} \, dt.$$

Definition

Given $x, y \in M$, the **sub-Riemannian distance** between x and y is defined by

$$d_{SR}(x,y) := \inf\Big\{\text{length}^g(\gamma) \,|\, \gamma \text{ hor.}, \gamma(0) = x, \gamma(1) = y\Big\}.$$

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Definition

We call **minimizing geodesic** between x and y any horizontal path $\gamma : [0,1] \to M$ joining x to y with contant speed such that

$$d_{SR}(x,y) = \mathrm{length}^g(\gamma).$$

Properties of minimizing geodesics

Given a sub-Riemannian structure (Δ, g) on M and a minimizing geodesic γ from x to y, two cases may happen:

- The geodesic γ is the projection of a normal extremal so it is smooth..
- The geodesic γ is a singular curve and could be non-smooth..

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Questions about singular curves:

When? How many? How?

Part II

II. A few open problems

A few open problems

Let (Δ, g) be a SR structure on M and $x \in M$ be fixed.

How many?

$$\mathcal{S}^x_{\Delta,min^g} = \{\gamma(1) | \gamma : [0,1] \to M, \gamma(0) = x, \gamma \text{ hor., sing., min.}\}.$$

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Conjecture (Sard Conjectures)

The sets $\mathcal{S}^x_{\Delta,min^g}$ and $\mathcal{S}^x_{\Delta}$ have Lebesgue measure zero.

How?

Conjecture (Regularity Conjecture)

Minimizing geodesics are of class C^1 or smooth.

Part II

III. A few partial results

The strong Sard Conjecture on Martinet surfaces

Let M be of dimension 3, Δ of rank 2 and g be fixed:

$$\mathcal{S}_{\Delta,g}^{x,L} = \left\{ \gamma(1) | \gamma \in \mathcal{S}_{\Delta}^{x} \text{ and } , \text{length}^{g}(\gamma) \leq L \right\}.$$

Conjecture (Strong Sard Conjecture)

The set $\mathcal{S}_{\Delta}^{x,L}$ has finite $\mathcal{H}^1$-measure.

Theorem (Belotto-Figalli-Parusinski-R, 2018)

Assume that M and Δ are analytic and that g is smooth and complete. Then any singular horizontal curve is a semianalytic curve in M. Moreover, for every $x \in M$ and every $L \geq 0$, the set $\mathcal{S}_{\Delta,g}^{x,L}$ is a finite union of singular horizontal curves, so it is a semianalytic curve.

Proof

Ingredients of the proof

- Resolution of singularities.

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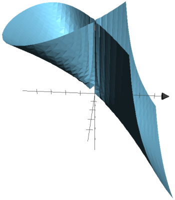
- The vector field which generates the trace of $\tilde{\Delta}$ over $\tilde{\Sigma}$ (after resolution) has singularities of type saddle.
- A result of Speissegger (following Ilyashenko) on the regularity of Poincaré transitions mappings.

An example

In $\mathbb{R}^3$,

$$X = \partial_y \quad \text{and} \quad Y = \partial_x + \left[\frac{y^3}{3} - x^2 y(x+z) \right] \partial_z .$$



Martinet Surface: $\Sigma_\Delta = \left\{ y^2 - x^2(x+z) = 0 \right\}.$

The Sard Conjecture on Martinet surfaces

As a consequence, thanks to a striking result by Hakavuori and Le Donne, we have:

Theorem (Belotto-Figalli-Parusinski-R, 2018)

Assume that M and Δ are analytic and that g is smooth and complete and let $\gamma : [0,1] \to M$ be a singular minimizing geodesic. Then γ is of class C^1 on $[0,1]$. Furthermore, $\gamma([0,1])$ is semianalytic, and therefore it consists of finitely many points and finitely many analytic arcs.

Thank you for your attention !!

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