Continuity of optimal transport maps and convexity of injectivity domains on small deformations of $\mathbb{S}^2$

A. Figalli* L. Rifford†

Abstract

Given a compact Riemannian manifold, we study the regularity of the optimal transport map between two probability measures with cost given by the squared Riemannian distance. Our strategy is to define a new form of the so-called Ma-Trudinger-Wang condition and to show that this condition, together with the strict convexity on the nonfocal domains, implies the continuity of the optimal transport map. Moreover our new condition, again combined with the strict convexity of the nonfocal domains, allows to prove that all injectivity domains are strictly convex too. These results apply for instance on any small C^4-deformation of the two-sphere.

1 Introduction

Let μ, ν be two probability measures on a smooth compact connected Riemannian manifold (M, g) equipped with its geodesic distance d. Given a cost function $c : M \times M \to \mathbb{R}$, the Monge-Kantorovich problem consists in finding a transport map $T : M \to M$ which sends μ onto ν (i.e. $T_\# \mu = \nu$) and which minimizes the functional

$$\min_{S_\# \mu = \nu} \int_M c(x, S(x))\, dx.$$

In [22] McCann (generalizing [2] from the Euclidean case) proved that, if μ gives zero mass to countably $(n-1)$-rectifiable sets, then there is a unique transport map T solving the Monge-Kantorovich problem with initial measure μ, final measure ν, and cost function $c = d^2/2$. The purpose of this paper is to study the regularity of T. This problem has been extensively investigated in the Euclidean space [3, 4, 5, 9, 25, 26], in the case of the flat torus or nearly flat metrics [8, 10], on the standard sphere and its perturbations [11, 17, 19], and on manifolds with nonfocal cut locus [20] (see [28, Chapter 12] for an introduction to the problem of the regularity of the optimal tranport map for a general cost function).

Definition 1.1. *Let (M, g) be a smooth compact connected Riemannian manifold. We say that (M, g) satisfies the* transport continuity property *(abbreviated $\mathcal{TCP}$) if, whenever μ and ν satisfy*

(i) $\lim_{r\to 0} \frac{\mu(B_r(x))}{r^{n-1}} = 0$ *for any* $x \in M$,

(ii) $\inf_{x\in M} \left(\liminf_{r\to 0} \frac{\nu(B_r(x))}{r^n}\right) > 0$,

*Centre de Mathématiques Laurent Schwartz, Ecole Polytechnique, Palaiseau, France (figalli@math.polytechnique.fr)

†Université de Nice-Sophia Antipolis, Labo. J.-A. Dieudonné, UMR 6621, Parc Valrose, 06108 Nice Cedex 02, France (rifford@unice.fr)

then the unique optimal transport map T between μ and ν is continuous.

Note that the above definition makes sense: by a standard covering argument one can prove that assumption (i) implies that μ gives zero mass to countably $(n-1)$-rectifiable sets. Thus, by McCann's Theorem, the optimal transport map T from μ to ν exists and is unique.

If (M, g) is a given Riemannian manifold, we call C^4-*deformation* of (M, g) any Riemannian manifold of the form (M, g^ε) with g^ε close to g in C^4-topology. Loeper [19] proved that the round sphere $(\mathbb{S}^n, g^{\mathrm{can}})$ satisfies $\mathcal{TCP}$. Then, Loeper and Villani [20] showed that any C^4-deformation of quotients of the sphere (like $\mathbb{RP}^n$) satisfies $\mathcal{TCP}$. Furthermore, Delanoe and Ge [11] proved a regularity result under restriction on the measures on C^4-deformation of the round spheres (see also [29]). The main aim of this paper is to prove the following result:

Theorem 1.2. *Any C^4-deformation of the round sphere $(\mathbb{S}^2, g^{\mathrm{can}})$ satisfies $\mathcal{TCP}$.*

We notice that the above theorem is the first regularity result for optimal transport maps allowing for perturbations of the standard metric on the sphere without any additional assumption on the measures. In particular this shows that, if we sligthly perturbs the sphere into an ellipsoid, then $\mathcal{TCP}$ holds true.

Furthermore, quite surprisingly the method of our proof allows to easily deduce as a byproduct the strict convexity of all injectivity domains on perturbations of the two sphere. This geometric result is to our knowledge completely new (see [20] where the authors deal with nonfocal manifolds):

Theorem 1.3. *On a C^4-deformation of the round sphere $(\mathbb{S}^2, g^{\mathrm{can}})$, all injectivity domains are strictly convex.*

It is known [18, 28] that a necessary condition to prove the continuity of optimal transport maps is the so-called Ma-Trudinger-Wang condition (in short MTW condition). This condition is expressed in terms of the fourth derivatives of the cost function, and so makes sense on the domain on smoothness of the distance function, that is outside the cut locus. Another important condition to prove regularity results is the so-called c-convexity of the target domain (see [21, 18]), which in the case of the squared Riemannian distance corresponds to the convexity of all injectivity domains (see (2.4)). So, to obtain regularity results on small deformations of the sphere, on the one hand one has to prove the stability of the MTW condition, and on the other hand one needs to show that the convexity of the injectivity domain is stable under small perturbations. Up to now it was not known whether the convexity of the injectivity domains is stable under small perturbations of the metric, except in the nonfocal case (see [7, 15, 20]). Indeed the boundaries of the injectivity domains depend on the global geometry of the manifold, and this makes the convexity issue very difficult. Theorem 1.3 above is the first general result in this direction.

Our strategy to deal with these problems is to introduce a variant of the MTW condition, which coincides with the usual one up to the cut locus, but that can be extended up to the first conjugate point (see Paragraph 2.2). In this way, since our extended MTW condition is defined up to the first conjugate time, all we really need is the convexity of the nonfocal domains (see (2.1)), which can be shown to be stable under small C^4-perturbation of the metric (see Paragraph 5.2). Thus, in Theorems 3.2 and 3.6 we prove that the strict convexity of nonfocal domains, together with our extended MTW condition, allows to adapt the argument in [20] (changing in a careful way the function to which one has to apply the MTW condition) to conclude the validity of $\mathcal{TCP}$. Moreover, as shown in Theorem 3.4 and Remark 3.5, the strategy of our proof of Theorem 3.2 allows to easily deduce the (strict) convexity of the injectivity domains. Since the assumptions of Theorem 3.2 are satisfied by C^4-deformation of $(\mathbb{S}^2, g^{\mathrm{can}})$, Theorems 1.2 and 1.3 follow.

The paper is organized as follows: in Section 2 we recall some basic facts in symplectic geometry, and we introduce what we call the extended Ma-Trudinger-Wang condition $\mathcal{MTW}(K,C)$. In Section 3 we show how $\mathcal{MTW}(K,C)$, together with the strict convexity of the cotangent nonfocal domains, allows to prove the strict convexity of the injectivity domains and $\mathcal{TCP}$ on a general Riemannian manifold. In Section 4 we prove the stability of $\mathcal{MTW}(K,C)$ under C^4-deformation of $(\mathbb{S}^2, g^{\mathrm{can}})$. Then, in Section 5 we collect several remarks showing other cases when our results apply, and explaining why our continuity result cannot be easily improved to higher regularity. Finally, in the appendix we show that the standard sphere $(\mathbb{S}^n, g^{\mathrm{can}})$ satisfies $\mathcal{MTW}(K_0, K_0)$ for some $K_0 > 0$.

Acknowledgements: we warmly thank Cédric Villani for stimulating our interest on the problem. We also acknowledge the anonimous referee for useful comments and for spotting a mistake in a preliminary version of the paper.

2 The extended MTW condition

2.1

In the sequel, (M,g) always denotes a smooth compact connected Riemannian manifold of dimension n, and we denote by d its Riemannian distance. We denote by TM the tangent bundle and by $\pi : TM \to M$ the canonical projection. A point in TM is denoted by (x,v), with $x \in M$ and $v \in T_xM = \pi^{-1}(x)$. For $v \in T_xM$, the norm $\|v\|_x$ is $g_x(v,v)^{1/2}$. For every $x \in M$, $\exp_x : T_xM \to M$ stands for the *exponential mapping* from x, and $\mathrm{cut}(x)$ for the *cut locus* from x (i.e. the closure of the set of points $y \neq x$ where the distance function from x $d(x,\cdot)$ is not differentiable). We denote by T^*M the cotangent bundle and by $\pi^* : T^*M \to M$ the canonical projection. A point in T^*M will be denoted by (x,p), with $x \in M$ and $p \in T^*M$ a linear form on the vector space T_xM. For every $p \in T_x^*M$ and $v \in T_xM$, we denote by $\langle p, v\rangle$ the action of p on v. The dual metric and norm on T^*M are respectively denoted by $g_x(\cdot,\cdot)$ and $\|\cdot\|_x$. The cotangent bundle is endowed with its standard symplectic structure ω. A local chart

$$\varphi : U \subset M \to \varphi(U) \subset \mathbb{R}^n$$

for M induces on T^*M a natural chart

$$T^*\varphi : T^*U \to T^*(\varphi(U)) = \varphi(U) \times (\mathbb{R}^n)^*.$$

This gives coordinates $(x_1,\ldots,x_n)$ on U, and so coordinates $(x_1,\ldots,x_n,p_1,\cdots,p_n)$ on T^*U such that the symplectic form is given by $\omega = dx \wedge dp$ on T^*U. Such a set of local coordinates on T^*M is called *symplectic*. Fix $\theta = (x,p) \in T^*M$. We recall that a subspace $E \subset T_\theta(T^*M)$ is called *Lagrangian* if it is a n-dimensional vector subspace where the symplectic bilinear form $\omega_\theta : T_\theta(T^*M) \times T_\theta(T^*M) \to \mathbb{R}$ vanishes. The tangent space $T_\theta(T^*M)$ splits as a direct sum of two Lagrangian subspaces: the *vertical subspace* $V_\theta = \ker(d_\theta\pi^*)$ and the *horizontal subspace* H_θ given by the kernel of the connection map $C_\theta : T_\theta(T^*M) \to T_x^*M$ defined as

$$C_\theta(\chi) := D_t\Gamma(0) \qquad \forall \chi \in T_\theta(T^*M),$$

where $t \in (-\varepsilon,\varepsilon) \mapsto (\gamma(t),\Gamma(t)) \in T^*M$ is a smooth curve satisfying $(\gamma(0),\Gamma(0)) = (x,p)$ and $(\dot\gamma(0),\dot\Gamma(0)) = \chi$, and where $D_t\Gamma$ denotes the covariant derivative of Γ along the curve γ. Using the isomorphism

$$\begin{array}{rcl} \mathcal{K}_\theta : T_\theta(T^*M) & \longrightarrow & T_xM \times T_x^*M \\ \chi & \longmapsto & \left(d_\theta\pi^*(\chi), C_\theta(\chi)\right), \end{array}$$

we can identify any tangent vector $\chi \in T_\theta(T^*M)$ with its coordinates $(\chi_h,\chi_v) := \mathcal{K}_\theta(\chi)$ in the splitting (H_θ, V_θ). Therefore we have

$$H_\theta \simeq T_xM \times \{0\} \simeq \mathbb{R}^n \times \{0\}$$

and

$$V_\theta \simeq \{0\} \times T_x^*M \simeq \{0\} \times \mathbb{R}^n,$$

so that

$$T_\theta(T^*M) \simeq H_\theta \oplus V_\theta \simeq \mathbb{R}^n \oplus \mathbb{R}^n.$$

If a given n-dimensional vector subspace $E \subset T_\theta(T^*M)$ is transversal to V_θ (i.e. $E \cap V_\theta = \{0\}$), then E is the graph of some linear map $S : H_\theta \to V_\theta$. It can be checked that E is Lagrangian if and only if S is symmetric in a symplectic set of local coordinates. The Hamiltonian vector field X_H of a smooth function $H : T^*M \to \mathbb{R}$ is the vector field on T^*M uniquely defined by $\omega_\theta\big(X_H(\theta), \cdot\big) = -d_\theta H$ for any $\theta \in T^*M$. In a symplectic set of local coordinates, the Hamiltonian equations (i.e. the equations satisfied by any solution of $(\dot{x}, \dot{p}) = X_H\big((x,p)\big)$) are given by $\dot{x} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial x}$. Finally, we recall that the Hamiltonian flow ϕ_t^H of X_H preserves the symplectic form ω. We refer the reader to [1, 6] for more details about the notions of symplectic geometry introduced above.

2.2

Let $H : T^*M \to \mathbb{R}$ be the Hamiltonian canonically associated with the metric g, i.e.

$$H(x,p) = \frac{1}{2}\|p\|_x^2 \qquad \forall\, (x,p) \in T^*M.$$

We denote by ϕ_t^H the Hamiltonian flow on T^*M, that is the flow of the vector field written in a symplectic set of local coordinates as

$$\begin{cases} \dot{x} &= \frac{\partial H}{\partial p}(x,p), \\ \dot{p} &= -\frac{\partial H}{\partial x}(x,p). \end{cases}$$

For every $(x,p) \in T^*M$, we define the Lagrangian subspace $J_{(x,p)} \subset T_{(x,p)}(T^*M) \simeq T_xM \times T_x^*M$ as the pullback of the vertical distribution at $\phi_1^H((x,p))$ by ϕ_1^H, that is

$$J_{(x,p)} := \left(\phi_1^H\right)^* \left(V_{\phi_1^H((x,p))}\right) = \left(\phi_{-1}^H\right)_* \left(V_{\phi_1^H((x,p))}\right) \qquad \forall\, (x,p) \in T^*M.$$

Let $x \in M$ be fixed. We call *cotangent nonfocal domain* of x the open subset of T_x^*M defined as

$$\mathcal{NF}^*(x) := \big\{p \in T_x^*M \mid J_{(x,tp)} \cap V_{(x,tp)} = \{0\} \quad \forall\, t \in (0,1]\big\}. \tag{2.1}$$

It is the set of covectors $p \in T_x^*M \setminus \{0\}$ such that the corresponding geodesic $\gamma : [0,1] \to M$ defined as $\gamma(t) := \pi^* \circ \phi_t^H(x,p)$ has no conjugate points on the interval $(0,1]$. By construction, for every $p \in \mathcal{NF}^*(x)$, the Lagrangian subspace $J_{(x,p)}$ is transversal to the vertical subspace $V_{(x,p)}$ in $T_{(x,p)}(T^*M)$. Hence, there is a linear operator $K(x,p) : T_xM \to T_x^*M$ such that

$$J_{(x,p)} = \big\{\big(h, K(x,p)h\big) \in T_xM \times T_x^*M \mid h \in T_xM\big\}.$$

We are now ready to define our extended Ma-Trudinger-Wang tensor.

Definition 2.1. *We call* extended Ma-Trudinger-Wang tensor *(abbreviated $\mathcal{MTW}$ tensor), the mixed tensor field given by*

$$\widehat{\mathfrak{S}}(x,p) \cdot (\xi, \eta) := \frac{3}{2} \frac{d^2}{ds^2} \langle K(x, p + s\eta)\xi, \xi\rangle_{|s=0} \qquad \forall\, \xi \in T_xM, \quad \forall\, \eta \in T_x^*M,$$

for every $(x,p) \in \mathcal{NF}^*(x)$.

The above definition extends the definition of the Ma-Trudinger-Wang tensor, which was first introduced in [21] and extensively studied in [11, 12, 13, 17, 18, 19, 20, 23, 24, 29]. Indeed, let $x \neq y \in M$ be such that $y \notin \mathrm{cut}(x)$, and take $\xi \in T_xM, \eta \in T_x^*M$. There is a unique $p \in T_x^*M \setminus \{0\}$ such that the curve $\gamma : [0,1] \to M$ defined by

$$\gamma(t) := \pi^* \circ \phi_t^H(x,p) \qquad \forall\, t \in [0,1]$$

is a minimizing geodesic between x and y. Since such a curve contains no conjugate points, the covector p necessarily belongs to $\mathcal{NF}^*(x)$. Let $v \in T_xM$ (resp. $\tilde{\eta} \in T_xM$) be the unique vector such that $\langle p, w\rangle = g_x(v,w)$ (resp. $\langle \eta, w\rangle = g_x(\tilde{\eta}, w)$) for any $w \in T_xM$. By the definition of $K(\cdot,\cdot)$, if we define $c(x,y) = d^2(x,y)/2$, then for s small one has[1]

$$\langle K(x,p+s\eta)\xi,\xi\rangle = -\frac{d^2}{dt^2} c\big(\exp_x(t\xi), \exp_x(v+s\tilde{\eta})\big)_{|t=0}. \tag{2.2}$$

Thus, differentiating both sides yields

$$\widehat{\mathfrak{S}}(x,p)\cdot(\xi,\eta) = -\frac{3}{2}\frac{d^2}{ds^2}\frac{d^2}{dt^2} c\big(\exp_x(t\xi), \exp_x(v+s\tilde{\eta})\big)_{|s=t=0} = \mathfrak{S}(x,y)\cdot(\xi,\tilde{\eta}), \tag{2.3}$$

where $\mathfrak{S}$ denotes the classical Ma-Trudinger-Wang tensor (see for instance [28, Chapter 12]). Observe that, although the $\mathcal{MTW}$ tensor is not defined at $(x,0)$, the above formula shows that $\widehat{\mathfrak{S}}(x,0)$ is well-defined by continuity, and it is a smooth function near $(x,0)$. Denote by $\mathcal{I}^*(x)$ the *cotangent injectivity domain* of x defined as

$$\mathcal{I}^*(x) := \left\{p \in T_x^*M \mid \pi^* \circ \phi_t^H(x,p) \notin \mathrm{cut}(x) \quad \forall\, t \in [0,1]\right\}, \tag{2.4}$$

and observe that $\mathcal{I}^*(x) \subset \mathcal{NF}^*(x) \cup \{0\}$. The discussion above shows that, up to identify $p, \eta \in T_x^*M$ with $v, \tilde{\eta} \in T_xM$ using the Riemannian metric g_x, the $\mathcal{MTW}$ tensor $\widehat{\mathfrak{S}}$ and the classical MTW tensor $\mathfrak{S}$ coincide on the injectivity domains. For this reason, our tensor can be seen as an extension of the MTW tensor beyond the injectivity domain until the boundary of the nonfocal domain.

It is worth mentioning that in Definition 2.1 it is not necessary to work with the horizontal spaces which are given by the Riemannian connection associated with the metric g. Let $\varphi : U \subset M \to \varphi(U) \subset \mathbb{R}^n$ be a local chart in M and $(x_1, \ldots, x_n, p_1, \cdots, p_n)$ be a symplectic set of local coordinates on T^*U. As we already said before, for every $\theta = (x,p) \in T^*U = \mathbb{R}^n \times (\mathbb{R}^n)^*$, the horizontal space H_θ canonically associated with g is defined as the set of pairs $\chi = (h,v) \in \mathbb{R}^n \times (\mathbb{R}^n)^*$ such that $v = \dot{\Gamma}(0)$, where $t \in (-\epsilon,\epsilon) \mapsto (\gamma(t), \Gamma(t))$ is the smooth curve satisfying $(\gamma(0), \Gamma(0)) = (x,p)$, $\dot{\gamma}(0) = h$, and $\Gamma(t)$ is obtained by parallel transport of the covector $\Gamma(0) = p$ along the curve γ. Writing the ordinary differential equations of parallel transport in local coordinates yields that there exists a bilinear mapping

$$\mathcal{L}_x : \mathbb{R}^n \times (\mathbb{R}^n)^* \longrightarrow (\mathbb{R}^n)^*$$

[1] The equality is a simple consequence of the following fact: for each $x_t := \exp_x(t\xi)$, denote by p_t, q_t the covectors in $T_{x_t}^*M$ and T_y^*M satisfying $\phi_{-1}^H(y,q_t) = (x_t,p_t)$ (with $y := \exp_x(v+s\tilde{\eta})$), $p_0 = p+s\eta$, and $\|p_t\|_{x_t}^2 = \|q_t\|_y^2 = d(x_t,y)^2$. Then

$$\frac{d}{d\tau} c(x_{t+\tau}, y)_{|\tau=0} = \langle d_x c(x_t,y), \dot{x}_t\rangle = -\langle p_t, \dot{x}_t\rangle,$$

so that differentiating again at $t=0$ we obtain

$$\frac{d^2}{dt^2} c(x_t,y)_{|t=0} = -\langle \dot{p}_0, \xi\rangle,$$

where $\dot{p}_0$ denotes the covariant derivate of p_t along the curve x_t, and we used that the covariant derivative of $\dot{x}_t$ along x_t is zero. Hence, since $\phi_1^H(x_t,p_t) = (y,q_t)$, by the definition of $K(x,p+s\eta)$ we easily get $\dot{p}_0 = K(x,p+s\eta)\xi$.

such that the horizontal space H_θ in local coordinates is given by

$$H_\theta = \big\{ \big(h, \mathcal{L}_x(h,p)\big) \mid h \in \mathbb{R}^n \big\}.$$

Denote by $\tilde{H}_\theta$ the horizontal space given by the base space in the symplectic set of local coordinates, that is

$$\tilde{H}_\theta := \mathbb{R}^n \times \{0\}.$$

Since $\tilde{H}_\theta$ is a Lagrangian subspace of $T_\theta(T^*M) = \mathbb{R}^n \times (\mathbb{R}^n)^*$, there is a linear operator $\tilde{K}(x,p) : \mathbb{R}^n \to (\mathbb{R}^n)^*$ such that

$$J_{(x,p)} = \big\{ \big(h, \tilde{K}(x,p)h\big) \in \mathbb{R}^n \times (\mathbb{R}^n)^* \mid h \in \mathbb{R}^n \big\}.$$

Then, for every $h \in \mathbb{R}^n$ we have

$$\tilde{K}(x,p)h = \mathcal{L}_x(h,p) + K(x,p)h.$$

Since $\mathcal{L}_x$ is linear in the p variable, this shows that for every $(x,p) \in \mathcal{NF}^*(x)$

$$\widehat{\mathfrak{S}}(x,p) \cdot (\xi,\eta) = \frac{3}{2} \frac{d^2}{ds^2} \langle \tilde{K}(x,p+s\eta)\xi, \xi \rangle_{|s=0} \qquad \forall\, \xi \in \mathbb{R}^n, \quad \forall\, \eta \in (\mathbb{R}^n)^*$$

(this argument is the symplectic analogous of [28, Remark 12.31]).

It has also to be noticed that the $\mathcal{MTW}$ tensor may be (locally) associated with an extended cost function through formulas like (2.2)-(2.3). More precisely, fix $\theta = (x,p) \in T^*M$ with $p \in \mathcal{NF}^*(x)$. Since the point $y := \pi^* \circ \phi_1^H(x,p)$ is not conjugated with x, thanks to the Inverse Function Theorem there exist an open neighborhood $\mathcal{V}$ of (x,p) in T^*M, and an open neighborhood $\mathcal{W}$ of (x,y) in $M \times M$, such that the function

$$\begin{array}{rccl} \Psi_\theta : \mathcal{V} \subset T^*M & \longrightarrow & \mathcal{W} \subset M \times M \\ (x',p') & \longmapsto & \big(x', \pi^* \circ \phi_1^H(x',p')\big), \end{array}$$

is a smooth diffeomorphism from $\mathcal{V}$ to $\mathcal{W}$. The extended cost function $\hat{c}_\theta : \mathcal{W} \to \mathbb{R}$ which can be (locally) associated with the $\mathcal{MTW}$ tensor at $\theta = (x,p)$ is (uniquely) defined by

$$\hat{c}_\theta(x',y') := \frac{1}{2} \big\| \Psi_\theta^{-1}(x',y') \big\|_{x'}^2 \qquad \forall (x',y') \in \mathcal{W}. \tag{2.5}$$

For the same reasons as before, we have for any $\xi \in T_xM$ and $\eta \in T_x^*M$,

$$\langle K(x,p+s\eta)\xi, \xi \rangle = -\frac{d^2}{dt^2} \hat{c}_\theta\big(\exp_x(t\xi), \pi^* \circ \phi_1^H(x,p+s\eta)\big)_{|t=0} \tag{2.6}$$

which yields

$$\widehat{\mathfrak{S}}(x,p) \cdot (\xi,\eta) = -\frac{3}{2} \frac{d^2}{ds^2} \frac{d^2}{dt^2} \hat{c}_\theta\big(\exp_x(t\xi), \pi^* \circ \phi_1^H(x,p+s\eta)\big)_{|s=t=0}. \tag{2.7}$$

Moreover, if instead we work in a symplectic set of local coordinates $(x_1,\ldots,x_n,p_1,\cdots,p_n)$ on T^*U, then for any $\theta = (x,p) \in T^*U$ with $p \in \mathcal{NF}^*(x)$, and any $\xi \in \mathbb{R}^n, \eta \in (\mathbb{R}^n)^*$, there holds[2]

$$\langle \tilde{K}(x,p+s\eta)\xi, \xi \rangle = -\frac{\partial^2}{\partial x_\xi^2} \hat{c}_\theta\big(x, \pi^* \circ \phi_1^H(x,p+s\eta)\big) \tag{2.8}$$

[2]Set, for t small, $x_t := x + t\xi$, and denote by p_t, q_t the covectors in $T_{x_t}^*M = (\mathbb{R}^n)^*$ and T_y^*M satisfying $\phi_{-1}^H(y,q_t) = (x_t,p_t)$ (with $y := \phi_1^H(x,p+s\eta)$), $p_0 = p + s\eta$, and $\|p_t\|_{x_t}^2 = \|q_t\|_y^2 = 2\hat{c}_\theta(x_t,y)$. Then

$$\frac{d}{d\tau} \hat{c}_\theta(x_{t+\tau},y)_{|\tau=0} = \langle d_x \hat{c}_\theta(x_t,y), \dot{x}_t \rangle = -\langle p_t, \dot{x}_t \rangle = -\langle p_t, \xi \rangle,$$

so that differentiating again at $t = 0$ we obtain

$$\frac{\partial^2}{\partial x_\xi^2} \hat{c}_\theta\big(x, \phi_1^H(x,p+s\eta)\big) = \frac{d^2}{dt^2} \hat{c}_\theta(x_t,y)_{|t=0} = -\langle \dot{p}_0, \xi \rangle,$$

where $\dot{p}_0$ denotes the classical derivative of p_t in the direction η (in local coordinates). Since $\phi_1^H(x_t,p_t) = (y,q_t)$, by the definition of $\tilde{K}(x,p+s\eta)$ we obtain $\dot{p}_0 = \tilde{K}(x,p+s\eta)\xi$.

and

$$\widehat{\mathfrak{S}}(x,p)\cdot(\xi,\eta) = -\frac{3}{2}\frac{\partial^2}{\partial p_\eta^2}\frac{\partial^2}{\partial x_\xi^2}\hat{c}_\theta\big(x, \pi^*\circ\phi_1^H(x,p)\big), \tag{2.9}$$

where $\frac{\partial^2}{\partial x_\xi^2}$ (resp. $\frac{\partial^2}{\partial p_\eta^2}$) denotes the classical second derivative (in coordinates) in the x variable in the direction ξ (resp. in the p variable in the direction η).

The following definition extends the definition of MTW(K,C) introduced in [20]:

Definition 2.2. *Let $K, C \geq 0$. We say that (M,g) satisfies $\mathcal{MTW}(K,C)$ if, for any $(x,p) \in T^*M$ with $p \in \mathcal{NF}^*(x)$,*

$$\widehat{\mathfrak{S}}(x,p)\cdot(\xi,\eta) \geq K\|\xi\|_x^2\|\eta\|_x^2 - C\big|\langle\eta,\xi\rangle\big|\|\xi\|_x\|\eta\|_x \qquad \forall\,\xi\in T_xM, \quad \forall\,\eta\in T_x^*M.$$

In [18] it was observed that, if ξ and $\tilde{\eta}$ are orthogonal unit vectors in T_xM and $\eta := g_x(\tilde{\eta},\cdot) \in T_x^*M$, then $\widehat{\mathfrak{S}}(x,0)\cdot(\xi,\eta) = \mathfrak{S}(x,x)\cdot(\xi,\tilde{\eta})$ coincides with the sectional curvature at x along the plane generated by ξ and $\tilde{\eta}$. More precisely one can prove that, for all $\xi \in T_xM$, $\eta\in T_x^*M$,

$$\widehat{\mathfrak{S}}(x,0)\cdot(\xi,\eta) = \sigma_x(P)\big(\|\xi\|_x^2\|\eta\|_x^2 - |\langle\eta,\xi\rangle|^2\big),$$

where $\sigma_x(P)$ denotes the sectional curvature at x along the plane generated by ξ and $\tilde{\eta}$. In particular, if (M,g) satisfies $\mathcal{MTW}(K,C)$, then its sectional curvature is bounded from below by K. Therefore, if (M,g) satisfies $\mathcal{MTW}(K,C)$ with $K>0$, by Bonnet-Myers Theorem its diameter is bounded, so that M is compact, and in addition the set $\cup_{x\in M}\big(x,\overline{\mathcal{NF}^*(x)}\big)\subset T^*M$ is compact. Furthermore, by the above formula we also see that $\widehat{\mathfrak{S}}(x,0)\cdot(\xi,\eta)=0$ whenever $g_x(\xi,\cdot)$ is parallel to η (since in this case $|\langle\eta,\xi\rangle| = \|\xi\|_x\|\eta\|_x$). Therefore, if (M,g) satisfies $\mathcal{MTW}(K,C)$, then $C\geq K$.

The round sphere $(\mathbb{S}^n, g^{\mathrm{can}})$ and its quotients satisfy $\mathcal{MTW}(K_0,K_0)$ for some $K_0>0$ (see Appendix). Moreover, since the $\mathcal{MTW}$ tensor depends only on the Hamiltonian geodesic flow, if a given Riemannian manifold (M,g) satisfies $\mathcal{MTW}(K,C)$, then its quotients as well as its coverings satisfy $\mathcal{MTW}(K,C)$. The aim of this paper is to show that the $\mathcal{MTW}(K,C)$ condition, together with the strict convexity of the cotangent nonfocal domains, allows to prove the strict convexity of all cotangent injectivity domains and $\mathcal{TCP}$. As a corollary, we will obtain Theorems 1.2 and 1.3.

3 Extended regularity, convexity of injectivity domains, and $\mathcal{TCP}$

3.1

Our strategy is to show that an extended version of the *uniform regularity property* introduced by Loeper and Villani in [20] is sufficient to obtain $\mathcal{TCP}$. Our definition of extended regularity is in some sense stronger than the one given by Loeper and Villani, as it takes into account what happens until the boundary of the cotangent nonfocal domain, and besides requires its convexity. On the other hand we do not require the uniform convexity of the injectivity domains, which is an assumption much more complicated to check than the convexity of the nonfocal domains (see [7]). Moreover our definition has the advantage that it allows to deduce the convexity of all injectivity domains as an immediate corollary (see Theorem 3.4). We notice that, since our definition of extended regularity involves the geodesic Hamiltonian flow (as we want to be able to cross the cut locus), it cannot be expressed only in term of the cost function $c = d^2/2$. For this reason we will say what means for a Riemannian manifold (M,g) to be *(stricly) regular*, while in [20] the authors defined what means $d^2/2$ being uniformly regular.

Definition 3.1 (Extended regularity). *We say that (M,g) is* regular *(resp.* strictly regular*) if there are $\rho,\kappa>0$, and a function $f\in C_c^\infty([0,1])$ with $f\geq 0$ and $\{f>0\}=(1/4,3/4)$, such that*

(a) for every $x\in M$, $\overline{\mathcal{NF}^*(x)}$ is convex (resp. strictly convex),

(b) for every $\bar x\in M$, let $(p_t)_{0\leq t\leq 1}$ be a C^2 curve drawn in $\mathcal{NF}^*(\bar x)\cup\{0\}$ with $p_0\neq p_1\in\overline{\mathcal{I}^*(\bar x)}$, and let $y_t:=\pi^*\circ\phi_1^H(\bar x,p_t)$. Then there exists $\lambda>0$ such that the following holds: let $x\in M$. If

$$\|\ddot p_t\|_{\bar x}\leq \rho\, d(\bar x,x)\,\|\dot y_t\|_{y_t}^2 \qquad \forall\, t\in(0,1), \tag{3.1}$$

then, for any $t\in[0,1]$,

$$d(x,y_t)^2-\|p_t\|_{\bar x}^2\geq \min\Big(d(x,y_0)^2-d(\bar x,y_0)^2,\, d(x,y_1)^2-d(\bar x,y_1)^2\Big)+\lambda f(t)\, d(\bar x,x)^2. \tag{3.2}$$

Moreover, given a family of curves $(p_t)_{0\leq t\leq 1}$ as above such that $(p_t)_{0\leq t\leq 1}$ vary inside a compact subset of $\mathcal{NF}^*(\bar x)\cup\{0\}$ and $\|p_1-p_0\|_{\bar x}$ is uniformly bounded away from 0, the constant $\lambda>0$ can be chosen to be the same for all curves.

One of the motivations of the above definition is that, roughly speaking, the extended regularity is an "integral" manifestation of the extended $\mathcal{MTW}$ condition:

Theorem 3.2. *Assume that there exist $K,C>0$ such that*

(i) for every $x\in M$, $\overline{\mathcal{NF}^*(x)}$ is convex,

(ii) (M,g) satisfies $\mathcal{MTW}(K,C)$.

Then (M,g) is regular.

The above theorem is indeed a simple consequence of the following lemma, combined with an approximation argument:

Lemma 3.3. *Let (M,g) be a Riemannian manifold satisfying $\mathcal{MTW}(K,C)$ for some $K,C>0$, and assume that $\overline{\mathcal{NF}^*(x)}$ is convex for all $x\in M$. Let $\bar x\in M$, and let $(p_t)_{0\leq t\leq 1}$ be a C^2 curve drawn in $\mathcal{NF}^*(\bar x)\cup\{0\}\subset T_{\bar x}^*M$, with $p_0\neq p_1\in\mathcal{I}^*(\bar x)$. For any $t\in(0,1)$, set $y_t:=\pi^*\circ\phi_1^H(\bar x,p_t)$, and suppose that*

$$\|\ddot p_t\|_{\bar x}\leq \frac{K}{6}d(\bar x,x)\|\dot y_t\|_{y_t}^2 \qquad \forall\, t\in(0,1). \tag{3.3}$$

Assume further that $x\in\mathrm{cut}(y_t)$ only for a finite set of times $0<t_1<\ldots<t_{N-1}<1$. Finally, let $f\in C_c^\infty([0,1])$ be as in Definition 3.1. Then, for any $t\in[0,1]$,

$$d(x,y_t)^2-\|p_t\|_{\bar x}^2\geq \min\Big(d(x,y_0)^2-d(\bar x,y_0)^2,\, d(x,y_1)^2-d(\bar x,y_1)^2\Big)+\lambda f(t) d(\bar x,x)^2, \tag{3.4}$$

where

$$\lambda:=\min\left\{\frac{K}{2C\,\mathrm{diam}(M)\|\dot f\|_\infty}\inf_{1/4\leq t\leq 3/4}\big(\|\dot y_t\|_{y_t}\big),\ \frac{K}{12\|\ddot f\|_\infty}\inf_{1/4\leq t\leq 3/4}\big(\|\dot y_t\|_{y_t}^2\big)\right\}.$$

Note that, since M has sectional curvature bounded from below by $K>0$ (thanks to $\mathcal{MTW}(K,C)$), then $\mathrm{diam}(M)$ is finite.

Proof of Theorem 3.2. Let $\bar{x}$, x, $(p_t)_{0\leq t\leq 1}$ and $(y_t)_{0\leq t\leq 1}$ be as in Definition 3.1. Up to slightly reduce ρ, by density and the approximation lemma proved in [14, Section 2] we may assume that $y_0, y_1 \notin \mathrm{cut}(x)$ and that y_t meets $\mathrm{cut}(x)$ only at finitely many times $t_1, \ldots, t_{N-1}$, all the other conditions in Definition 3.1 being unchanged.

Since $p_t \in \mathcal{NF}^*(\bar{x}) \cup \{0\}$ for all $t \in [0,1]$, we have

$$\mathrm{dist}\big(p_t, \partial\big(\mathcal{NF}^*(\bar{x}) \cup \{0\}\big)\big) > 0 \qquad \forall\, t \in [0,1]. \tag{3.5}$$

Moreover, as $\|\dot{y}_t\|_{y_t} \leq C_0 \|\dot{p}_t\|_{\bar{x}}$ for some constant $C_0 > 0$ depending only on M[3], thanks to (3.3) we deduce that $\dot{p}_t \neq 0$ for all $t \in (0,1)$. Indeed, if not, by (3.1) we would get

$$\left|\frac{d}{dt}\|\dot{p}_t\|_{\bar{x}}\right| \leq \rho\, d(\bar{x}, x) C_0^2 \|\dot{p}_t\|_{\bar{x}}^2,$$

and Gronwall Lemma would imply $\dot{p}_t \equiv 0$, which contradicts $p_0 \neq p_1$. Hence, combining (3.5) with the fact that $\dot{p}_t \neq 0$ for $t \in (0,1)$, we obtain

$$\|\dot{y}_t\|_{y_t} > 0 \qquad \forall\, t \in (0,1), \tag{3.6}$$

which by continuity implies $\inf_{1/4\leq t\leq 3/4}\big(\|\dot{y}_t\|_{y_t}\big) > 0$. Moreover, if we take a family of curves $(p_t)_{0\leq t\leq 1}$ inside a compact subset of $\mathcal{NF}^*(\bar{x}) \cup \{0\}$, with $\|p_1 - p_0\|_{\bar{x}}$ is uniformly bounded away from 0, it is easy to see by compactness that there exists a constant $\delta_0 > 0$ such that $\inf_{1/4\leq t\leq 3/4}\big(\|\dot{y}_t\|_{y_t}\big) \geq \delta_0$ for all curves $(p_t)_{0\leq t\leq 1}$. Then the theorem follows easily from Lemma 3.3. □

The proof of Lemma 3.3 is strongly inspired by the proof of [20, Theorem 3.1], which uses a variant of the tecniques introduced in [17, Section 4]. However the main difference with respect to the preceding proofs is in the fact that, since our curve $t \mapsto p_t$ can exit from $\mathcal{I}^*(\bar{x})$, we have to change carefully the function to which one applies the $\mathcal{MTW}(K,C)$ condition. The advantage of our choice of such a function is that it allows to deduce a stronger result, where we bound from below $d(x, y_t)^2 - \|p_t\|_{\bar{x}}^2$ instead of $d(x,y_t)^2 - d(\bar{x}, y_t)^2$. This fact is crucial to deduce the (strict) convexity of all cotangent injectivity domains.

Proof of Lemma 3.3. First of all, without loss of generality we can assume that $x \neq \bar{x}$. Indeed, if $x = \bar{x}$ we simply write (3.4) for a sequence $(x_k)_{k\in\mathbb{N}}$, with $x_k \neq \bar{x}$, and then let $x_k \to \bar{x}$. Thus, we suppose $d(\bar{x}, x) > 0$.

Since $p_t \in \mathcal{NF}^*(\bar{x}) \cup \{0\}$ for all $t \in [0,1]$, as in the proof of Theorem 3.2 we have

$$\mathrm{dist}\big(p_t, \partial\big(\mathcal{NF}^*(\bar{x}) \cup \{0\}\big)\big) > 0 \quad \text{and} \quad \|\dot{y}_t\|_{y_t} > 0 \qquad \forall\, t \in [0,1].$$

Set $t_0 = 0$, $t_N = 1$, and define $h : [0,1] \to \mathbb{R}$ by

$$h(t) := -c(x, y_t) + \frac{\|p_t\|_{\bar{x}}^2}{2} + \delta f(t) \qquad \forall t \in [0,1],$$

where $c(x,y) = d^2(x,y)/2$ and $\delta := \lambda d(\bar{x}, x)^2$. Let us first show that h cannot have a maximum point on an interval of the form (t_j, t_{j+1}). For every $t \in (t_j, t_{j+1})$, since $y_t \notin \mathrm{cut}(x)$, h is a smooth function of t. We fix $t \in (t_j, t_{j+1})$, and we compute $\dot{h}(t)$ and $\ddot{h}(t)$.

As in Paragraph 2.2, define the extended cost function $\hat{c} := \hat{c}_{(\bar{x}, p_t)}$ in an open set $\mathcal{W}$ of $M \times M$ containing $(\bar{x}, y_t)$ as

$$\hat{c}(z,y) := \frac{1}{2}\|\exp_z^{-1}(y)\|_z^2 = \frac{1}{2}\|\exp_y^{-1}(z)\|_y^2 \qquad \forall (z,y) \in \mathcal{W},$$

[3]Actually, since $\mathcal{MTW}(K,C)$ implies that the sectional curvature of M is bounded from below by $K > 0$ (see the discussion after Definition 2.2), $C_0 = 1$ would work.

where $\exp_z^{-1}$ (resp. $\exp_y^{-1}$) denotes a local inverse for $\exp_z$ (resp. $\exp_y$) near $\bar{x}$ (resp. y_t). Hence, for s close to t, we can write

$$h(s) = -c(x, y_s) + \hat{c}(\bar{x}, y_s) + \delta f(s).$$

Moreover the identity $p_s = -d_x \hat{c}(\bar{x}, y_s)$ holds. Take a local chart in an open set $U \subset M$ containing y_t and consider the associated symplectic set of local coordinates $(y^1, \cdots, y^n, q^1, \cdots, q^n)$ in T^*U. Then, as in [20, Proof of Theorem 3.1] we can easily compute $\dot{y}_t$ and $\ddot{y}_t$ at time t: using Einstein convention of summation over repeated indices, we get

$$\dot{y}^i = -\hat{c}^{x_i y_j} \dot{p}_j, \qquad \ddot{y}^i = -\hat{c}^{x_i y_k} \hat{c}_{x_k y_\ell y_j} \dot{y}^\ell \dot{y}^j - \hat{c}^{x_i y_j} \ddot{p}_j,$$

everything being evaluated at $(\bar{x}, y_t)$ and at time t. Here we used the notation $\hat{c}^{x_i y_j}$ for the inverse of $\hat{c}_{x_i y_j} = (d_{xy} \hat{c})_{ij}$, which denotes the second partial derivatives of $\hat{c}$ in the x_i and y_j variables. Let us define $q_t := -d_y c(x, y_t)$, $\bar{q}_t := -d_y \hat{c}(\bar{x}, y_t)$. Then we easily get

$$\dot{h}(t) = \langle \hat{q}_t, \dot{y}_t \rangle + \delta \dot{f}(t),$$

$$\begin{aligned}\ddot{h}(t) = -\Big(\big[c_{y_i y_j}(x, y_t) - \hat{c}_{y_i y_j}(\bar{x}, y_t)\big] + \hat{c}_{x_k y_i y_j}(\bar{x}, y_t) \hat{c}^{x_\ell y_k}(\bar{x}, y_t) \big(\hat{q}_t\big)_\ell\Big) \dot{y}^i \dot{y}^j& \\ - \hat{c}^{x_i y_j}(\bar{x}, y_t) \, \big(\hat{q}_t\big)_i \big(\ddot{p}_t\big)_j + \delta \ddot{f}(t),&\end{aligned}$$

where $\hat{q}_t := q_t - \bar{q}_t$. Now, using (2.8), we obtain

$$\begin{aligned}\ddot{h}(t) &= \langle \tilde{K}(y_t, q_t) \dot{y}_t, \dot{y}_t \rangle - \langle \tilde{K}(y_t, \bar{q}_t) \dot{y}_t, \dot{y}_t \rangle - \frac{d}{ds} \langle \tilde{K}(y_t, \bar{q}_t + s\hat{q}_t) \dot{y}_t, \dot{y}_t \rangle_{|s=0} \\ &\quad + \langle v_t, \ddot{p}_t \rangle + \delta \ddot{f}(t), \end{aligned} \tag{3.7}$$

where $v_t := -\big(d_{xy} \hat{c}(\bar{x}, y_t)\big)^{-1} \hat{q}_t$. Recalling (2.9), that is

$$\frac{d^2}{ds^2} \langle \tilde{K}(y_t, \bar{q}_t + s\hat{q}_t) \dot{y}_t, \dot{y}_t \rangle = \frac{d^2}{ds^2} \langle K(y_t, \bar{q}_t + s\hat{q}_t) \dot{y}_t, \dot{y}_t \rangle,$$

(3.7) can be written as

$$\begin{aligned}\ddot{h}(t) &= \int_0^1 (1-s) \left(\frac{d^2}{ds^2} \langle K(y_t, \bar{q}_t + s\hat{q}_t) \dot{y}_t, \dot{y}_t \rangle \right) ds + \langle v_t, \ddot{p}_t \rangle + \delta \ddot{f}(t) \\ &= \frac{2}{3} \int_0^1 (1-s) \, \widehat{\mathfrak{S}}(y_t, \bar{q}_t + s\hat{q}_t) \cdot \big(\dot{y}_t, \hat{q}_t\big) \, ds + \langle v_t, \ddot{p}_t \rangle + \delta \ddot{f}(t).\end{aligned}$$

By $\mathcal{MTW}(K, C)$ and $\int_0^1 (1-s) ds = 1/2$, we get

$$\ddot{h}(t) \geq \frac{1}{3} \Big(K \|\hat{q}_t\|_{y_t} \|\dot{y}_t\|_{y_t} - C |\langle \hat{q}_t, \dot{y}_t \rangle| \Big) \|\hat{q}_t\|_{y_t} \|\dot{y}_t\|_{y_t} + \langle v_t, \ddot{p}_t \rangle + \delta \ddot{f}(t).$$

We now claim that the function h cannot have any maximum on (t_j, t_{j+1}). Indeed, if $\dot{h}(t) = 0$ for some $t \in (t_j, t_{j+1})$, we have

$$0 = \dot{h}(t) = \langle \hat{q}_t, \dot{y}_t \rangle + \delta \dot{f}(t),$$

which implies $|\langle \hat{q}_t, \dot{y}_t \rangle| \leq \delta |\dot{f}(t)|$. Thus

$$\ddot{h}(t) \geq \frac{1}{3} \Big(K \|\hat{q}_t\|_{y_t} \|\dot{y}_t\|_{y_t} - C\delta |\dot{f}(t)| \Big) \|\hat{q}_t\|_{y_t} \|\dot{y}_t\|_{y_t} - |\langle v_t, \ddot{p}_t \rangle| - \delta |\ddot{f}(t)|,$$

and so by (3.3)

$$\ddot{h}(t) \geq \frac{1}{3}\Big(K\|\hat{q}_t\|_{y_t}\|\dot{y}_t\|_{y_t} - C\delta|\dot{f}(t)|\Big)\|\hat{q}_t\|_{y_t}\|\dot{y}_t\|_{y_t} - \frac{K}{6}\|v_t\|_{\bar{x}} d(\bar{x},x)\|\dot{y}_t\|^2_{y_t} - \delta|\ddot{f}(t)|.$$

Since $\mathcal{MTW}(K,C)$ implies that the sectional curvature of M is bounded below by $K>0$ (see the discussion after Definition 2.2), the exponential mapping $\exp_{y_t}$ is 1-Lipschitz, which implies that the norm of the operator $\big(d_{xy}\hat{c}(\bar{x},y_t)\big)^{-1} : T^*_{y_t}M \to T_{\bar{x}}M$ is bounded by 1. Hence, we have

$$\|v_t\|_{\bar{x}} \leq \|\hat{q}_t\|_{y_t}, \qquad d(\bar{x},x) \leq \|\hat{q}_t\|_{y_t}, \tag{3.8}$$

which give

$$\begin{aligned} \ddot{h}(t) &\geq \frac{1}{3}\Big(K\|\hat{q}_t\|_{y_t}\|\dot{y}_t\|_{y_t} - C\delta|\dot{f}(t)|\Big)\|\hat{q}_t\|_{y_t}\|\dot{y}_t\|_{y_t} - \frac{K}{6}\|\hat{q}_t\|^2_{y_t}\|\dot{y}_t\|^2_{y_t} - \delta|\ddot{f}(t)| \\ &\geq \Big(\frac{K}{6}\|\hat{q}_t\|_{y_t}\|\dot{y}_t\|_{y_t} - \frac{C}{3}\delta|\dot{f}(t)|\Big)\|\hat{q}_t\|_{y_t}\|\dot{y}_t\|_{y_t} - \delta|\ddot{f}(t)|. \end{aligned}$$

If $t \notin [1/4,3/4]$ then $\dot{f}(t) = \ddot{f}(t) = 0$, which combined with (3.8) implies

$$\ddot{h}(t) \geq \frac{K}{6}\|\hat{q}_t\|^2_{y_t}\|\dot{y}_t\|^2_{y_t} \geq \frac{K}{6} d(\bar{x},x)^2\|\dot{y}_t\|^2_{y_t}.$$

On the other hand, if $t \in [1/4,3/4]$, recalling that $\delta = \lambda d(\bar{x},x)^2$ and the definition of λ we obtain

$$\frac{C}{3}\delta|\dot{f}(t)| = \frac{C}{3}\lambda d(\bar{x},x)^2|\dot{f}(t)| \leq \frac{K}{12}d(\bar{x},x)\|\dot{y}_t\|_{y_t} \leq \frac{K}{12}\|\hat{q}_t\|_{y_t}\|\dot{y}_t\|_{y_t},$$

which using again (3.8) and the definition of λ yields

$$\ddot{h}(t) \geq \frac{K}{12}\|\hat{q}_t\|^2_{y_t}\|\dot{y}_t\|^2_{y_t} - \delta|\ddot{f}(t)| \geq \frac{K}{12}d(\bar{x},x)^2\|\dot{y}_t\|^2_{y_t} - \delta|\ddot{f}(t)| \geq \frac{K}{24}d(\bar{x},x)^2\|\dot{y}_t\|^2_{y_t}. \tag{3.9}$$

In any case, thanks to (3.6), we have $\ddot{h}(t) > 0$, which shows that h cannot have a maximum on any interval (t_j,t_{j+1}). Thus, as h is continuous on $[0,1]$, it has to achieve its maximum at one of the times t_j $(0 \leq j \leq N)$. The goal is to show that necessarily $j=0$ or $j=N$. Indeed, let $j \in \{1,\dots,N-1\}$. We first note that, since $t \to c(x,y_t) = d^2(x,y_t)/2$ is semiconcave and $t \mapsto \|p_t\|^2_{\bar{x}}$ is smooth, $h(t)$ is semiconvex. If $\dot{h}$ is continuous at t_j and $\dot{h}(t_j) \neq 0$, clearly t_j cannot be a maximum of h. The same is true if $\dot{h}$ is discontinuous at t_j, because by semiconvexity necessarily $\dot{h}(t_j^+) > \dot{h}(t_j^-)$. Finally, if $\dot{h}$ is continuous at t_j and $\dot{h}(t_j) = 0$, the same computations as before show that $\ddot{h}(t)$ is strictly positive when t is close to (but different from) t_j, which implies that h cannot have a maximum at t_j. The only possibility left out for h is to achieve its maximum at $t_0 = 0$ or $t_N = 1$, and we obtain (3.2). □

3.2

One main feature of our definition of regularity is that it immediately implies the convexity of all injectivity domains:

Theorem 3.4. *Let (M,g) be a regular Riemannian manifold. Then $\overline{\mathcal{I}^*(x)}$ is convex for all $x \in M$.*

Proof. It is sufficient to show that $\mathcal{I}^*(x)$ is convex for all $x \in M$. We fix $\bar{x} \in M$, and choose $x = \bar{x}$ in the definition of regularity. Then, considering $p_t := tp_1 + (1-t)p_0$ with $p_0 \neq p_1 \in \mathcal{I}^*(\bar{x}) \subset \mathcal{NF}^*(\bar{x})$, we get

$$d(\bar{x},y_t) \geq \|p_t\|_{\bar{x}} \qquad \forall\, t \in [0,1].$$

This gives $p_t \in \mathcal{I}^*(\bar{x})$, that is $\mathcal{I}^*(\bar{x})$ is convex. □

Remark 3.5. One can actually prove that, if (M, g) is strictly regular, then strict convexity of all injectivity domains holds. Indeed, let us assume by contradiction that there are $p_0 \neq p_1 \in \mathcal{NF}^*(\bar{x}) \cap \overline{\mathcal{I}^*(\bar{x})}$ such that that $tp_1 + (1-t)p_0 \notin \mathcal{I}^*(\bar{x})$ for all $t \in (0,1)$. Consider $x = \bar{x}$ in the proof of Lemma 3.3, and using the same notation we perturb the segment into a curve $(p_t)_{0 \le t \le 1}$ such that $p_t \in \mathcal{NF}^*(\bar{x}) \setminus \overline{\mathcal{I}^*(\bar{x})}$ for all $t \in (1/4, 3/4)$, and $\|\ddot{p}_t\|_{y_t} \le \frac{K}{6} \|q_t - \bar{q}_t\|_{y_t} \|\dot{y}_t\|_{y_t}^2$, where $q_t := -d_y c(\bar{x}, y_t) \neq \bar{q}_t$ for $t \in (1/4, 3/4)$ (this can always be done as the segment $tp_1 + (1-t)p_0$ lies at positive distance from $\partial\big(\mathcal{NF}^*(\bar{x})\big)$ for $t \in (1/4, 3/4)$). The function $h(t) = -c(\bar{x}, y_t) + \|p_t\|_{\bar{x}}^2/2$ is identically zero on $[0, 1/4] \cup [3/4, 1]$, and it is smooth on $(1/4, 3/4)$. Since now $\delta = 0$, by the first inequality in (3.9) we get

$$\ddot{h}(t) \ge \frac{K}{12} \|q_t - \bar{q}_t\|_{y_t}^2 \|\dot{y}_t\|_{y_t}^2 > 0 \qquad \forall\, t \in (1/4, 3/4)$$

whenever $\dot{h}(t) = 0$. This fact implies that h cannot attain a maximum on $(1/4, 3/4)$. Hence, for any $t \in (1/4, 3/4)$,

$$0 = d(\bar{x}, y_t)^2 - d(\bar{x}, y_t)^2 < \|p_t\|_{\bar{x}}^2 - d(\bar{x}, y_t)^2 \le 2 \max_{s \in [0,1]} h(s) = 0,$$

a contradiction.

3.3

Here is another main motivation for our definition of extended regularity:

Theorem 3.6. *Any Riemannian manifold (M, g) which is stricly regular satisfies $\mathcal{TCP}$.*

The proof of this theorem closely follows the proof of [20, Theorem 5.1].

Proof. Condition (i) in the definition of $\mathcal{TCP}$ insures that μ gives no mass to set with σ-finite $(n-1)$-dimensional Hausdorff measure. Thanks to McCann's Theorem read in the Hamiltonian formalism, there exists a unique optimal transport map between μ and ν, which is given by $T(x) = \pi^* \circ \phi_1^H(x, d_x\psi)$, where ψ is a semiconvex function. Moreover $d_x\psi \in \overline{\mathcal{I}^*(x)} \subset \overline{\mathcal{NF}^*(x)}$ at all point of differentiability of ψ. Since $\overline{\mathcal{NF}^*(x)}$ is convex for all x, the subdifferential of ψ satisfies $\partial\psi(x) \subset \overline{\mathcal{NF}^*(x)}$ for all $x \in M$. To prove that ψ is C^1, we need to show that $\partial\psi(x)$ is everywhere a singleton. The proof is by contradiction.

Assume that there is $\bar{x} \in M$ such that $p_0 \neq p_1 \in \partial\psi(\bar{x})$. Let $y_0 = \exp_{\bar{x}} p_0$, $y_1 = \exp_{\bar{x}} p_1$. Thus $y_i \in \partial^c\psi(\bar{x})$, i.e.

$$\psi(\bar{x}) + \frac{1}{2} d^2(\bar{x}, y_i) = \min_{x \in M} \left\{ \psi(x) + \frac{1}{2} d^2(x, y_i) \right\}, \qquad i = 0, 1.$$

In particular

$$\frac{1}{2} d^2(x, y_i) - \frac{1}{2} d^2(\bar{x}, y_i) \ge \psi(\bar{x}) - \psi(x), \qquad \forall x \in M, \forall i = 0, 1. \tag{3.10}$$

Fix $\eta_0 > 0$ small (the smallness to be chosen later). For $\varepsilon \in (0,1)$, we define $D_\varepsilon \subset \overline{\mathcal{NF}^*(\bar{x})}$ as follows: D_ε consists of the set of points $p \in T_{\bar{x}}^* M$ such that there exists a path $(p_t)_{0 \le t \le 1} \subset \overline{\mathcal{NF}^*(\bar{x})}$ from p_0 to p_1 such that, if we set $y_t := \pi^* \circ \phi_1^H(\bar{x}, p_t)$, we have $\ddot{p}_t = 0$ for $t \notin [1/4, 3/4]$, $\|\ddot{p}_t\|_{y_t} \le \varepsilon \eta_0 \|\dot{y}_t\|_{y_t}^2$ for $t \in [1/4, 3/4]$, and $p = p_t$ for some $t \in [1/4, 3/4]$.

By the strict convexity of $\overline{\mathcal{NF}^*(\bar{x})}$, if η_0 is sufficiently small then D_ε lies a positive distance σ away from $\partial\big(\overline{\mathcal{NF}^*(\bar{x})}\big)$ for all $\varepsilon \in (0,1)$. Thus all paths $(p_t)_{0 \le t \le 1}$ used in the definition of D_ε satisfy

$$\|\dot{y}_t\|_{y_t} \ge c \|p_0 - p_1\|_{\bar{x}} \qquad \forall\, t \in [1/4, 3/4],$$

with c independent of $\varepsilon \in (0,1)$. Moreover condition (3.1) is satisfied if $\eta_0 \le \rho$ and $d(\bar{x}, x) \ge \varepsilon$. By simple geometric consideration, we see that D_ε contains a parallelepiped E_ε centered at $(p_0 + p_1)/2$ with one side of length $\sim \|p_0 - p_1\|_{\bar{x}}$, and the other sides of length $\sim \varepsilon \|p_0 - p_1\|_{\bar{x}}^2$, such that all points y in such parallelepiped can be written as y_t for some $t \in [1/3, 2/3]$, with y_t as in the definition of D_ε. Therefore

$$\mathscr{L}^n(E_\varepsilon) \ge c\varepsilon^{n-1},$$

with $\mathscr{L}^n$ denoting the Lebesgue measure on $T_{\bar{x}}M$. Since E_ε lies a positive distance from $\partial\big(\overline{\mathcal{NF}^*(\bar{x})}\big)$, we obtain

$$\mathrm{vol}\,(Y_\varepsilon) \sim \mathscr{L}^n(E_\varepsilon) \ge c\varepsilon^{n-1}, \qquad Y_\varepsilon := \pi^* \circ \phi_1^H(\bar{x}, E_\varepsilon).$$

We then apply Theorem 3.2 to the paths $(p_t)_{0 \le t \le 1}$ used in the definition of D_ε to obtain that, for any $y \in Y_\varepsilon$ and $x \in M \setminus B_\varepsilon(\bar{x})$,

$$d(x,y)^2 - d(\bar{x},y)^2 \ge \min\Big(d(x,y_0)^2 - d(\bar{x},y_0)^2,\, d(x,y_1)^2 - d(\bar{x},y_1)^2\Big) + \lambda \inf_{t \in [1/3,2/3]} \big(f(t)\big) d(\bar{x},x)^2,$$

with $\inf_{t \in [1/3,2/3]} \big(f(t)\big) > 0$. Combining this inequality with (3.10), we conclude that

$$\text{for any } y \in Y_\varepsilon, \qquad y \notin \partial^c \psi(x) \qquad \forall\, x \in M \setminus B_\varepsilon(\bar{x}).$$

This implies that all the mass brought into Y_ε by the optimal map comes from $B_\varepsilon(\bar{x})$, and so

$$\mu(B_\varepsilon(\bar{x})) \ge \nu(Y_\varepsilon).$$

We now remark that by condition (ii) in the definition of $\mathcal{TCP}$ and a standard covering argument, there exists a constant $c_1 > 0$ such that $\nu(A) \ge c_1 \mathrm{vol}\,(A)$ for all Borel sets $A \subset M$. Thus, as $\mu(B_\varepsilon(\bar{x})) \le o(1)\varepsilon^{n-1}$ and $\nu(Y_\varepsilon) \ge c_1 \mathrm{vol}\,(Y_\varepsilon) \ge c\varepsilon^{n-1}$, we obtain a contradiction as $\varepsilon \to 0$. □

4 Stability of $\mathcal{MTW}$ near the sphere

In this section, we show that any C^4-deformation of the standard 2-sphere satisfies $\mathcal{MTW}(K,C)$ for some $K, C > 0$. Let (M,g) be a smooth, compact and positively curved surface. It is easy to show that, for every $x \in M$, the set $\overline{\mathcal{NF}^*(x)} \subset T_x^*M$ is a compact set with smooth boundary (see [7]). In fact it can even be shown that, if $M = \mathbb{S}^2$ and g is C^4-close to the round metric g^{can}, then all the $\overline{\mathcal{NF}^*(x)}$ are uniformly convex. Thus Theorems 1.2 and 1.3 are a consequence of Theorems 3.2, 3.6, 3.4 and Remark 3.5, together with the following result:

Theorem 4.1. *There exist $K, C > 0$ such that any C^4-deformation of $(\mathbb{S}^2, g^{\mathrm{can}})$ satisfies $\mathcal{MTW}(K,C)$.*

Proof. For $\varepsilon \ge 0$, let g^ε be a smooth metric on $\mathbb{S}^2$ such that $\|g^\varepsilon - g^{\mathrm{can}}\|_{C^4} \le \varepsilon$ (so that $g^0 = g^{\mathrm{can}}$). We see $\mathbb{S}^2$ as the sphere centered at the origin with radius one in $\mathbb{R}^3$, so that we can identify covectors with vectors. Let $x \in \mathbb{S}^2$; we observe that for g^0 the set $\mathcal{NF}^*(x) \cup \{0\}$ corresponds to the open ball centered at x with radius π intersected with the hyperplan tangent to $\mathbb{S}^2$ at x, while for g^ε the nonfocal domain $\mathcal{NF}^*_\varepsilon(x) \cup \{0\}$ is a C^2-perturbation of the ball. Our aim is to show that there exist $K, C > 0$ such that, if $\varepsilon > 0$ is sufficiently small, then for every $x \in \mathbb{S}^2$ and every $p \in \mathcal{NF}^*_\varepsilon(x)$ one has

$$\widehat{\mathfrak{S}}(x,p) \cdot (\xi, \eta) \ge K g_x^\varepsilon(\xi,\xi) g_x^\varepsilon(\eta,\eta) - C \big|\langle \xi, \eta \rangle\big| g_x^\varepsilon(\xi,\xi)^{1/2} g_x^\varepsilon(\eta,\eta)^{1/2}$$

for all $\xi \in T_x\mathbb{S}^n$, $\eta \in T_x^*\mathbb{S}^n$. Since the property holds true on $(\mathbb{S}^2, g^{\mathrm{can}})$ with $K = C = K_0$ for some $K_0 > 0$ (see Appendix), the above inequality holds by continuity on $(\mathbb{S}^2, g^\varepsilon)$ with

$K = K_0/2$ and $C = 2K_0$ when p is uniformly away from the boundary of $\mathcal{NF}^*_\varepsilon(x) \cup \{0\}$, and ε is sufficiently small. Thus all we have to prove is that the above inequality remains true when p is close to $\partial\big(\mathcal{NF}^*_\varepsilon(x) \cup \{0\}\big)$. Moreover, by the homogeneity of $(\mathbb{S}^2, g^{\mathrm{can}})$, it will suffice to prove the estimate only for a fixed point $x \in \mathbb{S}^2$ and along a fixed geodesic $t \mapsto \pi^* \circ \phi^H_t(x,p)$.

Consider the stereographic projection of the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ from the north pole $N = (0,0,1)$ onto the space $\mathbb{R}^2 \simeq \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. The projection of some point $x = (x_1, x_2, x_3) \in \mathbb{S}^2$ is given by

$$\sigma(x) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right).$$

The function σ is a smooth diffeomorphism from $\mathbb{S}^2 \setminus \{N\}$ onto $\mathbb{R}^2$, whose inverse is

$$\sigma^{-1}(y) = \left(\frac{2y_1}{1+|y|^2}, \frac{2y_2}{1+|y|^2}, \frac{|y|^2-1}{1+|y|^2}\right) \qquad \forall\, y = (y_1, y_2) \in \mathbb{R}^2,$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^2$. The pushforward of the metric g^ε under σ induces a metric on $\mathbb{R}^2$, that we still denote by g^ε, and which for $\varepsilon = 0$ is given by

$$g^0_y(v,v) = \frac{4}{(1+|y|^2)^2}|v|^2 \qquad \forall\, y, v \in \mathbb{R}^2.$$

Note that since we work in $\mathbb{R}^2$, we can identify covectors with vectors. We denote by $H^\varepsilon(y,p)$ the Hamiltonian canonically associated to g^ε, which for $\varepsilon = 0$ is given by

$$H^0(y,p) = \frac{(1+|y|^2)^2}{8}|p|^2 \qquad \forall\, y, p \in \mathbb{R}^2.$$

We observe that $\|H^\varepsilon - H^0\|_{C^4} \lesssim \varepsilon$. The Hamiltonian system associated to H^ε is

$$\left\{\begin{array}{lcl} \dot y^\varepsilon & = & \frac{\partial H^\varepsilon}{\partial p}(y^\varepsilon, p^\varepsilon) \\ \dot p^\varepsilon & = & -\frac{\partial H^\varepsilon}{\partial y}(y^\varepsilon, p^\varepsilon), \end{array}\right.$$

and the linearized Hamiltonian system along a given solution $(y^\varepsilon(t), p^\varepsilon(t))$ is

$$\left\{\begin{array}{lcl} \dot h^\varepsilon & = & \frac{\partial^2 H^\varepsilon}{\partial p^2}(y^\varepsilon, p^\varepsilon)q^\varepsilon + \frac{\partial^2 H^\varepsilon}{\partial y \partial p}(y^\varepsilon, p^\varepsilon)h^\varepsilon \\ \dot q^\varepsilon & = & -\frac{\partial^2 H^\varepsilon}{\partial y \partial p}(y^\varepsilon, p^\varepsilon)q^\varepsilon - \frac{\partial^2 H^\varepsilon}{\partial y^2}(y^\varepsilon, p^\varepsilon)h^\varepsilon \end{array}\right.$$

We note that h^ε is a Jacobi vector field along the geodesic $t \mapsto y^\varepsilon(t)$.

Set $Y = (-1,0) \in \mathbb{R}^2$, and consider the geodesic $\theta^\varepsilon_\alpha$ starting from Y with velocity of norm 1 and making angle α (computed with respect to g^ε) with the line $\{x_2 = 0\}$. For $\varepsilon = 0$ this geodesic is given by

$$\theta^0_\alpha(t) = \left(\frac{\cos(t-\pi)}{1-\cos(\alpha)\sin(t-\pi)}, \frac{\sin(\alpha)\sin(t-\pi)}{1-\cos(\alpha)\sin(t-\pi)}, 0\right),$$

and it is a minimizing geodesic between Y and $(1,0)$. Since the first conjugate time for θ_α is $t = \pi$ for all α, and we are perturbing the metric in the C^4 topology, there exists a smooth function $\alpha \mapsto t^\varepsilon_c(\alpha)$ such that $t^\varepsilon_c(\alpha)$ is the first conjugate time of $\theta^\varepsilon_\alpha$, and $\|t^\varepsilon_c(\alpha) - \pi\|_{C^2} \lesssim \varepsilon$. Fix $\bar t^\varepsilon(\alpha) \in (0, t^\varepsilon_c(\alpha))$, and set $V^\varepsilon_\alpha := \bar t^\varepsilon(\alpha)\dot\theta^\varepsilon_\alpha(0)$.

As we notice in Paragraph 2.2, in order to compute the $\mathcal{MTW}$ tensor at $(Y, V^\varepsilon_\alpha)$, we can use the horizontal space given by any choice of a symplectic set of local coordinates. Therefore, we can work with the standard splitting $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$ and take as horizontal vertical spaces

$$H_{(Y,V^\varepsilon_\alpha)} = \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^4 \quad \text{and} \quad V_{(Y,V^\varepsilon_\alpha)} = \{0\} \times \mathbb{R}^2 \subset \mathbb{R}^4.$$

In the sequel, we shall denote by $K^\varepsilon(Y,V_\alpha^\varepsilon)$ (which corresponds to the operator $\tilde{K}^\varepsilon(Y,V_\alpha^\varepsilon)$ of Paragraph 2.2) the 2×2 matrix such that

$$J_{(Y,V_\alpha^\varepsilon)} = \left\{ (h, K^\varepsilon(Y,V_\alpha^\varepsilon)h) \in \mathbb{R}^2\times\mathbb{R}^2 \mid h\in\mathbb{R}^2 \right\}.$$

By an easy rescaling, it is not difficult to see that $K^\varepsilon(Y,V_\alpha^\varepsilon)$ is given by

$$K^\varepsilon(Y,V_\alpha^\varepsilon) = \bar{t}^\varepsilon(\alpha) S_\alpha^\varepsilon(\bar{t}^\varepsilon(\alpha))$$

where $S_\alpha^\varepsilon(\bar{t}^\varepsilon(\alpha))$ is the 2×2 symmetric matrix such that any solution of the linearized Hamiltonian system along $\theta_\alpha^\varepsilon$ starting from $(h, S_\alpha^\varepsilon(\bar{t})h)$ satisfies $h^\varepsilon(\bar{t}^\varepsilon(\alpha))=0$. Let us compute $S_\alpha^\varepsilon(\bar{t}^\varepsilon(\alpha))$.

Let $E_1^\varepsilon(t,\alpha) := \dot\theta_\alpha^\varepsilon(t)$ and $E_2^\varepsilon(t,\alpha)$ be a basis of parallel vector fields along $\theta_\alpha^\varepsilon$ such that $g^\varepsilon\big(E_1^\varepsilon(0,\alpha), E_2^\varepsilon(0,\alpha)\big)=0$. For $\varepsilon=0$ they are given by

$$E_1^0(t,\alpha) := \dot\theta_\alpha(t) = \left(\frac{\cos(\alpha)-\sin(t-\pi)}{(1-\cos(\alpha)\sin(t-\pi))^2}, \frac{\sin(\alpha)\cos(t-\pi)}{(1-\cos(\alpha)\sin(t-\pi))^2}, 0 \right),$$

$$E_2^0(t,\alpha) := \left(\frac{-\sin(\alpha)\cos(t-\pi)}{(1-\cos(\alpha)\sin(t-\pi))^2}, \frac{\cos(\alpha)-\sin(t-\pi)}{(1-\cos(\alpha)\sin(t-\pi))^2}, 0 \right),$$

Let $(h_\alpha^\varepsilon, q_\alpha^\varepsilon)$ be a solution of the linearized Hamiltonian system along $\theta_\alpha^\varepsilon$ such that $h_\alpha^\varepsilon(\bar{t}^\varepsilon(\alpha))=0$ for some $\bar{t}^\varepsilon(\alpha)\in(0,t_c^\varepsilon(\alpha))$. Since $E_1^\varepsilon(t,\alpha), E_2^\varepsilon(t,\alpha)$ form a basis of parallel vector fields along θ_α, there are two smooth functions $u_{\alpha,1}^\varepsilon(t), u_{\alpha,2}^\varepsilon(t)$ such that

$$h_\alpha^\varepsilon(t) = u_{\alpha,1}^\varepsilon(t)E_1^\varepsilon(t,\alpha) + u_{\alpha,2}^\varepsilon(t)E_2^\varepsilon(t,\alpha).$$

If we denote by $u_\alpha^\varepsilon(t)\in\mathbb{R}^2$ the vector $(u_{\alpha,1}^\varepsilon(t), u_{\alpha,2}^\varepsilon(t))$, and by $A_\alpha^\varepsilon(t)$ the 2×2 matrix having $E_1^\varepsilon(t,\alpha)$ and $E_2^\varepsilon(t,\alpha)$ as column vectors, we can write

$$h_\alpha^\varepsilon(t) = A_\alpha^\varepsilon(t)u_\alpha^\varepsilon(t).$$

As $h_\alpha^\varepsilon(t)$ is a Jacobi vector field along $\theta_\alpha^\varepsilon$ we have

$$\ddot h_\alpha^\varepsilon + R^\varepsilon\big(h_\alpha^\varepsilon, \dot\theta_\alpha^\varepsilon\big)\dot\theta_\alpha^\varepsilon = 0,$$

where R^ε denotes the Riemann tensor, and using the symmetries of R^ε we get

$$\begin{aligned} R^\varepsilon\big(h_\alpha^\varepsilon,\dot\theta_\alpha^\varepsilon\big)\dot\theta_\alpha^\varepsilon = R^\varepsilon\big(h_\alpha^\varepsilon, E_1^\varepsilon\big)E_1^\varepsilon &= R^\varepsilon\big(u_{\alpha,1}^\varepsilon E_1^\varepsilon + u_{\alpha,2}^\varepsilon E_2^\varepsilon, E_1^\varepsilon\big)E_1^\varepsilon = R^\varepsilon\big(u_{\alpha,2}^\varepsilon E_2^\varepsilon, E_1^\varepsilon\big)E_1^\varepsilon \\ &= u_{\alpha,2}^\varepsilon\big\langle R^\varepsilon(E_2^\varepsilon,E_1^\varepsilon)E_1^\varepsilon, E_1^\varepsilon\big\rangle E_1^\varepsilon + u_{\alpha,2}^\varepsilon\big\langle R^\varepsilon(E_2^\varepsilon,E_1^\varepsilon)E_1^\varepsilon, E_2^\varepsilon\big\rangle E_2^\varepsilon \\ &= u_{\alpha,2}^\varepsilon\big\langle R^\varepsilon(E_2^\varepsilon,E_1^\varepsilon)E_1^\varepsilon, E_2^\varepsilon\big\rangle E_2^\varepsilon. \end{aligned}$$

This gives

$$\left\{ \begin{array}{rcl} \ddot u_{\alpha,1}^\varepsilon(t,\alpha) &=& 0 \\ \ddot u_{\alpha,2}^\varepsilon(t,\alpha) &=& -g^\varepsilon\big(R^\varepsilon(E_2^\varepsilon,E_1^\varepsilon)E_1^\varepsilon, E_2^\varepsilon\big)u_{\alpha,2}^\varepsilon(t,\alpha), \end{array}\right.$$

so that

$$\left\{ \begin{array}{rcl} u_{\alpha,1}^\varepsilon(t,\alpha) &=& \lambda_1+\lambda_2 t \\ \ddot u_{\alpha,2}^\varepsilon(t,\alpha) &=& -g^\varepsilon\big(R^\varepsilon(E_2^\varepsilon,E_1^\varepsilon)E_1^\varepsilon, E_2^\varepsilon\big)u_{\alpha,2}^\varepsilon(t,\alpha) = -a^\varepsilon(t,\alpha)u_{\alpha,2}^\varepsilon(t,\alpha), \end{array}\right.$$

where $(t,\alpha)\mapsto a^\varepsilon(t,\alpha)$ is close to 1 in C^2-topology. Hence we can write

$$u_\alpha^\varepsilon(t) = U_1^\varepsilon(t,\alpha)u_\alpha^\varepsilon(0) + U_2^\varepsilon(t,\alpha)\dot u_\alpha^\varepsilon(0),$$

with

$$U_1^\varepsilon(t,\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & f_1^\varepsilon(t,\alpha) \end{pmatrix}, \qquad U_2^\varepsilon(t,\alpha) = \begin{pmatrix} t & 0 \\ 0 & f_2^\varepsilon(t,\alpha) \end{pmatrix},$$

with $f_1^\varepsilon(t,\alpha)$ and $f_2^\varepsilon(t,\alpha)$ are close to $\cos(t)$ and $\sin(t)$ in the C^2-norm, respectively. Recalling that $h_\alpha^\varepsilon(\bar{t}^\varepsilon(\alpha)) = 0$, we have

$$0 = U_1^\varepsilon(t_\alpha^\varepsilon,\alpha)u_\alpha^\varepsilon(0) + U_2^\varepsilon(t_\alpha^\varepsilon,\alpha)\dot{u}_\alpha^\varepsilon(0) \quad \Longrightarrow \quad \dot{u}_\alpha^\varepsilon(0) = -\big[U_2^\varepsilon(t_\alpha^\varepsilon,\alpha)\big]^{-1}U_1^\varepsilon(t_\alpha^\varepsilon,\alpha)u_\alpha^\varepsilon(0).$$

and as $u_\alpha^\varepsilon(0) = \big(A_\alpha^\varepsilon(0)\big)^{-1}h_\alpha^\varepsilon(0)$ we get

$$\begin{aligned}
\dot{h}_\alpha^\varepsilon(0) &= \dot{A}_\alpha^\varepsilon(0)u_\alpha^\varepsilon(0) + A_\alpha^\varepsilon(0)\dot{u}_\alpha^\varepsilon(0) = \Big[\dot{A}_\alpha^\varepsilon(0) - A_\alpha^\varepsilon(0)\big[U_2^\varepsilon(t_\alpha^\varepsilon,\alpha)\big]^{-1}U_1^\varepsilon(t_\alpha^\varepsilon,\alpha)\Big]u_\alpha^\varepsilon(0)\\
&= \Big[\dot{A}_\alpha^\varepsilon(0) - A_\alpha^\varepsilon(0)\big[U_2^\varepsilon(t_\alpha^\varepsilon,\alpha)\big]^{-1}U_1^\varepsilon(t_\alpha^\varepsilon,\alpha)\Big]\big(A_\alpha^\varepsilon(0)\big)^{-1}h_\alpha^\varepsilon(0).
\end{aligned}$$

Hence from the linearized Hamiltonian system we finally obtain

$$\begin{aligned}
q_\alpha^\varepsilon(0) &= \Big[\frac{\partial^2 H^\varepsilon}{\partial p^2}(Y,\dot{\theta}_\alpha^\varepsilon(0))\Big]^{-1}\dot{h}_\alpha^\varepsilon(0) - \Big[\frac{\partial^2 H^\varepsilon}{\partial p^2}(Y,\dot{\theta}_\alpha^\varepsilon(0))\Big]^{-1}\frac{\partial^2 H^\varepsilon}{\partial y\partial p}(Y,\dot{\theta}_\alpha^\varepsilon(0))h_\alpha^\varepsilon(0)\\
&= S_\alpha^\varepsilon(t)h_\alpha^\varepsilon(0),
\end{aligned}$$

with

$$S_\alpha^\varepsilon(t) = C_\alpha^\varepsilon - \Big[\frac{\partial^2 H^\varepsilon}{\partial p^2}(Y,\dot{\theta}_\alpha^\varepsilon(0))\Big]^{-1}A_\alpha^\varepsilon(0)\big[U_2^\varepsilon(t_\alpha^\varepsilon,\alpha)\big]^{-1}U_1^\varepsilon(t_\alpha^\varepsilon,\alpha)\big(A_\alpha^\varepsilon(0)\big)^{-1},$$

where

$$C_\alpha^\varepsilon = \Big[\frac{\partial^2 H^\varepsilon}{\partial p^2}(Y,\dot{\theta}_\alpha^\varepsilon(0))\Big]^{-1}\Big[\dot{A}_\alpha^\varepsilon(0)\big(A_\alpha^\varepsilon(0)\big)^{-1} - \frac{\partial^2 H^\varepsilon}{\partial y\partial p}(Y,\dot{\theta}_\alpha^\varepsilon(0))\Big].$$

Defining

$$N_\alpha^\varepsilon(t) := tf_2^\varepsilon(t,\alpha)\big[U_2^\varepsilon(t,\alpha)\big]^{-1}U_1^\varepsilon(t,\alpha) = \begin{pmatrix} f_2^\varepsilon(t,\alpha) & 0 \\ 0 & tf_1^\varepsilon(t,\alpha)\end{pmatrix}$$

we can write

$$S_\alpha^\varepsilon(t) = C_\alpha^\varepsilon - \frac{1}{tf_2^\varepsilon(t,\alpha)}\Big[\frac{\partial^2 H^\varepsilon}{\partial p^2}(Y,\dot{\theta}_\alpha^\varepsilon(0))\Big]^{-1}A_\alpha^\varepsilon(0)N_\alpha^\varepsilon(t)\big(A_\alpha^\varepsilon(0)\big)^{-1}.$$

We observe that $N_\alpha^\varepsilon(t)$ is smooth up to $t = t_c^\varepsilon(\alpha)$. As a matter of fact we remark that, for $\varepsilon = 0$,

$$C_\alpha^0 = I\Big[-R_{2\alpha}R_{-\alpha} - 2\begin{pmatrix} -\cos(\alpha) & 0 \\ \sin(\alpha) & 0\end{pmatrix}\Big] = 2\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ -\sin(\alpha) & -\cos(\alpha)\end{pmatrix},$$

and $A_\alpha^0(0) = R_\alpha$, where we used the notation

$$R_\alpha = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha)\end{pmatrix}.$$

Let us now focus on the matrix

$$\Big[\frac{\partial^2 H^\varepsilon}{\partial p^2}(Y,\dot{\theta}_\alpha^\varepsilon(0))\Big]^{-1}A_\alpha^\varepsilon(0)N_\alpha^\varepsilon(t)\big(A_\alpha^\varepsilon(0)\big)^{-1}.$$

Denoting by G^ε the matrix associated to the metric g^ε at the point Y, we have $\big[\frac{\partial^2 H^\varepsilon}{\partial p^2}(Y,\dot{\theta}_\alpha^\varepsilon(0))\big]^{-1} = G^\varepsilon$. Moreover

$$\begin{aligned}
G^\varepsilon A_\alpha^\varepsilon(0)N_\alpha^\varepsilon(t)\big(A_\alpha^\varepsilon(0)\big)^{-1} &= (G^\varepsilon)^{1/2}\big[(G^\varepsilon)^{1/2}A_\alpha^\varepsilon(0)\big]N_\alpha^\varepsilon(t)\big(A_\alpha^\varepsilon(0)\big)^{-1}\\
&= (G^\varepsilon)^{1/2}\big[(G^\varepsilon)^{1/2}A_\alpha^\varepsilon(0)\big]N_\alpha^\varepsilon(t)\big[(G^\varepsilon)^{1/2}A_\alpha^\varepsilon(0)\big]^{-1}(G^\varepsilon)^{1/2}.
\end{aligned}$$

Recalling that $A_\alpha^\varepsilon(0) = \Big(E_1^\varepsilon(0,\alpha), E_2^\varepsilon(0,\alpha)\Big)$, we have

$$(G^\varepsilon)^{1/2}A_\alpha^\varepsilon(0) = \Big((G^\varepsilon)^{1/2}E_1^\varepsilon(0,\alpha), (G^\varepsilon)^{1/2}E_2^\varepsilon(0,\alpha)\Big),$$

and since

$$1 = g^\varepsilon\big(E_1^\varepsilon(0,\alpha), E_1^\varepsilon(0,\alpha)\big) = g^\varepsilon\big(E_2^\varepsilon(0,\alpha), E_2^\varepsilon(0,\alpha)\big), \qquad 0 = g^\varepsilon\big(E_1^\varepsilon(0,\alpha), E_2^\varepsilon(0,\alpha)\big),$$

we immediately get that $(G^\varepsilon)^{1/2} A_\alpha^\varepsilon(0)$ is an orthogonal matrix for all α. Thus, there exists $\bar{\alpha}^\varepsilon$ and α^ε such that

$$(G^\varepsilon)^{1/2} A_\alpha^\varepsilon(0) = (G^\varepsilon)^{1/2} A_0^\varepsilon(0) \begin{pmatrix} \cos(\alpha^\varepsilon) & \sin(\alpha^\varepsilon) \\ -\sin(\alpha^\varepsilon) & \cos(\alpha^\varepsilon). \end{pmatrix} =: R_{\bar{\alpha}^\varepsilon} R_{\alpha^\varepsilon}$$

(that is, $\bar{\alpha}^\varepsilon$ is the angle between $(1,0) = \dot{\theta}_0^\varepsilon$ and $(G^\varepsilon)^{1/2}(1,0) = (G^\varepsilon)^{1/2}\dot{\theta}_0^\varepsilon$), and we obtain

$$\begin{aligned}
&\frac{1}{tf_2^\varepsilon(t,\alpha)}\Big(G^\varepsilon A_\alpha^\varepsilon(0) N_\alpha^\varepsilon(t) \big(A_\alpha^\varepsilon(0)\big)^{-1}\Big) \\
&\qquad = (G^\varepsilon)^{1/2}\left[\frac{1}{tf_2^\varepsilon(t,\alpha)}\Big((G^\varepsilon)^{1/2} A_\alpha^\varepsilon(0) N_\alpha^\varepsilon(t) \big((G^\varepsilon)^{1/2} A_\alpha^\varepsilon(0)\big)^{-1}\Big)\right](G^\varepsilon)^{1/2} \\
&\qquad\qquad = (G^\varepsilon)^{1/2} R_{\bar{\alpha}^\varepsilon}\left[\frac{1}{tf_2^\varepsilon(t,\alpha)}\Big(R_{\alpha^\varepsilon} N_\alpha^\varepsilon(t) R_{-\alpha^\varepsilon}\Big)\right] R_{-\bar{\alpha}^\varepsilon}(G^\varepsilon)^{1/2}.
\end{aligned}$$

A simple computations gives that $R_{\alpha^\varepsilon} N^\varepsilon(t,\alpha^\varepsilon) R_{-\alpha^\varepsilon}$ is equal to the matrix

$$\begin{pmatrix} \cos^2(\alpha^\varepsilon) f_2^\varepsilon(t,\alpha^\varepsilon) + t\sin^2(\alpha^\varepsilon) f_1^\varepsilon(t,\alpha^\varepsilon) & -\cos(\alpha^\varepsilon)\sin(\alpha^\varepsilon)\big(f_2^\varepsilon(t,\alpha^\varepsilon) - tf_1^\varepsilon(t,\alpha^\varepsilon)\big) \\ -\cos(\alpha^\varepsilon)\sin(\alpha^\varepsilon)\big(f_2^\varepsilon(t,\alpha^\varepsilon) - tf_1^\varepsilon(t,\alpha^\varepsilon)\big) & \sin^2(\alpha^\varepsilon) f_2^\varepsilon(t,\alpha^\varepsilon) + t\cos^2(\alpha^\varepsilon) f_1^\varepsilon(t,\alpha^\varepsilon) \end{pmatrix},$$

so that

$$\begin{aligned}
\frac{1}{tf_2^\varepsilon(t,\alpha^\varepsilon)}\big(R_{\alpha^\varepsilon} N^\varepsilon(t,\alpha^\varepsilon) R_{-\alpha^\varepsilon}\big) &= \frac{1}{t}\begin{pmatrix} \cos^2(\alpha^\varepsilon) & -\cos(\alpha^\varepsilon)\sin(\alpha^\varepsilon) \\ -\cos(\alpha^\varepsilon)\sin(\alpha^\varepsilon) & \sin^2(\alpha^\varepsilon) \end{pmatrix} \\
&\quad + \frac{1}{f_2^\varepsilon(t,\alpha^\varepsilon)}\begin{pmatrix} \sin^2(\alpha^\varepsilon) f_1^\varepsilon(t,\alpha^\varepsilon) & \cos(\alpha^\varepsilon)\sin(\alpha^\varepsilon) f_1^\varepsilon(t,\alpha^\varepsilon) \\ \cos(\alpha^\varepsilon)\sin(\alpha^\varepsilon) f_1^\varepsilon(t,\alpha^\varepsilon) & \cos^2(\alpha^\varepsilon) f_1^\varepsilon(t,\alpha^\varepsilon) \end{pmatrix}.
\end{aligned}$$

We now define $T^\varepsilon(s)$ as the g^ε-norm at Y of the vector $(v,0)+s\eta$, and $\alpha^\varepsilon(s)$ is the angle (computed with respect to g^ε) between $(v,0)$ and $(v,0)+s\eta$. In the sequel we denote by $\hat{f}_1^\varepsilon$ (resp. $\hat{f}_2^\varepsilon$) the function $s \mapsto f_1^\varepsilon(T^\varepsilon(s), \alpha^\varepsilon(s))$ (resp. $s \mapsto f_2^\varepsilon(T^\varepsilon(s), \alpha^\varepsilon(s))$), and by $\dot{\hat{f}}_1^\varepsilon, \ddot{\hat{f}}_1^\varepsilon$ (resp. $\dot{\hat{f}}_2^\varepsilon, \ddot{\hat{f}}_2^\varepsilon$) its first and second derivative. We want to compute the second derivative of $K^\varepsilon(s) := K^\epsilon(Y,(v,0)+s\eta)$ for $T^\varepsilon(0)$ close to $t_c^\varepsilon(\alpha) \sim \pi$, so that $1/\hat{f}_2^\varepsilon(0) \sim 1/\sin(t_c^\varepsilon(\alpha))$ will be dominant with respect to all other terms. Thanks to the computations made above, we have

$$\begin{aligned}
\frac{d^2}{ds^2}\left\{K^\varepsilon(s)\right\}_{|s=0} &= \frac{d^2}{ds^2}\left\{T^\varepsilon(s) C^\varepsilon_{\alpha^\varepsilon(s)}\right\}_{|s=0} \\
&\quad + (G^\varepsilon)^{1/2} R_{\bar{\alpha}^\varepsilon}\left[M_0^\varepsilon + \frac{1}{\hat{f}_2^\varepsilon(0)} M_1^\varepsilon + \frac{1}{\hat{f}_2^\varepsilon(0)^2} M_2^\varepsilon + \frac{1}{\hat{f}_2^\varepsilon(0)^3} M_3^\varepsilon\right] R_{-\bar{\alpha}^\varepsilon}(G^\varepsilon)^{1/2}
\end{aligned}$$

with

$$M_i^\varepsilon = \begin{pmatrix} M_i^\varepsilon(1) & M_i^\varepsilon(2) \\ M_i^\varepsilon(2) & M_i^\varepsilon(3) \end{pmatrix} \qquad \forall\, i = 0,1,2,3,$$

and

$$M_0^\varepsilon(1) = 2\left(\dot{\alpha}^\varepsilon(0)\right)^2, \qquad M_0^\varepsilon(2) = \ddot{\alpha}^\varepsilon(0), \qquad M_0^\varepsilon(3) = -2\left(\dot{\alpha}^\varepsilon(0)\right)^2,$$

and

$$M_1^\varepsilon(1) = -2T^\varepsilon(0)\hat{f}_1^\varepsilon(0)\left(\dot{\alpha}^\varepsilon(0)\right)^2,$$

$$M_1^\varepsilon(2) = -2\hat{f}_1^\varepsilon(0)\dot{T}^\varepsilon(0)\dot{\alpha}^\varepsilon(0) - 2T^\varepsilon(0)\dot{\hat{f}}_1^\varepsilon(0)\dot{\alpha}^\varepsilon(0) - T^\varepsilon(0)\hat{f}_1^\varepsilon(0)\ddot{\alpha}^\varepsilon(0),$$

$$M_1^\varepsilon(3) = -\hat f_1^\varepsilon(0)\ddot T^\varepsilon(0) - 2\dot{\hat f}_1^\varepsilon(0)\dot T^\varepsilon(0) - T^\varepsilon(0)\ddot{\hat f}_1^\varepsilon(0) + 2T^\varepsilon(0)\hat f_1^\varepsilon(0)\left(\dot\alpha^\varepsilon(0)\right)^2,$$

and

$$M_2^\varepsilon(1) = 0, \qquad M_2^\varepsilon(2) = 2T^\varepsilon(0)\hat f_1^\varepsilon(0)\dot{\hat f}_2^\varepsilon(0)\dot\alpha^\varepsilon(0),$$

$$M_2^\varepsilon(3) = 2T^\varepsilon(0)\dot{\hat f}_1^\varepsilon(0)\dot{\hat f}_2^\varepsilon(0) + 2\hat f_1^\varepsilon(0)\dot{\hat f}_2^\varepsilon(0)\dot T^\varepsilon(0) + T^\varepsilon(0)\hat f_1^\varepsilon(0)\ddot{\hat f}_2^\varepsilon(0),$$

and

$$M_3^\varepsilon(1) = M_3^\varepsilon(2) = 0, \qquad M_3^\varepsilon(3) = -2T^\varepsilon(0)\hat f_1^\varepsilon(0)\left(\dot{\hat f}_2^\varepsilon(0)\right)^2.$$

We now observe that $\alpha^\varepsilon(s)$ is given by the angle between the two vectors

$$(G^\varepsilon)^{1/2}\begin{pmatrix} v \\ 0 \end{pmatrix} \quad \text{and} \quad (G^\varepsilon)^{1/2}\left[\begin{pmatrix} v \\ 0 \end{pmatrix} + s\eta\right].$$

Therefore, if we define $\begin{pmatrix} v^\varepsilon \\ 0 \end{pmatrix} = R_{-\bar\alpha^\varepsilon}(G^\varepsilon)^{1/2}\begin{pmatrix} v \\ 0 \end{pmatrix}$ and $\eta^\varepsilon = R_{-\bar\alpha^\varepsilon}(G^\varepsilon)^{1/2}\eta$, we get

$$\alpha^\varepsilon(s) = -\arctan\left(\frac{s\eta_2^\varepsilon}{v^\varepsilon + s\eta_1^\varepsilon}\right)$$

which implies

$$\alpha^\varepsilon(0) = 0, \qquad \dot\alpha^\varepsilon(0) = -\frac{\eta_2^\varepsilon}{v^\varepsilon}, \qquad \ddot\alpha^\varepsilon(0) = \frac{2\eta_1^\varepsilon\eta_2^\varepsilon}{(v^\varepsilon)^2}.$$

Regarding $T^\varepsilon(s)$, we have

$$\begin{aligned} T^\varepsilon(s) &= \left|(G^\varepsilon)^{1/2}\begin{pmatrix} v + s\eta_1 \\ s\eta_2 \end{pmatrix}\right| = \left|R_{-\bar\alpha^\varepsilon}(G^\varepsilon)^{1/2}\begin{pmatrix} v + s\eta_1 \\ s\eta_2 \end{pmatrix}\right| \\ &= \left|\begin{pmatrix} v^\varepsilon + s\eta_1^\varepsilon \\ s\eta_2^\varepsilon \end{pmatrix}\right| = \sqrt{(v^\varepsilon + s\eta_1^\varepsilon)^2 + s^2(\eta_2^\varepsilon)^2}, \end{aligned}$$

hence

$$T^\varepsilon(0) = v^\varepsilon, \qquad \dot T^\varepsilon(0) = \eta_1^\varepsilon, \qquad \ddot T^\varepsilon(0) = \frac{(\eta_2^\varepsilon)^2}{v^\varepsilon}.$$

Moreover $|v^\varepsilon - v| \lesssim \varepsilon|v|$, $|\eta^\varepsilon - \eta| \lesssim \varepsilon|\eta|$. Thus we finally obtain

$$M_0^\varepsilon = \frac{2}{(v^\varepsilon)^2}\begin{pmatrix} (\eta_2^\varepsilon)^2 & \eta_1^\varepsilon\eta_2^\varepsilon \\ \eta_1^\varepsilon\eta_2^\varepsilon & -(\eta_2^\varepsilon)^2 \end{pmatrix},$$

$$M_1^\varepsilon = \begin{pmatrix} -\frac{2\hat f_1^\varepsilon(0)}{v^\varepsilon}(\eta_2^\varepsilon)^2 & 2\dot{\hat f}_1^\varepsilon(0)\eta_2^\varepsilon \\ 2\dot{\hat f}_1^\varepsilon(0)\eta_2^\varepsilon & \frac{\hat f_1^\varepsilon(0)}{v^\varepsilon}(\eta_2^\varepsilon)^2 - 2\dot{\hat f}_1^\varepsilon(0)\eta_1^\varepsilon - v^\varepsilon\ddot{\hat f}_1^\varepsilon(0) \end{pmatrix},$$

$$M_2^\varepsilon = \begin{pmatrix} 0 & -2\hat f_1^\varepsilon(0)\dot{\hat f}_2^\varepsilon(0)\eta_2^\varepsilon \\ -2\hat f_1^\varepsilon(0)\dot{\hat f}_2^\varepsilon(0)\eta_2^\varepsilon & 2v^\varepsilon\dot{\hat f}_1^\varepsilon(0)\dot{\hat f}_2^\varepsilon(0) + 2\hat f_1^\varepsilon(0)\dot{\hat f}_2^\varepsilon(0)\eta_1^\varepsilon + v^\varepsilon\hat f_1^\varepsilon(0)\ddot{\hat f}_2^\varepsilon(0) \end{pmatrix},$$

$$M_3^\varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & -2v^\varepsilon\hat f_1^\varepsilon(0)\left(\dot{\hat f}_2^\varepsilon(0)\right)^2 \end{pmatrix}.$$

We note that for $\varepsilon = 0$

$$\begin{aligned} &\frac{d^2}{ds^2}\left\{T^0(s)C^0_{\alpha^0(s)}\right\}_{|s=0} \\ &\qquad = \begin{pmatrix} \ddot T^0(0) - T^0(0)\left(\dot\alpha^0(0)\right)^2 & -2\dot T^0(0)\dot\alpha^0(0) - T^0(0)\ddot\alpha^0(0) \\ -2\dot T^0(0)\dot\alpha^0(0) - T^0(0)\ddot\alpha^0(0) & -\ddot T^0(0) + T^0(0)\left(\dot\alpha^0(0)\right)^2 \end{pmatrix} = 0, \end{aligned}$$

which implies

$$\left| \frac{d^2}{ds^2} \left\{ T^\varepsilon(s) C^\varepsilon_{\alpha^\varepsilon(s)} \right\}_{|s=0} \right| \lesssim \varepsilon |\eta|^2.$$

Therefore, defining $\xi^\varepsilon = R_{-\bar{\alpha}^\varepsilon}(G^\varepsilon)^{1/2}\xi$, we end up with

$$\begin{aligned}\langle \xi, \frac{d^2}{ds^2} \left\{ K^\varepsilon(s) \right\}_{|s=0} \xi \rangle &= \langle \xi^\varepsilon, M_0^\varepsilon \xi^\varepsilon \rangle + \frac{1}{\hat{f}_2^\varepsilon(0)} \langle \xi^\varepsilon, M_1^\varepsilon \xi^\varepsilon \rangle \\ &\quad + \frac{1}{\hat{f}_2^\varepsilon(0)^2} \langle \xi^\varepsilon, M_2^\varepsilon \xi^\varepsilon \rangle + \frac{1}{\hat{f}_2^\varepsilon(0)^3} \langle \xi^\varepsilon, M_3^\varepsilon \xi^\varepsilon \rangle + O(\varepsilon)|\eta|^2|\xi|^2.\end{aligned}$$

We now observe that, as $\hat{f}_1^\varepsilon(0) \sim \cos(v^\varepsilon)$, we have $\hat{f}_1^\varepsilon(0) \le 0$ for $v^\varepsilon \ge 2\pi/3$, and so in this case

$$\begin{aligned}&\frac{1}{\hat{f}_2^\varepsilon(0)^3} M_3^\varepsilon(3)(\xi_2^\varepsilon)^2 + 2\frac{1}{\hat{f}_2^\varepsilon(0)^2} M_2^\varepsilon(2)\xi_1^\varepsilon\xi_2^\varepsilon + \frac{1}{\hat{f}_2^\varepsilon(0)} M_1^\varepsilon(1)(\xi_1^\varepsilon)^2 \\ &\qquad = -\frac{2\hat{f}_1^\varepsilon(0)}{\hat{f}_2^\varepsilon(0)} \left[\sqrt{v^\varepsilon} \frac{\dot{\hat{f}}_2^\varepsilon(0)}{\hat{f}_2^\varepsilon(0)} \xi_2^\varepsilon + \frac{1}{\sqrt{v^\varepsilon}} \eta_2^\varepsilon \xi_1^\varepsilon \right]^2 \ge 0\end{aligned}$$

Therefore for $v^\varepsilon \ge 2\pi/3$

$$\begin{aligned}\langle \xi, \frac{d^2}{ds^2} \left\{ K^\varepsilon(s) \right\}_{|s=0} \xi \rangle &\ge \langle \xi^\varepsilon, M_0^\varepsilon \xi^\varepsilon \rangle \\ &\quad + \frac{1}{\hat{f}_2^\varepsilon(0)} \left[2M_1^\varepsilon(2)\xi_1^\varepsilon\xi_2^\varepsilon + M_1^\varepsilon(3)(\xi_2^\varepsilon)^2 \right] + \frac{1}{\hat{f}_2^\varepsilon(0)^2} M_2^\varepsilon(3)(\xi_2^\varepsilon)^2 + O(\varepsilon)|\eta|^2|\xi|^2.\end{aligned}$$

Now, easy computations give for $i = 1, 2$

$$\dot{\hat{f}}_i^\varepsilon(0) = \frac{\partial f_i^\varepsilon}{\partial t}(v^\varepsilon, 0)\eta_1^\varepsilon - \frac{1}{v^\varepsilon} \frac{\partial f_i^\varepsilon}{\partial \alpha}(v^\varepsilon, 0)\eta_2^\varepsilon,$$

$$\begin{aligned}\ddot{\hat{f}}_i^\varepsilon(0) &= \left[\frac{\partial^2 f_i^\varepsilon}{\partial t^2}(v^\varepsilon, 0) \right] (\eta_1^\varepsilon)^2 + \left[\frac{1}{(v^\varepsilon)^2} \frac{\partial^2 f_i^\varepsilon}{\partial \alpha^2}(v^\varepsilon, 0) + \frac{1}{v^\varepsilon} \frac{\partial f_i^\varepsilon}{\partial t}(v^\varepsilon, 0) \right] (\eta_2^\varepsilon)^2 \\ &\quad + \left[\frac{2}{(v^\varepsilon)^2} \frac{\partial f_i^\varepsilon}{\partial \alpha}(v^\varepsilon, 0) - \frac{2}{v^\varepsilon} \frac{\partial^2 f_i^\varepsilon}{\partial \alpha \partial t}(v^\varepsilon, 0) \right] \eta_1^\varepsilon \eta_2^\varepsilon.\end{aligned}$$

Let us now observe the following: since (as functions of v^ε) $\hat{f}_1^\varepsilon(0)$ and $\hat{f}_2^\varepsilon(0)$ are close to $\cos(v^\varepsilon)$ and $\sin(v^\varepsilon)$ in the C^2-norm respectively, if we define $\ell^\varepsilon := t_c^\varepsilon(0) - v^\varepsilon$ we easily get

$$|\hat{f}_1^\varepsilon(0) + 1| \lesssim \varepsilon + \ell^\varepsilon, \qquad 0 \le \hat{f}_2^\varepsilon(0) \lesssim \varepsilon + \ell^\varepsilon,$$

$$|\dot{\hat{f}}_1^\varepsilon(0)| \lesssim (\varepsilon + \ell^\varepsilon)|\eta^\varepsilon|, \qquad |\dot{\hat{f}}_2^\varepsilon(0) + \eta_1| \lesssim (\varepsilon + \ell^\varepsilon)|\eta^\varepsilon|,$$

$$|\ddot{\hat{f}}_1^\varepsilon(0) - \eta_1^2| \lesssim (\varepsilon + \ell^\varepsilon)|\eta^\varepsilon|^2, \qquad |\ddot{\hat{f}}_2^\varepsilon(0) + \frac{1}{v^\varepsilon}(\eta_2^\varepsilon)^2| \lesssim (\varepsilon + \ell^\varepsilon)|\eta^\varepsilon|^2.$$

From these estimates it is easy to see that

$$|M_2^\varepsilon(3) - 2(\eta_1^\varepsilon)^2 - (\eta_2^\varepsilon)^2| \lesssim (\varepsilon + \ell^\varepsilon)|\eta^\varepsilon|^2 \quad \Longrightarrow \quad M_2^\varepsilon(3) \ge \big(1 - C(\varepsilon + \ell^\varepsilon)\big)\big[2(\eta_1^\varepsilon)^2 + (\eta_2^\varepsilon)^2\big],$$

and

$$|M_1^\varepsilon(2)| \lesssim (\varepsilon + \ell^\varepsilon)|\eta^\varepsilon|\eta_2^\varepsilon.$$

Since $\frac{1}{\hat{f}_2^\varepsilon(0)} \to +\infty$ as $\ell^\varepsilon \to 0$ (i.e. $v^\varepsilon \to t_c^\varepsilon(\alpha)$), we obtain

$$M_0^\varepsilon(3) + \frac{1}{\hat{f}_2^\varepsilon(0)} M_1^\varepsilon(3) + \frac{1}{\hat{f}_2^\varepsilon(0)^2} M_2^\varepsilon(3) \geq \frac{\big(1-\overline{C}(\varepsilon+\ell^\varepsilon)\big)}{\hat{f}_2^\varepsilon(0)^2} |\eta^\varepsilon|^2,$$

$$\left| M_0^\varepsilon(2) + \frac{1}{\hat{f}_2^\varepsilon(0)} M_1^\varepsilon(2) \right| \leq \frac{\overline{C}(\varepsilon+\ell^\varepsilon)}{\hat{f}_2^\varepsilon(0)} |\eta^\varepsilon||\eta_2^\varepsilon|,$$

(from now on, $\overline{C}$ is a positive constant, independent of ε for $\varepsilon > 0$ sufficiently small, which may change from line to line). Hence, combining all together,

$$\begin{aligned}
\langle \xi, \frac{d^2}{ds^2}\left\{K^\varepsilon(s)\right\}_{|s=0}\xi\rangle \quad &\geq \quad \frac{2}{(v^\varepsilon)^2}(\eta_2^\varepsilon)^2(\xi_1^\varepsilon)^2 - \frac{\overline{C}(\varepsilon+\ell^\varepsilon)}{\hat{f}_2^\varepsilon(0)}|\eta_2^\varepsilon||\eta^\varepsilon||\xi_1^\varepsilon||\xi_2^\varepsilon| \\
&\quad + \quad \frac{\big(1-\overline{C}(\varepsilon+\ell^\varepsilon)\big)}{\hat{f}_2^\varepsilon(0)^2}\big[2(\eta_1^\varepsilon)^2 + (\eta_2^\varepsilon)^2\big](\xi_2^\varepsilon)^2 + O(\varepsilon)|\eta|^2|\xi|^2 \\
&\geq \quad \frac{2}{(v^\varepsilon)^2}(\eta_2^\varepsilon)^2(\xi_1^\varepsilon)^2 - \frac{\overline{C}(\varepsilon+\ell^\varepsilon)}{\hat{f}_2^\varepsilon(0)}|\eta_2^\varepsilon||\eta^\varepsilon||\xi_1^\varepsilon||\xi_2^\varepsilon| \\
&\quad + \quad \frac{\big(1-\overline{C}(\varepsilon+\ell^\varepsilon)\big)}{\hat{f}_2^\varepsilon(0)^2}|\eta^\varepsilon|^2(\xi_2^\varepsilon)^2 + O(\varepsilon)|\eta|^2|\xi|^2 \\
&\geq \quad \frac{\big(2-\overline{C}(\varepsilon+\ell^\varepsilon)\big)}{\pi^2}(\eta_2^\varepsilon)^2(\xi_1^\varepsilon)^2 + \frac{\big(1-\overline{C}(\varepsilon+\ell^\varepsilon)\big)}{\hat{f}_2^\varepsilon(0)^2}|\eta^\varepsilon|^2(\xi_2^\varepsilon)^2 + O(\varepsilon)|\eta|^2|\xi|^2 \\
&\geq \quad \frac{\big(2-\overline{C}(\varepsilon+\ell^\varepsilon)\big)}{\pi^2}\big[(\eta_2^\varepsilon)^2(\xi_1^\varepsilon)^2 + |\eta^\varepsilon|^2(\xi_2^\varepsilon)^2\big] + O(\varepsilon)|\eta|^2|\xi|^2.
\end{aligned}$$

From this formula, since $|\eta^\varepsilon - \eta| \leq \overline{C}\varepsilon|\eta|$ and $|\xi^\varepsilon - \xi| \leq \overline{C}\varepsilon|\xi|$, we finally get

$$\begin{aligned}
\frac{3}{2}\langle \xi, \frac{d^2}{ds^2}\left\{K^\varepsilon(s)\right\}_{|s=0}\xi\rangle \quad &\geq \quad \frac{3\big(2-\overline{C}(\varepsilon+\ell^\varepsilon)\big)}{2\pi^2}\big[\eta_2^2\xi_1^2 + |\eta|^2\xi_2^2\big] + O(\varepsilon)|\eta|^2|\xi|^2 \\
&\geq \quad \frac{3\big(2-\overline{C}(\varepsilon+\ell^\varepsilon)\big)}{2\pi^2}\big[\eta_2^2\xi_1^2 + \eta_1^2\xi_2^2\big] + O(\varepsilon)|\eta|^2|\xi|^2 \\
&\geq \quad \frac{3\big(1-\overline{C}(\varepsilon+\ell^\varepsilon)\big)}{2\pi^2}\big[|\eta|^2|\xi|^2 - \langle \eta, \xi\rangle^2\big] + O(\varepsilon)|\eta|^2|\xi|^2 \\
&\geq \quad \frac{3\big(1-\overline{C}(\varepsilon+\ell^\varepsilon)\big)}{2\pi^2}|\eta|^2|\xi|^2 - \frac{3\big(1+\overline{C}(\varepsilon+\ell^\varepsilon)\big)}{2\pi^2}\langle \eta, \xi\rangle^2.
\end{aligned}$$

Fix $\delta > 0$. From the above estimate we deduce that, if $\varepsilon \leq \delta\pi^2/(3\overline{C})$ and $\ell^\varepsilon \leq \delta\pi^2/(3\overline{C})$, then $\mathcal{MTW}(3/(2\pi^2) - \delta, 3/(2\pi^2) + \delta)$ holds for all $v \in \mathcal{NF}_\varepsilon^*(x) \cup \{0\}$ such that $\mathrm{dist}\big(v, \partial\big(\mathcal{NF}_\varepsilon^*(x) \cup \{0\}\big)\big) \leq \delta/\overline{C}$. Since as we already said $\mathcal{MTW}(K, C)$ trivially holds if v is uniformly away from $\partial\big(\mathcal{NF}_\varepsilon^*(x) \cup \{0\}\big)$ for $\varepsilon > 0$ small enough, the result follows. $\square$

5 Final comments

5.1

Our approach applies to more general situations than the one we chose to present. In particular, we do not necessarily need the strict convexity of the cotangent nonfocal domains: let (M, g) be a compact Riemannian manifold, and define $M^* > 0$ and $m^* \in (0, +\infty]$ by

$$M^* := \max\Big\{\|p\|_x \mid p \in \mathcal{I}^*(x), x \in M\Big\} \quad \text{and} \quad m^* := \min\Big\{\|p\|_x \mid p \notin \overline{\mathcal{NF}^*(x)}, x \in M\Big\}.$$

Assume that the two following conditions are satisfied:

(i) $M^* < m^*$,

(ii) there is $K > 0$ such that for every $x \in M$,

$$\forall \xi \in T_x M, \forall \eta \in T_x^* M, \qquad \langle \eta, \xi \rangle = 0 \Longrightarrow \widehat{\mathfrak{S}}(x,p) \cdot (\xi, \eta) \geq K \|\xi\|_x^2 \|\eta\|_x^2$$

for any $p \in T_x^* M$ satisfying $\|p\|_x \in (0, M^*]$.

Following the proof of [20, Lemma 2.3], it is not difficult to show that under these assumptions there exists $C > 0$ such that, for every $x \in M$ and every $p \in T_x^* M$ satisfying $\|p\|_x \in (0, M^*]$,

$$\widehat{\mathfrak{S}}(x,p) \cdot (\xi, \eta) \geq K \|\xi\|_x^2 \|\eta\|_x^2 - C \big| \langle \eta, \xi \rangle \big| \|\xi\|_x \|\eta\|_x \qquad \forall \xi \in T_x M, \quad \forall \eta \in T_x^* M.$$

Then, one can easily check that both the proof of Lemma 3.3 and the proof of Theorem 3.6 still work, and so (M, g) satisfies $\mathcal{TCP}$, and all its injectivity domains are strictly convex. In particular, this allows to recover in a simple way the result (proved independently in [11, 20]) that any C^4-deformation of a quotient of the standard sphere $\mathbb{S}^n$ (say for instance $\mathbb{RP}^n$) satisfies $\mathcal{TCP}$. Indeed (ii) follows from the fact that our extended MTW condition is stable far from the boundary of $\mathcal{NF}^*(x) \cup \{0\}$ (while the classical MTW condition is a priori stable only far from the boundary of $\mathcal{I}^*(x)$).

5.2

It can be shown [7] that the cotangent injectivity domains of any smooth complete Riemannian manifold have locally semiconcave boundaries. In fact, if g is a smooth Riemannian metric which is C^4-close to the round metric on the sphere $\mathbb{S}^n$, then for all $x \in \mathbb{S}^n$ the sets $\overline{\mathcal{NF}^*(x)}$ are uniformly convex [7] (while it is not known whether the sets $\overline{\mathcal{I}^*(x)}$ are convex or not). As a consequence, if $(\mathbb{S}^n, g)$ is a C^4-deformation of the standard sphere which satisfies $\mathcal{MTW}(K, C)$ for some $K, C > 0$, then it satisfies $\mathcal{TCP}$, and all its injectivity domains are strictly convex.

5.3

In [11, 20] the authors can improve $\mathcal{TCP}$ to higher regularity thanks to the stay-away property of the optimal transport map T. More precisely, in [20] the authors assume that the cut locus is nonfocal (which is for example that case if one considers C^4-deformation of a quotient of the standard sphere $\mathbb{S}^n$), and combining this hypothesis with $\mathcal{TCP}$ one gets the existence of a constant $\sigma > 0$ such that $d(T(x), \mathrm{cut}(x)) \geq \sigma$ for all $x \in M$. On the other hand, in [11] the authors show that if (M, g) is C^4-deformation of $(\mathbb{S}^n, g^{\mathrm{can}})$, and one imposes some boundedness constraint on the measures μ and ν (the constraint depending on the size of the perturbation), then the stay-away property of the optimal map holds. Once the stay-away property is established, $\mathcal{TCP}$ allows to localize the problem and to apply the a priori estimates of Ma, Trudinger and Wang [21], obtaining C^∞ regularity on T (under C^∞ assumptions on the measures). In our case it is not clear whether the stay-away property is true or not, and this is why our result cannot be easily improved to higher regularity.

5.4

As we already said, if a Riemannian manifold (M, g) satisfies $\mathcal{MTW}(K, C)$ for some $K, C > 0$, then its sectional curvature is bounded below by K. As it was shown by Kim [16], the converse result is false. Describing the positively curved and simply connected Riemannian manifolds which satisfy $\mathcal{MTW}(K, C)$ for some $K, C > 0$ is a formidable challenge.

A Appendix : The round sphere

The purpose of the appendix is to provide a proof of the following result.

Theorem A.1. *There exists $K_0 > 0$ such that, for every $n \geq 2$, the round sphere $(\mathbb{S}, g^{\mathrm{can}})$ satisfies $\mathcal{MTW}(K_0, K_0)$.*

Proof. Let us see the round sphere $\mathbb{S}^n$ as a submanifold of $\mathbb{R}^{n+1}$ equipped with the Riemannian metric induced by the Euclidean metric. More precisely, we see $\mathbb{S}^n$ as the sphere centered at the origin with radius one in $\mathbb{R}^{n+1}$. Since we work in $\mathbb{R}^{n+1}$, we can identify covectors with vectors. For every $x \in \mathbb{S}^n$, the set $\mathcal{NF}^*(x) \cup \{0\}$ corresponds to the open ball centered at x with radius π intersected with the hyperplan tangent to $\mathbb{S}^n$ at x. Our aim is to show that there exists a constant $K_0 > 0$ such that, for every $x \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ and $p \in \mathcal{NF}^*(x)$, one has

$$\widehat{\mathfrak{S}}(x,p) \cdot (\xi, \eta) = \frac{3}{2} \frac{d^2}{ds^2} \big(\langle K(x, p+s\eta)\xi, \xi \rangle \big)_{|s=0} \geq K_0 \|\xi\|_x^2 \|\eta\|_x^2 - K_0 \big| \langle \xi, \eta \rangle \big| \|\xi\| \|\eta\|,$$

for all $\xi \in T_x\mathbb{S}^n$, $\eta \in T_x^*\mathbb{S}^n$. This is equivalent to show that, for every $x \in \mathbb{S}^n$ and every $v \in T_x\mathbb{S}^n$,

$$-\frac{3}{2} \frac{d^2}{ds^2} \frac{d^2}{dt^2} c\big(\exp_x(t\xi), \exp_x(v+s\eta)\big)_{|t=s=0} \geq K_0 \|\xi\|_x^2 \|\eta\|_x^2 - K_0 \big| \langle \xi, \eta \rangle \big| \|\xi\| \|\eta\| \qquad \forall\, \xi, \eta \in T_x\mathbb{S}^n,$$

where $c := d^2/2$. Since the function $(t,s) \mapsto c\big(\exp_x(t\xi), \exp_x(v+s\eta)\big)$ depends only on the behavior of the Riemannian distance in the affine space containing x and spanned by the three vectors v, ξ, η, we just have to prove Theorem A.1 for $n = 3$. Moreover, by the homogeneity of $(\mathbb{S}, g^{\mathrm{can}})$, it suffices to prove the estimate only for a fixed point $x \in \mathbb{S}^n$ and along a fixed geodesic $t \mapsto \exp_x(tv)$.

Consider the stereographic projection of the sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ centered at the origin and of radius 1 from the north pole $N = (0,0,0,1)$ onto the space $\mathbb{R}^3 \simeq \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$. The projection of some point $x = (x_1, x_2, x_3, x_4) \in \mathbb{S}^3$ is given by

$$\sigma(x) = \left(\frac{x_1}{1-x_4}, \frac{x_2}{1-x_4}, \frac{x_3}{1-x_4} \right).$$

The function σ is a smooth diffeomorphism from $\mathbb{S}^3 \setminus \{N\}$ onto $\mathbb{R}^3$, whose inverse is

$$\sigma^{-1}(y) = \left(\frac{2y_1}{1+|y|^2}, \frac{2y_2}{1+|y|^2}, \frac{2y_3}{1+|y|^2}, \frac{|y|^2-1}{1+|y|^2} \right) \qquad \forall\, y = (y_1, y_2, y_3) \in \mathbb{R}^3,$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^3$. The pushforward of the round metric on $\mathbb{S}^3$ is given by

$$g_y(v,v) = \frac{4}{(1+|y|^2)^2} |v|^2 \qquad \forall\, y, v \in \mathbb{R}^3,$$

and the Hamiltonian canonically associated to g is

$$H(y,p) = \frac{(1+|y|^2)^2}{8} |p|^2 \qquad \forall\, y, p \in \mathbb{R}^3.$$

The Hamiltonian system associated to H is

$$\begin{cases} \dot{y} &= \frac{\partial H}{\partial p}(y,p) &= \frac{(1+|y|^2)^2}{4} p \\ \dot{p} &= -\frac{\partial H}{\partial y}(y,p) &= -\frac{(1+|y|^2)|p|^2}{2} y, \end{cases}$$

and the linearized Hamiltonian system along a given solution $(y(t), p(t))$ is

$$\left\{\begin{array}{lcl} \dot{h} & = & (1+|y|^2)\langle y,h\rangle p + \frac{(1+|y|^2)^2}{4} q \\ \dot{q} & = & -\frac{(1+|y|^2)^2|p|^2}{2} h - |p|^2\langle y,h\rangle y - (1+|y|^2)(p\cdot q) y \end{array}\right.$$

We note that h is a Jacobi vector field along the geodesic $t \mapsto y(t)$.

Set $x^1 = (-1,0,0,0)$ and $x^2 = (1,0,0,0)$, and for $|\alpha|$ small, let γ_α be the minimizing geodesic on $\mathbb{S}^3$ joining x^1 to x^2 defined by

$$\gamma_\alpha(t) := (\cos(t-\pi), \sin(\alpha)\sin(t-\pi), 0, \cos(\alpha)\sin(t-\pi)) \qquad \forall\, t \in [0,\pi].$$

Its image by the stereographic projection is given by

$$\theta_\alpha(t) := \sigma(\gamma_\alpha(t)) = \left(\frac{\cos(t-\pi)}{1-\cos(\alpha)\sin(t-\pi)}, \frac{\sin(\alpha)\sin(t-\pi)}{1-\cos(\alpha)\sin(t-\pi)}, 0 \right).$$

It is a minimizing geodesic between $Y := \sigma(x^1) = (-1,0,0)$ and $\sigma(x^2) = (1,0,0)$. Fix $\bar{t} \in (0,\pi)$, and set $V := \bar{t}\dot{\theta}_\alpha(0)$. We need to compute the matrix $K(Y,V)$. By an easy rescaling argument, we have

$$K(Y,V) = \bar{t} S_\alpha(\bar{t}),$$

where $S_\alpha(\bar{t})$ is the 3×3 symmetric matrix such that any solution of the linearized Hamiltonian system along θ_α starting from $(h, S_\alpha(\bar{t})h)$, satisfies $h(\bar{t}) = 0$. Let us compute $S_\alpha(\bar{t})$.

Define three vector fields E_1, E_2, E_3 along θ_α by

$$E_1(t) := \dot{\theta}_\alpha(t) = \left(\frac{\cos(\alpha) - \sin(t-\pi)}{(1-\cos(\alpha)\sin(t-\pi))^2}, \frac{\sin(\alpha)\cos(t-\pi)}{(1-\cos(\alpha)\sin(t-\pi))^2}, 0 \right),$$

$$E_2(t) := \left(\frac{-\sin(\alpha)\cos(t-\pi)}{(1-\cos(\alpha)\sin(t-\pi))^2}, \frac{\cos(\alpha) - \sin(t-\pi)}{(1-\cos(\alpha)\sin(t-\pi))^2}, 0 \right),$$

$$E_3(t) := (0,0,1).$$

The vectors $E_1(t), E_2(t), E_3(t)$ form a basis of parallel vector fields along θ_α. Let (h,q) be a solution of the linearized Hamiltonian system along θ_α such that $h(t) = 0$ for some $\bar{t} > 0$. Since $E_1(t), E_2(t), E_3(t)$ form a basis of parallel vector fields along θ_α, there are three smooth functions u_1, u_2, u_3 such that

$$h(t) = u_1(t)E_1(t) + u_2(t)E_2(t) + u_3(t)E_3(t) \qquad \forall\, t.$$

Hence, as h is a Jacobi vector field along θ_α, its second covariant derivative along θ_α is given by

$$D_t^2 h(t) = \ddot{u}_1(t)E_1(t) + \ddot{u}_2(t)E_2(t) + \ddot{u}_3(t)E_3(t).$$

Therefore, since $(\mathbb{R}^3, g)$ has constant curvature, we have

$$\begin{array}{lcl} 0 & = & D_t^2 h + R(h,\dot{\theta}_\alpha)\dot{\theta}_\alpha \\ & = & D_t^2 h + g\left(\dot{\theta}_\alpha, \dot{\theta}_\alpha\right) h - g\left(h, \dot{\theta}_\alpha\right)\dot{\theta}_\alpha \\ & = & \ddot{u}_1(t)E_1(t) + \ddot{u}_2(t)E_2(t) + \ddot{u}_3(t)E_3(t) + u_1(t)E_1(t) + u_2(t)E_2(t) + u_3(t)E_3(t) - u_1(t)\dot{\theta}_\alpha(t) \\ & = & \ddot{u}_1(t)E_1(t) + [\ddot{u}_2(t) + u_2(t)]E_2(t) + [\ddot{u}_3(t) + u_3(t)]E_3(t). \end{array}$$

We deduce that there are six constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ such that

$$\left\{\begin{array}{lcl} u_1(t) & = & \lambda_1 + \lambda_2 t \\ u_2(t) & = & \lambda_3\cos(t) + \lambda_4\sin(t) \\ u_3(t) & = & \lambda_5\cos(t) + \lambda_6\sin(t). \end{array}\right.$$

Since

$$E_1(0) = (\cos(\alpha), -\sin(\alpha), 0), \qquad E_2(0) = (\sin(\alpha), \cos(\alpha), 0), \qquad E_3(0) = (0, 0, 1).$$

the equality $h(0) = u_1(0)E_1(0) + u_2(0)E_2(0) + u_3(0)E_3(0)$ yields

$$\begin{cases} h_1(0) &= u_1(0)\cos(\alpha) + u_2(0)\sin(\alpha) \\ h_2(0) &= -u_1(0)\sin(\alpha) + u_2(0)\cos(\alpha) \\ h_3(0) &= u_3(0), \end{cases}$$

which gives

$$\lambda_1 = \cos(\alpha)h_1(0) - \sin(\alpha)h_2(0), \qquad \lambda_3 = \sin(\alpha)h_1(0) + \cos(\alpha)h_2(0), \qquad \lambda_5 = h_3(0).$$

Furthermore,

$$\dot{E}_1(0) = \left(-\cos(2\alpha), \sin(2\alpha)\right), \qquad \dot{E}_2(0) = \left(-\sin(2\alpha), -\cos(2\alpha)\right), \qquad \dot{E}_3(0) = 0.$$

Differentiating $h(t) = u_1(t)E_1(t) + u_2(t)E_2(t) + u_3(t)E_3(t)$ at $t = 0$, we obtain

$$\begin{cases} \dot{h}_1(0) &= \lambda_2\cos(\alpha) + \lambda_4\sin(\alpha) - \lambda_1\cos(2\alpha) - \lambda_3\sin(2\alpha) \\ \dot{h}_2(0) &= -\lambda_2\sin(\alpha) + \lambda_4\cos(\alpha) + \lambda_1\sin(2\alpha) - \lambda_3\cos(2\alpha) \\ \dot{h}_3(0) &= \lambda_6. \end{cases}$$

From the linearized Hamiltonian system, since $Y(0) = (-1, 0, 0)$ and $P(0) = V(0) = E_1(0)$, we have

$$q(0) = \dot{h}(0) + 2h_1(0)E_1(0).$$

Recalling that $h(\bar{t}) = 0 \Rightarrow u(t) = 0$, we get

$$\lambda_2 = -\frac{\lambda_1}{\bar{t}}, \qquad \lambda_4 = -\lambda_3\frac{\cos(\bar{t})}{\sin(\bar{t})}, \qquad \lambda_6 = -\lambda_5\frac{\cos(\bar{t})}{\sin(\bar{t})}.$$

Thus we finally obtain

$$\begin{pmatrix} q_1(0) \\ q_2(0) \\ q_3(0) \end{pmatrix} = q(0) = S_\alpha(\bar{t})h(0) = \begin{pmatrix} a_\alpha(\bar{t}) & b_\alpha(\bar{t}) & 0 \\ b_\alpha(\bar{t}) & c_\alpha(\bar{t}) & 0 \\ 0 & 0 & d(\bar{t}) \end{pmatrix} \begin{pmatrix} h_1(0) \\ h_2(0) \\ h_3(0) \end{pmatrix},$$

where

$$a_\alpha(\bar{t}) := -\frac{\cos^2(\alpha)}{\bar{t}} - \frac{\cos(\bar{t})\sin^2(\alpha)}{\sin(\bar{t})} + \cos(\alpha),$$

$$b_\alpha(\bar{t}) := \frac{\cos(\alpha)\sin(\alpha)}{\bar{t}} - \frac{\cos(\alpha)\sin(\alpha)\cos(\bar{t})}{\sin(\bar{t})} - \sin(\alpha),$$

$$c_\alpha(\bar{t}) := -\frac{\sin^2(\alpha)}{\bar{t}} - \frac{\cos^2(\alpha)\cos(\bar{t})}{\sin(\bar{t})} - \cos(\alpha), \qquad d(\bar{t}) = -\frac{\cos(\bar{t})}{\sin(\bar{t})}.$$

Hence

$$K(Y,V) = \bar{t}S_\alpha(\bar{t}) = \begin{pmatrix} k_1(Y,V) & k_2(Y,V) & 0 \\ k_2(Y,V) & k_3(Y,V) & 0 \\ 0 & 0 & k_4(Y,V) \end{pmatrix},$$

with

$$k_1(Y,V) = \bar{t}a_\alpha(\bar{t}), \qquad k_2(Y,V) = \bar{t}b_\alpha(\bar{t}), \qquad k_3(Y,V) = \bar{t}c_\alpha(\bar{t}), \qquad k_4(Y,V) = \bar{t}d(\bar{t}).$$

Let us now show that $\mathcal{MTW}(K_0, K_0)$ holds. For that, it is sufficient to show that for for every V of the form $V = (r, 0, 0)$ with $r \in (0, \pi)$, every $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ with $|\xi| = 1$ and every $\eta = (\eta_1, \eta_2, 0) \in \mathbb{R}^3$ with $|\eta| = 1$, one has

$$\langle \xi, \ddot{\tilde{K}}(0)\xi \rangle \geq \frac{2}{3} K_0 \left(1 - \left|\langle \xi, \eta \rangle\right|\right),$$

where $\tilde{K}(s)$ is defined as

$$\tilde{K}(s) := K(Y, V + s\eta) = \begin{pmatrix} \tilde{k}_1(s) & \tilde{k}_2(s) & 0 \\ \tilde{k}_2(s) & \tilde{k}_3(s) & 0 \\ 0 & 0 & \tilde{k}_4(s) \end{pmatrix}.$$

By the discussion above, we have

$$\tilde{K}(s) = T(s) S_{\alpha(s)}(T(s)),$$

where

$$\alpha(s) = -\arctan\left(\frac{s\eta_2}{r + s\eta_1}\right), \qquad T(s) = \left|\begin{pmatrix} r + s\eta_1 \\ s\eta_2 \end{pmatrix}\right| = \sqrt{(r + s\eta_1)^2 + s^2\eta_2^2}.$$

We note that

$$\alpha(0) = 0, \qquad \dot{\alpha}(0) = -\frac{\eta_2}{r} \qquad \ddot{\alpha}(0) = \frac{2\eta_1\eta_2}{r^2},$$

and

$$T(0) = r, \qquad \dot{T}(0) = \eta_1, \qquad \ddot{T}(0) = \frac{\eta_2^2}{r}.$$

The second derivatives at $s = 0$ of the functions $\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4$ are given by

$$\ddot{\tilde{k}}_1(0) = 2\left[\frac{1}{r^2} - \frac{\cos(r)}{r\sin(r)}\right]\eta_2^2, \qquad \ddot{\tilde{k}}_2(0) = 2\left[\frac{1}{r^2} - \frac{1}{\sin^2(r)}\right]\eta_1\eta_2,$$

$$\ddot{\tilde{k}}_3(0) = 2\left[\frac{1}{\sin^2(r)} - \frac{r\cos(r)}{\sin^3(r)}\right]\eta_1^2 + \left[-\frac{2}{r^2} + \frac{\cos(r)}{r\sin(r)} + \frac{1}{\sin^2(r)}\right]\eta_2^2,$$

$$\ddot{\tilde{k}}_4(0) = 2\left[\frac{1}{\sin^2(r)} - \frac{r\cos(r)}{\sin^3(r)}\right]\eta_1^2 + \left[\frac{1}{\sin^2(r)} - \frac{\cos(r)}{r\sin(r)}\right]\eta_2^2,$$

and so

$$\begin{aligned}
\langle \xi, \ddot{\tilde{K}}(0)\xi \rangle &= \ddot{\tilde{k}}_1(0)\xi_1^2 + 2\ddot{\tilde{k}}_2(0)\xi_1\xi_2 + \ddot{\tilde{k}}_3(0)\xi_2^2 + \ddot{\tilde{k}}_4(0)\xi_3^2 \\
&= 2\left[\frac{1}{r^2} - \frac{\cos(r)}{r\sin(r)}\right]\xi_1^2\eta_2^2 - 4\left[\frac{1}{\sin^2(r)} - \frac{1}{r^2}\right]\xi_1\xi_2\eta_1\eta_2 \\
&+ 2\left[\frac{1}{\sin^2(r)} - \frac{r\cos(r)}{\sin^3(r)}\right]\xi_2^2\eta_1^2 + \left[-\frac{2}{r^2} + \frac{\cos(r)}{r\sin(r)} + \frac{1}{\sin^2(r)}\right]\xi_2^2\eta_2^2 \\
&+ 2\left[\frac{1}{\sin^2(r)} - \frac{r\cos(r)}{\sin^3(r)}\right]\xi_3^2\eta_1^2 + \left[\frac{1}{\sin^2(r)} - \frac{\cos(r)}{r\sin(r)}\right]\xi_3^2\eta_2^2.
\end{aligned}$$

Define the functions c_1, c_2, c_3, c_4, c_5 on $[0, \pi)$ by

$$c_1(r) := \frac{1}{r^2} - \frac{\cos(r)}{r\sin(r)}, \qquad c_2(r) := \frac{1}{\sin^2(r)} - \frac{1}{r^2}, \qquad c_3(r) := \frac{1}{\sin^2(r)} - \frac{r\cos(r)}{\sin^3(r)},$$

$$c_4(r) := -\frac{2}{r^2} + \frac{\cos(r)}{r\sin(r)} + \frac{1}{\sin^2(r)}, \qquad c_5(r) := \frac{1}{\sin^2(r)} - \frac{\cos(r)}{r\sin(r)}.$$

We need the following two lemmas, whose proof is postponed to the end.

Lemma A.2. *One has*

$$c_i(r) \geq c_i(0) = \frac{1}{3} \qquad \forall r \in [0,\pi), \quad \forall i = 1,2,3.$$

Lemma A.3. *The function $c_4 = c_2 - c_1$ is nonnegative on $[0,\pi)$, and there exists $\alpha \in (0, 2/\pi^2)$ such that*

$$\big(c_1(r) - \alpha\big)\big(c_3(r) - \alpha\big) - \big(c_2(r) - \alpha\big)^2 \geq 0 \qquad \forall r \in [0,\pi).$$

We note that $c_4 = c_2 - c_1$ and $c_5 = c_1 + c_2$. Set $\tilde{c}_i(r) := c_i(r) - \frac{1}{3}$ for every $r \in (0,\pi)$ and $i = 1,2,3$. By Lemmas A.2 and A.3, together with the fact that $|\eta|^2 = 1 = \eta_1^2 + \eta_2^2$, we get

$$\begin{aligned}
\langle \xi, \ddot{\tilde{K}}(0)\xi\rangle &= 2c_1(r)\xi_1^2\eta_2^2 - 4c_2(r)\xi_1\xi_2\eta_1\eta_2 + 2c_3(r)\xi_2^2\eta_1^2 + c_4(r)\xi_2^2\eta_2^2 + 2c_3(r)\xi_3^2\eta_1^2 + c_5(r)\xi_3^2\eta_2^2\\
&\geq 2c_1(r)\xi_1^2\eta_2^2 - 4c_2(r)\xi_1\xi_2\eta_1\eta_2 + 2c_3(r)\xi_2^2\eta_1^2 + c_4(r)\xi_2^2\eta_2^2 + \frac{2}{3}\xi_3^2\eta_1^2 + \frac{2}{3}\xi_3^2\eta_2^2\\
&= 2c_1(r)\xi_1^2\eta_2^2 - 4c_2(r)\xi_1\xi_2\eta_1\eta_2 + 2c_3(r)\xi_2^2\eta_1^2 + \big(c_2(r) - c_1(r)\big)\xi_2^2\eta_2^2 + \frac{2}{3}\xi_3^2\\
&\geq 2c_1(r)\xi_1^2\eta_2^2 - 4c_2(r)\xi_1\xi_2\eta_1\eta_2 + 2c_3(r)\xi_2^2\eta_1^2 + \frac{2}{3}\xi_3^2.
\end{aligned}$$

For any $r \in (0,\pi)$ and $i = 1,2,3$, set $\hat{c}_i(r) := c_i(r) - \alpha$, with α given by Lemma A.3. Then

$$\begin{aligned}
\langle \xi, \ddot{\tilde{K}}(0)\xi\rangle &\geq 2\hat{c}_1(r)\xi_1^2\eta_2^2 - 4\hat{c}_2(r)\xi_1\xi_2\eta_1\eta_2 + 2\hat{c}_3(r)\xi_2^2\eta_1^2 + 2\alpha\left(\xi_1^2\eta_2^2 - 2\xi_1\xi_2\eta_1\eta_2 + \xi_2^2\eta_1^2\right) + 2\alpha\xi_3^2\\
&\geq 2\hat{c}_1(r)\xi_1^2\eta_2^2 - 4\hat{c}_2(r)\xi_1\xi_2\eta_1\eta_2 + 2\hat{c}_3(r)\xi_2^2\eta_1^2 + 2\alpha\left(1 - \langle\xi,\eta\rangle^2\right)\\
&\geq 2\hat{c}_1(r)\xi_1^2\eta_2^2 - 4\hat{c}_2(r)\xi_1\xi_2\eta_1\eta_2 + 2\hat{c}_3(r)\xi_2^2\eta_1^2 + 2\alpha\left(1 - |\langle\xi,\eta\rangle|\right)\\
&\geq 2\hat{c}_1(r)\xi_1^2\eta_2^2 - 4\hat{c}_2(r)|\xi_1\xi_2\eta_1\eta_2| + 2\hat{c}_3(r)\xi_2^2\eta_1^2 + 2\alpha\left(1 - |\langle\xi,\eta\rangle|\right)\\
&\geq 2\hat{c}_1(r)\xi_1^2\eta_2^2 - 4\sqrt{\hat{c}_1(r)\hat{c}_3(r)}|\xi_1\xi_2\eta_1\eta_2| + 2\hat{c}_3(r)\xi_2^2\eta_1^2 + 2\alpha\left(1 - |\langle\xi,\eta\rangle|\right)\\
&\geq 2\left(\sqrt{\hat{c}_1(r)}|\xi_1\eta_2| - \sqrt{\hat{c}_3(r)}|\xi_2\eta_1|\right)^2 + 2\alpha\left(1 - |\langle\xi,\eta\rangle|\right)\\
&\geq 2\alpha\left(1 - |\langle\xi,\eta\rangle|\right),
\end{aligned}$$

where we used again Lemma A.3. Thus we finally obtain

$$\frac{3}{2}\langle \xi, \ddot{\tilde{K}}(0)\xi\rangle \geq 3\alpha\left(1 - |\langle\xi,\eta\rangle|\right),$$

which shows that the round sphere satisfies $\mathcal{MTW}(K_0, K_0)$ with $K_0 := 3\alpha$. □

Proof of Lemma A.2. Define $f : [0,\pi) \to \mathbb{R}$ by $f(r) := 1 - \frac{r\cos(r)}{\sin(r)}$. The Taylor expansion of f at $r = 0$ is given by

$$f(r) = \frac{r^2}{3} + \frac{r^4}{45} + o(r^4).$$

This means that $f(r) > \frac{r^2}{3}$ for small r in $(0,\pi)$. Define $g : [0,\pi) \to \mathbb{R}$ by $g(r) := f(r) - \frac{r^2}{3}$. By the latter remark, g is strictly positive for small r in $(0,\pi)$. One has

$$c_1(r) = \frac{f(r)}{r^2} \quad \text{and} \quad c_3(r) = \frac{f(r)}{\sin^2(r)} \qquad \forall r \in (0,\pi).$$

Therefore showing that $c_1, c_3 \geq 1/3$ is equivalent to showing that $g \geq 0$. Define $h : [0,\pi) \to \mathbb{R}$ by $h(r) := r^2 c_2(r) - r^2/3$. The derivatives of g and h are respectively given by

$$g'(r) = \frac{r}{\sin^2(r)} - \frac{\cos(r)}{\sin(r)} - \frac{2r}{3} \qquad \forall r \in (0,\pi).$$

and

$$
\begin{aligned}
h'(r) &= \frac{2r}{\sin^2(r)} - \frac{2r^2\cos(r)}{\sin^3(r)} - \frac{2r}{3} \\
&= \frac{2r}{\sin^2(r)}\left(f(r) - \frac{\sin^2(r)}{3}\right) \geq \frac{2rg(r)}{\sin^2(r)}.
\end{aligned}
$$

This shows that if $g(r) > 0$ for every $r \in (0, \bar{r})$, then $h(r) > 0$ on $(0, \bar{r}]$. But if $\bar{r} \in (0, \pi)$ is such that $g(\bar{r}) = 0$, then $\frac{\cos(\bar{r})}{\sin(\bar{r})} = \frac{1}{\bar{r}} - \frac{\bar{r}}{3}$, so that

$$
g'(\bar{r}) = \frac{\bar{r}}{\sin^2(\bar{r})} - \frac{1}{\bar{r}} - \frac{\bar{r}}{3} = \bar{r}\left(c_2(\bar{r}) - \frac{1}{3}\right) = \frac{h(\bar{r})}{\bar{r}} > 0.
$$

Since g is strictly positive for r small, we conclude easily that $g, h \geq 0$ on $[0, \pi)$, which proves the lemma. $\square$

Proof of Lemma A.3. First of all we observe that the Taylor expansions of c_1, c_2, c_3 at $r = 0$ are given by

$$
c_1(r) = \frac{1}{3} + \frac{r^2}{45} + o(r^2), \quad c_2(r) = \frac{1}{3} + \frac{r^2}{15} + o(r^2), \quad c_3(r) = \frac{1}{3} + \frac{2r^2}{15} + o(r^2). \tag{A.1}
$$

Define $\ell : [0, \pi) \to \mathbb{R}$ by $\ell(r) := r^2 c_4(r)$. Its derivative is given by

$$
\ell'(r) = r\big(2c_3(r) - c_2(r) - c_1(r)\big).
$$

We first reamark that obviously $c_3 \geq c_1$ on $[0, \pi)$. Moreover the derivative of the function $m : [0, \pi) \to \mathbb{R}$ defined as $m(r) := r^2(c_3(r) - c_2(r))$ is given by

$$
m'(r) = \frac{r^3}{\sin^2(r)}\big(3(c_2(r) + c_1(r)) - 2\big),
$$

and it is nonnegative by Lemma A.2. Since by (A.1) $\lim_{r\to 0^+} m(r) = 1/15 > 0$, we obtain that $m(r) \geq 0$ on $[0, \pi)$. This gives $\ell'(r) \geq 0$ for every $r \in [0, \pi)$, and so $c_4 = c_2 - c_1$ is nonnegative on $[0, \pi)$.

Let us now prove the second assertion of the lemma. We first want to show that the function $c_1 c_3 - c_2^2$ is strictly positive on $(0, \pi)$. With the notation of Lemma A.2, we have

$$
c_1(r)c_3(r) = \frac{f^2(r)}{r^2\sin^2(r)}.
$$

Thus we need to prove that

$$
\frac{f(r)}{r\sin(r)} > c_2(r) \qquad \forall\, r \in (0, \pi),
$$

or equivalently

$$
F(r) := f(r) - r\sin(r)c_2(r) = f(r) - \frac{r}{\sin(r)} + \frac{\sin(r)}{r} > 0 \qquad \forall\, r \in (0, \pi).
$$

It is easily seen that $F(r) = \frac{r^4}{90} + o(r^4)$, so that $F(r) > 0$ for $r > 0$ small. Differentiating the above expression we get

$$\begin{aligned}
F'(r) &= \frac{r}{\sin^2(r)} - \frac{\cos(r)}{\sin(r)} - \frac{1}{\sin(r)} + \frac{r\cos(r)}{\sin^2(r)} + \frac{\cos(r)}{r} - \frac{\sin(r)}{r^2} \\
&= \frac{1}{r}\left(\frac{r^2}{\sin^2(r)} - 1 + f(r)\right) - \frac{1}{\sin(r)}f(r) - \frac{\sin(r)}{r^2}f(r) \\
&= \frac{1}{r}\left(\frac{r^2}{\sin^2(r)} - 1\right) + \left(\frac{1}{r} - \frac{1}{\sin(r)} - \frac{\sin(r)}{r^2}\right)f(r) \\
&= \frac{1}{r}\left[\left(\frac{r^2}{\sin^2(r)} - 1\right) + \frac{r\sin(r) - r^2 - \sin^2(r)}{r\sin(r)}f(r)\right].
\end{aligned}$$

Assume by contradiction that there exists $\bar r > 0$ such that $F(\bar r) = 0$. Then

$$f(\bar r) = \frac{\bar r}{\sin(\bar r)} + \frac{\sin(\bar r)}{\bar r} = \frac{\sin(\bar r)}{\bar r}\left(\frac{\bar r^2}{\sin^2(\bar r)} - 1\right)$$

which gives

$$F'(\bar r) = \frac{1}{\bar r}\left(\frac{\bar r^2}{\sin^2(\bar r)} - 1\right)\left[1 - \frac{\bar r^2 + \sin^2(\bar r) - \bar r\sin(\bar r)}{\bar r^2}\right].$$

Since

$$r^2 > \sin^2(r) \quad \text{and} \quad r^2 + \sin^2(r) - r\sin(r) < r^2 \qquad \forall\, r \in (0,\pi),$$

we get $F'(\bar r) > 0$, absurd. Thus $c_1c_3 - c_2^2 > 0$ on $(0,\pi)$. We now observe that, thanks to (A.1), for every $\alpha > 0$ we have

$$\big(c_1(r)-\alpha\big)\big(c_3(r)-\alpha\big) - \big(c_2(r)-\alpha\big)^2 = \left(\frac{1}{3} - \alpha\right)\frac{r^2}{45} + o(r^2).$$

On the other hand, for r close to π and every $\alpha > 0$,

$$\big(c_1(r)-\alpha\big)\big(c_3(r)-\alpha\big) - \big(c_2(r)-\alpha\big)^2 \sim \frac{\left(\frac{2}{\pi} - \pi\alpha\right)}{\sin^3(r)}.$$

Combining all together, we conclude easily that there exists $\alpha \in (0, 2/\pi^2)$ such that

$$\big(c_1(r)-\alpha\big)\big(c_3(r)-\alpha\big) - \big(c_2(r)-\alpha\big)^2 \geq 0 \qquad \forall r \in [0,\pi).$$

$\square$

Remark A.4. Starting from the formula for $\langle \xi, \ddot{K}(0)\xi\rangle$ given just after Lemma A.3, it is not difficult to see that $(\mathbb{S}^n, g^{\mathrm{can}})$ satisfies $\mathcal{MTW}(1,1)$ if and only if the quantity

$$2\tilde c_1(r)\hat\xi_1^2\hat\eta_2^2 - 4\tilde c_2(r)\hat\xi_1\hat\xi_2\hat\eta_1\hat\eta_2 + 2\tilde c_3(r)\hat\xi_2^2\hat\eta_1^2 + \big(\tilde c_2(r) - \tilde c_1(r)\big)\hat\xi_2^2\hat\eta_2^2 + \frac{2}{3}\big|\langle\hat\xi,\hat\eta\rangle\big|\left(1 - \big|\langle\hat\xi,\hat\eta\rangle\big|\right)$$

is nonnegative for any $\hat\xi, \hat\eta \in \mathbb{S}^1$ and any $r \in (0,\pi)$, where $\tilde c_i(r) := c_i(r) - \frac{1}{3}$. Numerical simulations suggest that the above inequality should be true.

References

[1] R. Abraham and J. E. Marsden. Foundations of Mechanics. Benjamin, London, 1978.

[2] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44:375–417, 1991.

[3] L. A. Caffarelli. The regularity of mappings with a convex potential. *J. Amer. Math. Soc.*, 5(1):99–104, 1992.

[4] L. A. Caffarelli. Boundary regularity of maps with convex potentials. *Comm. Pure Appl. Math.*, 45(9):1141–1151, 1992.

[5] L. A. Caffarelli. Boundary regularity of maps with convex potentials. II. *Ann. of Math. (2)*, 144(3):453–496, 1996.

[6] A. Cannas da Silva. Lectures on symplectic geometry. *Lecture Notes in Mathematics*, Vol. 1764. Springer-Verlag, Berlin, 2001.

[7] M. Castelpietra and L. Rifford. Regularity properties of the distance function to conjugate and cut loci for viscosity solutions of Hamilton-Jacobi equations and applications in Riemannian geometry. *ESAIM Control Optim. Calc. Var.*, to appear.

[8] D. Cordero-Erausquin. Sur le transport de mesures périodiques. *C. R. Acad. Sci. Paris Sèr. I Math.*, 329(3):199–202, 1999.

[9] P. Delanoë. Classical solvability in dimension two of the second boundary-value problem associated with the Monge-Ampère operator. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 8(5):443–457, 1991.

[10] P. Delanoë. Gradient rearrangement for diffeomorphisms of a compact manifold. *Differential Geom. Appl.*, 20(2):145–165, 2004.

[11] P. Delanoë and Y. Ge. Regularity of optimal transportation maps on compact, locally nearly spherical, manifolds. *J. Reine Angew. Math.*, to appear.

[12] A. Figalli, Y.-H. Kim and R. J. McCann. Hölder continuity of optimal maps for non-negatively cross-curved costs. In preparation.

[13] A. Figalli and G. Loeper. C^1 regularity of solutions of the Monge-Ampère equation for optimal transport in dimension two. *Calc. Var. Partial Differential Equations*, 35(4): 537–550, 2009.

[14] A. Figalli and C. Villani. An approximation lemma about the cut locus, with applications in optimal transport theory. *Methods Appl. Anal.*, 15(2):149–154, 2008.

[15] J. Itoh and M. Tanaka. The Lipschitz continuity of the distance function to the cut locus. *Trans. Amer. Math. Soc.*, 353(1):21–40, 2001.

[16] Y.-H. Kim. Counterexamples to continuity of optimal transport maps on positively curved Riemannian manifolds. *Int. Math. Res. Not. IMRN*, 2008, Art. ID rnn120, 15 pp.

[17] Y.-H. Kim and R. J. McCann. Continuity, curvature, and the general covariance of optimal transportation. *J. Eur. Math. Soc.*, to appear.

[18] G. Loeper. On the regularity of solutions of optimal transportation problems. *Acta Math.*, to appear.

[19] G. Loeper. Regularity of optimal maps on the sphere: The quadratic cost and the reflector antenna. *Arch. Ration. Mech. Anal.*, to appear.

[20] G. Loeper and C. Villani. Regularity of optimal transport in curved geometry: the nonfocal case. Preprint, 2008.

[21] X.-N. Ma, N. S. Trudinger and X.-J. Wang. Regularity of potential functions of the optimal transportation problem. *Arch. Ration. Mech. Anal.*, 177(2):151–183, 2005.

[22] R. J. McCann. Polar factorization of maps in Riemannian manifolds. *Geom. Funct. Anal.*, 11(3):589–608, 2001.

[23] N. S. Trudinger and X.-J. Wang. On the second boundary value problem for Monge-Ampère type equations and optimal transportation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 8(1):143–174, 2009.

[24] N. S. Trudinger and X.-J. Wang. On strict convexity and continuous differentiability of potential functions in optimal transportation. *Arch. Ration. Mech. Anal.*, 192(3):403–418, 2009.

[25] J. I. E. Urbas. Regularity of generalized solutions of Monge-Ampére equations. *Math. Z.*, 197(3):365–393, 1988.

[26] J. I. E. Urbas. On the second boundary value problem for equations of Monge-Ampére type. *J. Reine Angew. Math.*, 487:115–124, 1997.

[27] C. Villani. Topics in Mass Transportation. *Graduate Studies in Mathematics Surveys*, Vol. 58. American Mathematical Society, Providence, RI, 2003.

[28] C. Villani. Optimal transport, old and new. Grundlehren des mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 338, Springer-Verlag, Berlin-New York, 2009.

[29] C. Villani. Stability of a 4th-order curvature condition arising in optimal transport theory. *J. Funct. Anal.*, 255(9):2683–2708, 2008.