# On the minimal rank Sard Conjecture in sub-Riemannian geometry

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# Outline

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I. Reminder on singular curves

II. Characterization of abnormal lifts

III. Minimal rank Sard Conjecture

IV. A Partial result

# Part I

I. Reminder on singular curves

# The Setting

- $M$ is a smooth connected manifold of dimension $n$.
- $\Delta$ is a **totally nonholonomic distribution** of rank $m \leq n$ on $M$, also called **bracket-generating** of rank $m$.
- We call **horizontal path** any $\gamma \in W^{1,2}([0,1];M)$ such that
$$\dot{\gamma}(t) \in \Delta(\gamma(t)) \qquad \text{a.e. } t \in [0,1].$$
- By the Chow-Rashevsky Theorem, $M$ is **horizontally connected**, that is, every pair of points can be joined by an horizontal path.

# Singular horizontal paths

Consider a family $X^1, \ldots, X^k$ of smooth vector fields on $M$ such that

$$\Delta(z) = \mathrm{Span}\Big\{X^1(z), \ldots, X^m(z)\Big\} \qquad \forall z \in M$$

and given $x \in M$, define the **End-Point mapping**

$$\begin{array}{rcl} E^{x,1} \; : \; \mathcal{U} \subset L^2([0,1];\mathbb{R}^k) & \longrightarrow & M \\ u & \longmapsto & \gamma_u(1) \end{array}$$

where $\gamma_u : [0,1] \to M$ is solution to the Cauchy problem

$$\begin{cases} \dot{\gamma}(t) = \sum_{i=1}^k u_i(t) X^i\left(\gamma(t)\right) & \text{for a.e. } t \in [0,1] \\ \gamma(0) = x. \end{cases}$$

## Definition

An horizontal path is called **singular** if it is, through the "correspondence" $\gamma \leftrightarrow u$, a critical point of $E^{x,1}$.

# Examples of singular horizontal paths

**Example 1:** Riemannian case
Let $\Delta(x) = T_x M$, any path in $W^{1,2}$ is horizontal. There are no singular curves.

**Example 2:** Heisenberg, fat distributions
In $\mathbb{R}^3$, $\Delta$ given by $X^1 = \partial_x, X^2 = \partial_y + x\partial_z$ does not admin nontrivial singular horizontal paths. The same result is true for any contact or more generally fat distribution.

**Example 3:** Martinet distribution
In $\mathbb{R}^3$, let $\Delta = \mathrm{Vect}\{X^1, X^2\}$ with $X^1, X^2$ given by

$$X^1 = \partial_{x_1} \quad \text{and} \quad X^2 = \partial_{x_2} + x_1^2 \partial_{x_3}.$$

The singular horizontal paths are pieces of orbit of the line field given by the trace of $\Delta$ over the plane $\{x_1 = 0\}$.

# Characterization of singular curves

The **annihilator** of $\Delta$ in $T^*M$ is defined by

$$\Delta^\perp := \left\{(x,p) \in T^*M \,|\, p \perp \Delta(x),\, p \neq 0\right\} \subset T^*M$$

and its **Hamiltonian distribution** is given by

$$\vec{\Delta}(x,p) := \mathrm{Span}\left\{\vec{h}^1(x,p), \ldots, \vec{h}^m(x,p)\right\} \quad \forall (x,p) \in T^*M,$$

where $\vec{h^i}$ is the Hamiltonian vector field of $h^i(x,p) = p \cdot X^i(x)$ on $T^*M$ w.r.t. the canonical symplectic form $\omega$.

**Proposition**

*An horizontal path $\gamma : [0,1] \to M$ is singular if and only if it is the projection of a path $\psi : [0,1] \to \Delta^\perp$ which is horizontal w.r.t. $\vec{\Delta}$ or equivalently such that $\dot{\psi}(t) \in \ker \left(\omega_{|\Delta^\perp}\right)_{\psi(t)}$ for a.e. $t \in [0,1]$.*

# Part II

II. Characterization of abnormal lifts

# Characterization of abnormal lifts I

From now on, we assume that $M$ **and** $\Delta$ **are real-analytic** and we set

$$\omega^{\perp} = \omega_{|\Delta^{\perp}}.$$

In this setting, we proved in a work in collaboration with A. Belotto and A. Parusinski that:

There exist

- an open and dense set $\mathcal{S}_0 \subset \Delta^{\perp}$ whose complement is an analytic set,
- a subanalytic Whitney stratification $\mathcal{S} = (\mathcal{S}_{\alpha})$ which is invariant by dilation and with $\mathcal{S}_0$ as a stratum,
- a subanalytic distribution $\vec{\mathcal{K}}$ compatible with $\mathcal{S}$ and invariant by dilation,

such that the following properties are satisfied:

# Characterization of abnormal lifts II

(i) There holds

$$\vec{\mathcal{K}}(\mathfrak{a}) = \ker\left(\omega_{\mathfrak{a}}^{\perp}\right) \qquad \forall \mathfrak{a} \in \mathcal{S}_0,$$

$\vec{\mathcal{K}}_{|\mathcal{S}_0}$ has constant rank $k_0$ with $k_0 \equiv m(2)$ and $k_0 \leq m-2$ and $\vec{\mathcal{K}}_{|\mathcal{S}_\alpha}$ is isotropic and integrable.

(ii) For each stratum $\mathcal{S}_\alpha$, we have

$$\vec{\mathcal{K}}(\mathfrak{a}) = \ker\left(\omega_{\mathfrak{a}}^{\perp}\right) \cap T_{\mathfrak{a}}\mathcal{S}_\alpha \qquad \forall \mathfrak{a} \in \mathcal{S}_\alpha$$

and $\vec{\mathcal{K}}_{|\mathcal{S}_\alpha}$ is isotropic with constant rank $k_\alpha$ verifying $k_\alpha \leq m-1$ and $k_\alpha \geq k_0 + 2$.

(iii) A path $\gamma : [0,1] \to M$ is singular horizontal if and only if it admits a lift $\psi : [0,1] \to \Delta^{\perp}$ which is horizontal w.r.t. $\vec{\mathcal{K}}$.

# Examples

**Example 1:** Rank 2 distributions in dimension 3
$\Delta^\perp$ has dimension 4 with fibers of dimension 1 so it can be seen as a graph over $M$, $k_0 = 0$ and the complement of $\mathcal{S}_0$ is the lift of the so-called Martinet surface

$$\Sigma_\Delta := \{x \in M \,|\, [\Delta, \Delta](x) \in \Delta(x)\}\,.$$

Singular horizontal paths are given by orbits of the trace of $\Delta$ over $\Sigma_\Delta$.

**Example 2:** Corank 1 distributions
$\Delta^\perp$ has dimension $2n - (n-1) = n+1$ with fibers of dimension 1 so it can be seen as a graph over $M$ and everything can be projected down to $M$.

# Examples

**Example 3:** Rank 2 distributions in dimension $n$
$\Delta^\perp$ has dimension $2n-2$, $k_0 = 0$ and for every $\alpha \neq 0$, we have $k_\alpha \in \{0,1\}$.

**Example 4:** Rank 3 distributions in dimension 4
$\Delta^\perp$ has dimension 5 and $k_0 = 1$ so $\vec{\mathcal{K}}_{|\mathcal{S}_0}$ is a line field.

**Example 5:** Rank 4 distributions in dimension 5
$\Delta^\perp$ has dimension 6 and $k_0 = 0$ or $k_0 = 2$.

# Part III

III. Minimal rank Sard Conjecture

# The Sard Conjecture

Given $x \in M$, we denote by $\mathrm{Sing}_\Delta^x$ the set of points $y \in M$ for which there is a singular horizontal path joining $x$ to $y$, it is a closed subset of $M$ containing $x$.

**Conjecture (Sard Conjecture)**

*The set $Sing_\Delta^x$ has Lebesgue measure zero in $M$.*

The result is known in very few cases:

- Rank 2 in dimension 3 (much stronger result by Belotto, Figalli, Parusinski, R).
- Cases where the stratification $(\mathcal{S}_\alpha)$ consists in only one stratum.
- Some cases of Carnot groups.

# Rank of an horizontal path

The rank of an horizontal path $\gamma$ is defined by

$$\mathrm{rank}_\Delta(\gamma) := \dim\left(\mathrm{Im}\left(D_u E^{x,1}\right)\right),$$

where $u$ is a control such that $\gamma = \gamma_u$.

In fact, given an horizontal path $\gamma$ and $p \in T_y^* M_y \setminus \{0\}$ with $y := \gamma(1)$, the two following properties are equivalent:

(i) $p \in \left(\mathrm{Im}\left(D_u E^{x,1}\right)\right)^\perp$.

(ii) There is $\psi : [0,1] \to \Delta^\perp$ which is horizontal w.r.t. $\vec{\Delta}$ such that $\pi(\psi) = \gamma$ and $\psi(1) = (y,p)$.

There always holds

$$m \leq \mathrm{rank}_\Delta(\gamma) \leq n.$$

Given $x \in M$ and an integer $r \in [m, n-1]$, we denote by $\mathrm{Sing}_\Delta^{x,r}$ the set of points $y \in M$ for which there is a singular horizontal path of rank $r$ joining $x$ to $y$.

# Minimal Rank Sard Conjecture

**Conjecture (Sard Conjecture)**

*For every* $x \in M$ *and every integer* $r \in [m, n-1]$, *the set* $\mathrm{Sing}_\Delta^{x,r}$ *has Lebesgue measure zero in* $M$.

**Conjecture (Minimal Rank Sard Conjecture)**

*For every* $x \in M$, *the set* $\mathrm{Sing}_\Delta^{x,m}$ *has Lebesgue measure zero in* $M$.

**Remark**

*In the case of corank* 1 *distributions the two above conjectures are equivalent.*

**Example**

The Minimal Rank Sard Conjecture is satisfied in Carnot groups.

# Part IV

IV. A Partial result

# MRS Conjecture in the splittable case

**Theorem (Belotto-Parusinski-R, 2022)**

*Assume that both $M$ and $\Delta$ are real-analytic. If the integrable distribution $\vec{\mathcal{K}}_{\mathcal{S}_0}$ is* **splittable**, *then the Minimal Rank Sard Conjecture holds true.*

**Corollary (Belotto-Parusinski-R, 2022)**

*Assume that both $M$ and $\Delta$ are real-analytic. If $\Delta$ has rank 3 then the Minimal Rank Sard Conjecture holds true.*

**Corollary (Belotto-Parusinski-R, 2022)**

*Assume that both $M$ and $\Delta$ are real-analytic. If $\Delta$ has corank 1 ($m = n - 1$) and $\vec{\mathcal{K}}_{\mathcal{S}_0}$ is* **splittable** *then the Sard Conjecture holds true.*

# Splittable foliations I

Setting:

- $N$ is a real-analytic manifold of dimension $n \geq 2$ equipped with a smooth Riemannian metric $h$.
- $\mathcal{F}$ is a regular analytic foliation of constant rank $d \in [1, n-1]$.

## Definition

Given $\ell > 0$, we say that $x$ and $y$ in $N$ are $(\mathcal{F}, \ell)$**-related** if there exists a smooth path $\varphi : [0,1] \rightarrow N$ with length $\in [0, \ell]$ (w.r.t. $h$) which is horizontal w.r.t. $\mathcal{F}$ and joins $x$ to $y$.

## Definition

Given $\bar{x} \in N$, we call **local transverse section at** $\bar{x}$ any set $S \subset N$ containing $\bar{x}$ which is a smooth submanifold diffeomorphic to the open disc of dimension $n - d$ and transverse to the leaves of $\mathcal{F}$.

# Splittable foliations II

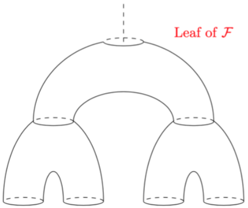
**Definition**

We say that the foliation $\mathcal{F}$ is **splittable** in $(N,h)$ if for every $\bar{x} \in N$, every local transverse section $S$ at $\bar{x}$ and every $\ell > 0$, the following property is satisfied:
For every Lebesgue measurable set $E \subset S$ with $\mathcal{L}^{n-d}(E) > 0$, there is a Lebesgue measurable set $F \subset E$ such that:

- $\mathcal{L}^{n-d}(F) > 0$,
- for any $x \neq y$ in $F$, $x$ and $y$ are not $(\mathcal{F},\ell)$-related.

# Examples

- Every foliation of rank 1 is splittable.
- If $\mathcal{F}$ has rank $\geq 2$ and the Ricci curvature (w.r.t. $h$) of all its leaves is uniformly bounded from below then it is splittable.
- By modifying a construction due to Hirsch, we can roughly speaking construct a smooth pair $(N, h)$ together with a rank 2 foliation which is not splittable.



# Sketch of proof

Thank you for your attention !!