Generic Aubry sets on surfaces

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Let $M$ be a smooth compact manifold of dimension $n \geq 2$ be fixed. Let $\mathcal{H} : T^* M \to \mathbb{R}$ be a Hamiltonian of class $C^k$, with $k \geq 2$, satisfying the following properties:

(H1) **Superlinear growth:**
For every $K \geq 0$, there is $C^*(K) \in \mathbb{R}$ such that

$$H(x, p) \geq K|p| + C^*(K) \quad \forall (x, p) \in T^* M.$$ 

(H2) **Uniform convexity:**
For every $(x, p) \in T^* M$, $\frac{\partial^2 H}{\partial p^2}(x, p)$ is positive definite.
We call **critical value** of $H$ the constant $c = c[H]$ defined as

$$c[H] := \inf_{u \in C^1(M; \mathbb{R})} \left\{ \max_{x \in M} \{ H(x, du(x)) \} \right\}.$$ 

In other terms, $c[H]$ is the infimum of numbers $c \in \mathbb{R}$ such that there is a $C^1$ function $u : M \rightarrow \mathbb{R}$ satisfying

$$H(x, du(x)) \leq c \quad \forall x \in M.$$
Critical value of $H$

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Note that

$$\min_{x \in M} \{ H(x, 0) \} \leq c[H] \leq \max_{x \in M} \{ H(x, 0) \}.$$
We call **critical subsolution** any Lipschitz function $u : M \to \mathbb{R}$ such that $H(x, du(x)) \leq c[H]$ for a.e. $x \in M$. 

Let $L : TM \to \mathbb{R}$ be the Tonelli Lagrangian associated with $H$ by Legendre-Fenchel duality, that is $L(x, v) := \max_{p \in T^* x M} \{ p \cdot v - H(x, p) \}$ for all $(x, v) \in TM$.

**Proposition** A Lipschitz function $u : M \to \mathbb{R}$ is a critical subsolution if and only if for every Lipschitz curve $\gamma : [a, b] \to M$, $u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(t), \dot{\gamma}(t)) \, dt + c[H](b-a)$. 

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Critical subsolutions

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The weak KAM Theorem

Definition

Given \( u : M \to \mathbb{R} \) and \( t \geq 0 \), \( T_t u : M \to \mathbb{R} \) is defined by

\[
T_t u(x) := \min_{y \in M} \{ u(y) + A_t(y, x) \},
\]

with

\[
A_t(z, z') := \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s)) \, ds + c[H] t \right\},
\]

where the infimum is taken over the Lipschitz curves \( \gamma : [0, t] \to M \) such that \( \gamma(0) = z \) and \( \gamma(t) = z' \).

Theorem (Fathi, 1997)

There is a critical subsolution \( u : M \to \mathbb{R} \) such that

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T_t u = u \quad \forall t \geq 0.
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It is called a critical or a weak KAM solution of \( H \).
The weak KAM Theorem

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More on critical solutions

Given a critical solution $u: M \rightarrow \mathbb{R}$, for every $x \in M$, there is a curve

$$\gamma: (-\infty, 0] \rightarrow M \quad \text{with} \quad \gamma(0) = x$$

such that, for any $a < b \leq 0$,

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b L(\gamma(s), \dot{\gamma}(s)) \, ds + c(b - a).$$

Therefore, any restriction of $\gamma$ minimizes the action between its end-points. Then, it satisfies the Euler-Lagrange equations.
The \textbf{projected Aubry set} of $H$ defined as

$$\mathcal{A}(H) = \{ x \in M \mid A_t(x, x) = 0 \}.$$ 

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Any critical subsolution $u$ is $C^1$ at any point of $\mathcal{A}(H)$ and satisfies $H(x, du(x)) = c[H], \forall x \in \mathcal{A}(H).$
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For every $x \in \mathcal{A}(H)$, the differential of a critical subsolution at $x$ does not depend on $u$. 

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The **Aubry set** of $H$ defined by

$$\tilde{\mathcal{A}}(H) := \{(x, du(x)) \mid x \in \mathcal{A}(H), u \text{ crit. subsol.}\} \subset T^*M$$

is compact, invariant by $\phi_H^t$, and is a Lipschitz graph over $\mathcal{A}(H)$. 

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Questions

- Uniqueness (up to constants) of critical solutions?
- Regularity of critical solutions?
- Structure of the Aubry sets?
- Size of the (quotiented) Aubry set?
- Dynamics of the Aubry set?
The Mañé Conjecture

Conjecture (Mañé, 96)

For every Tonelli Hamiltonian $H : T^* M \to \mathbb{R}$ of class $C^k$ (with $k \geq 2$), there is a residual subset (i.e., a countable intersection of open and dense subsets) $\mathcal{G}$ of $C^k(M)$ such that, for every $V \in \mathcal{G}$, the Aubry set of the Hamiltonian $H_V := H + V$ is either an equilibrium point or a periodic orbit.
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Strategy of proof:

- Density result.
- Stability result.
Partial results

**Theorem (Figalli-LR, 2011)**

Let $H : T^* M \to \mathbb{R}$ be a Tonelli Hamiltonian of class $C^2$. If there is a critical subsolution sufficiently regular on a neighborhood of $A(H)$, then for every $\epsilon > 0$, there exists $V \in C^2(M)$, with $\|V\|_{C^2} < \epsilon$ such that the Aubry set of $H + V$ is a hyperbolic periodic orbit.
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Theorem (Contreras-Figalli-LR, 2013)

Let $H : T^* M \to \mathbb{R}$ be a Tonelli Hamiltonian of class $C^2$, and assume that $\dim M = 2$. Then there is an open dense set of potentials $\mathcal{V} \subset C^2(M)$ such that, for every $V \in \mathcal{V}$, the Aubry set of $H + V$ is hyperbolic in its energy level.
Key ingredients of the proof

- Green bundles
- Nonsmooth analysis
- Techniques from closing lemmas
- Geometric control theory
- Geometric measure theory
For every $\theta \in T^* M$ and every $t \in \mathbb{R}$, we define the Lagrangian subspace $G^t_\theta \subset T_\theta T^* M$ by

$$G^t_\theta := (\phi^t_\theta)^* \left( V_{\phi^H_{-t}(\theta)} \right).$$
For every $\theta \in \tilde{\mathcal{A}}(H)$, we define the positive and negative Green bundles at $\theta$ as

$$G^+_\theta := \lim_{t \to +\infty} G^t_\theta \quad \text{and} \quad G^-_\theta := \lim_{t \to -\infty} G^t_\theta$$

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A dichotomy

Two cases may appear:

- For every $\theta \in \tilde{\mathbb{A}}(H)$ the Green bundles $G^-_{\theta}$ and $G^+_{\theta}$ are transverse.

- There is $\tilde{\theta} \in \tilde{\mathbb{A}}(H)$ such that $G^-_{\tilde{\theta}} = G^+_{\tilde{\theta}}$.
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- For every $\theta \in \tilde{\mathcal{A}}(H)$ the Green bundles $G^-_\theta$ and $G^+_\theta$ are transverse $\Rightarrow$ hyperbolicity of $\tilde{\mathcal{A}}(H)$

- There is $\bar{\theta} \in \tilde{\mathcal{A}}(H)$ such that $G^-_{\bar{\theta}} = G^+_{\bar{\theta}}$ $\Rightarrow$ further regularity for critical solutions
Further regularity (after Arnaud)

Definition

Let $S \subset \mathbb{R}^k$ be a compact set which has the origin as a cluster point. The **paratingent cone** to $S$ at 0 is the cone defined as

$$C_0(S) := \left\{ \lambda \lim_{i \to \infty} \frac{x_i - y_i}{|x_i - y_i|} \mid \lambda \in \mathbb{R}, x_i \neq y_i \xrightarrow{S} 0 \right\}.$$
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**Proposition**

For every $\theta \in \tilde{A}(H)$, there holds

$$G^-_\theta \leq C_\theta \left( \tilde{A}(H) \right) \leq G^+_\theta.$$
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Proposition

For every $\theta \in \tilde{A}(H)$, there holds

$$G^-_\theta \leq C_\theta (\tilde{A}(H)) \leq G^+_\theta.$$

As a consequence, if $G^-_\theta = G^+_\theta$ for some $\theta \in \tilde{A}(H)$, then $\tilde{A}(H)$ is locally contained in the graph of a Lipschitz 1-form which is $C^1$ at $\theta$.  

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Closing the Aubry set

Under this additional regularity, given $\epsilon > 0$, we are able to

- a $C^2$ potential $V : M \rightarrow \mathbb{R}$ with $\|V\|_{C^2} < \epsilon$,
- a periodic orbit $\gamma : [0, T] \rightarrow M$ ($\gamma(0) = \gamma(T)$),
- a Lipschitz function $v : M \rightarrow \mathbb{R}$,

in such a way that the following properties are satisfied:

- $H(x, dv(x)) + V(x) \leq 0$ for a.e. $x \in M$,
- $\int_0^T L(\gamma(t), \dot{\gamma}(t)) - V(\gamma(t)) \, dt = 0$. 
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- a $C^2$ potential $V : M \to \mathbb{R}$ with $\|V\|_{C^2} < \epsilon$,
- a periodic orbit $\gamma : [0, T] \to M$ ($\gamma(0) = \gamma(T)$),
- a Lipschitz function $\nu : M \to \mathbb{R}$,

in such a way that the following properties are satisfied:

- $H(x, d\nu(x)) + V(x) \leq 0$ for a.e. $x \in M$,
- $\int_0^T L(\gamma(t), \dot{\gamma}(t)) - V(\gamma(t))\,dt = 0$.

This shows that the Aubry set of $H + V$ contains a periodic orbit.
Thank you for your attention !!