The Sard Conjecture on Martinet Surfaces

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Let $M$ be a smooth connected manifold of dimension $n$.

**Definition**

A sub-Riemannian structure of rank $m$ in $M$ is given by a pair $(\Delta, g)$ where:

- $\Delta$ is a **totally nonholonomic distribution** of rank $m \leq n$ on $M$ which is defined locally by
  $$\Delta(x) = \text{Span}\left\{X^1(x), \ldots, X^m(x)\right\} \subset T_x M,$$
  where $X^1, \ldots, X^m$ is a family of $m$ linearly independent smooth vector fields satisfying the **Hörmander condition**.

- $g_x$ is a **scalar product** over $\Delta(x)$. 
We say that a family of smooth vector fields $X^1, \ldots, X^m$, satisfies the **Hörmander condition** if

$$\text{Lie}\{X^1, \ldots, X^m\}(x) = T_xM \quad \forall x,$$

where $\text{Lie}\{X^1, \ldots, X^m\}$ denotes the Lie algebra generated by $X^1, \ldots, X^m$, i.e. the smallest subspace of smooth vector fields that contains all the $X^1, \ldots, X^m$ and which is stable under Lie brackets.

**Reminder**

*Given smooth vector fields $X, Y$ in $\mathbb{R}^n$, the Lie bracket $[X, Y]$ at $x \in \mathbb{R}^n$ is defined by*

$$[X, Y](x) = DY(x)X(x) - DX(x)Y(x).$$
Lie Bracket: Dynamic Viewpoint

\[ e^{tX}(x) \]
Lie Bracket: Dynamic Viewpoint

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The Sard Conjecture on Martinet Surfaces
Exercise

There holds

\[
[X, Y](x) = \lim_{t \downarrow 0} \frac{(e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX})(x) - x}{t^2}.
\]
The Chow-Rashevsky Theorem

**Definition**

We call **horizontal path** any $\gamma \in W^{1,2}([0, 1]; M)$ such that

$$\dot{\gamma}(t) \in \Delta(\gamma(t)) \quad \text{a.e. } t \in [0, 1].$$
The Chow-Rashevsky Theorem

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The following result is the cornerstone of the sub-Riemannian geometry. (Recall that $M$ is assumed to be connected.)

**Theorem (Chow-Rashevsky, 1938)**

*Let $\Delta$ be a totally nonholonomic distribution on $M$, then every pair of points can be joined by an horizontal path.*
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The following result is the cornerstone of the sub-Riemannian geometry. (Recall that $M$ is assumed to be connected.)

**Theorem (Chow-Rashevsky, 1938)**

*Let $\Delta$ be a totally nonholonomic distribution on $M$, then every pair of points can be joined by an horizontal path.*

Since the distribution is equipped with a metric, we can measure the lengths of horizontal paths and consequently we can associate a metric with the sub-Riemannian structure.
Examples of sub-Riemannian structures

Example (Riemannian case)

*Every Riemannian manifold* $(M, g)$ *gives rise to a sub-Riemannian structure with* $\Delta = TM$. 
Examples of sub-Riemannian structures

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Every Riemannian manifold \((M, g)\) gives rise to a sub-Riemannian structure with \(\Delta = TM\).

Example (Heisenberg)

In \(\mathbb{R}^3\), \(\Delta = \text{Span}\{X^1, X^2\}\) with

\[
X^1 = \partial_x, \quad X^2 = \partial_y + x\partial_z \quad \text{et} \quad g = dx^2 + dy^2.
\]
Examples of sub-Riemannian structures

Example (Martinet)

In $\mathbb{R}^3$, $\Delta = \text{Span}\{X^1, X^2\}$ with

$$X^1 = \partial_x, \quad X^2 = \partial_y + x^2 \partial_z.$$ 

Since $[X^1, X^2] = 2x \partial_z$ and $[X^1, [X^1, X^2]] = 2 \partial_z$, only one bracket is sufficient to generate $\mathbb{R}^3$ if $x \neq 0$, however we needs two brackets if $x = 0$. 

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Since \( [X^1, X^2] = 2x \partial_z \) and \( [X^1, [X^1, X^2]] = 2 \partial_z \), only one bracket is sufficient to generate \( \mathbb{R}^3 \) if \( x \neq 0 \), however we needs two brackets if \( x = 0 \).

Example (Rank 2 distribution in dimension 4)

*In* \( \mathbb{R}^4 \), \( \Delta = \text{Span}\{X^1, X^2\} \) with

\[
X^1 = \partial_x, \quad X^2 = \partial_y + x \partial_z + z \partial_w
\]

satisfies \( \text{Vect}\{X^1, X^2, [X^1, X^2], [[X^1, X^2], X^2]\} = \mathbb{R}^4 \).
The sub-Riemannian distance

The length of an horizontal path $\gamma$ is defined by

$$\text{length}^g(\gamma) := \int_0^T |\dot{\gamma}(t)|^g_{\gamma(t)} \, dt.$$ 

Definition

Given $x, y \in M$, the **sub-Riemannian distance** between $x$ and $y$ is defined by

$$d_{SR}(x, y) := \inf \left\{ \text{length}^g(\gamma) \mid \gamma \text{ hor.}, \gamma(0) = x, \gamma(1) = y \right\}.$$
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The **length** of an horizontal path $\gamma$ is defined by

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**Proposition**

The manifold $M$ equipped with the distance $d_{SR}$ is a metric space whose topology coincides the one of $M$ (as a manifold).
**Definition**

Given $x, y \in M$, we call **minimizing horizontal path** between $x$ and $y$ any horizontal path $\gamma : [0, 1] \to M$ joining $x$ to $y$ satisfying $d_{SR}(x, y) = \text{length}^g(\gamma)$. 

The energy of the horizontal path $\gamma : [0, 1] \to M$ is given by $\text{ener}^g(\gamma) := \int_0^1 \left(\left|\dot{\gamma}(t)\right|^g\right)^2 dt$. 

**Definition**

We call **minimizing geodesic** between $x$ and $y$ any horizontal path $\gamma : [0, 1] \to M$ joining $x$ to $y$ such that $d_{SR}(x, y)^2 = \text{ener}^g(\gamma)$. 

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**The Sard Conjecture on Martinet Surfaces**
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Definition

We call **minimizing geodesic** between $x$ and $y$ any horizontal path $\gamma : [0, 1] \rightarrow M$ joining $x$ to $y$ such that

$$d_{SR}(x, y)^2 = \text{ener}^g(\gamma).$$
Let $x, y \in M$ and $\tilde{\gamma}$ be a minimizing geodesic between $x$ and $y$ be fixed. The SR structure admits an orthonormal parametrization along $\tilde{\gamma}$, which means that there exists a neighborhood $\mathcal{V}$ of $\tilde{\gamma}([0,1])$ and an orthonormal family of $m$ vector fields $X^1, \ldots, X^m$ such that

$$\Delta(z) = \text{Span}\left\{X^1(z), \ldots, X^m(z) \right\} \quad \forall z \in \mathcal{V}.$$
There exists a control $\bar{u} \in L^2([0, 1]; \mathbb{R}^m)$ such that

$$\dot{\gamma}(t) = \sum_{i=1}^{m} \bar{u}_i(t) X^i(\bar{\gamma}(t)) \quad \text{a.e. } t \in [0, 1].$$
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$$\dot{\gamma}(t) = \sum_{i=1}^{m} \bar{u}_i(t) X^i(\tilde{\gamma}(t)) \quad \text{a.e. } t \in [0, 1].$$

Moreover, any control $u \in U \subset L^2([0, 1]; \mathbb{R}^m)$ ($u$ sufficiently close to $\bar{u}$) gives rise to a trajectory $\gamma_u$ solution of

$$\dot{\gamma}_u = \sum_{i=1}^{m} u^i X^i(\gamma_u) \quad \text{sur } [0, T], \quad \gamma_u(0) = x.$$
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Furthermore, for every horizontal path $\gamma : [0, 1] \to \mathcal{V}$ there exists a unique control $u \in L^2([0, 1]; \mathbb{R}^m)$ for which the above equation is satisfied.
Consider the **End-Point mapping**

\[ E^{x,1} : L^2([0, 1]; \mathbb{R}^m) \to M \]

defined by

\[ E^{x,1}(u) := \gamma_u(1), \]

and set \( C(u) = \|u\|_{L^2}^2 \), then \( \bar{u} \) is a solution to the following **optimization problem with constraints**:
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$$\bar{u} \text{ minimize } C(u) \text{ among all } u \in \mathcal{U} \text{ s.t. } E^{x,1}(u) = y.$$
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(Since the family \( X^1, \ldots, X^m \) is orthonormal, we have

\[ \text{ener}^g(\gamma_u) = C(u) \quad \forall u \in \mathcal{U}. \)
Proposition (Lagrange Multipliers)

There exist $p \in T^*_y M \cong (\mathbb{R}^n)^*$ and $\lambda_0 \in \{0, 1\}$ with $(\lambda_0, p) \neq (0, 0)$ such that

$$p \cdot d_{\tilde{u}} E^{x,1} = \lambda_0 d_{\tilde{u}} C.$$
Proposition (Lagrange Multipliers)

There exist $p \in T^*_yM \simeq (\mathbb{R}^n)^*$ and $\lambda_0 \in \{0, 1\}$ with $(\lambda_0, p) \neq (0, 0)$ such that

$$p \cdot d\bar{u}E^{x,1} = \lambda_0 d\bar{u}C.$$

As a matter of fact, the function given by

$$\Phi(u) := (C(u), E^{x,1}(u))$$

cannot be a submersion at $\bar{u}$. Otherwise $D\bar{u}\Phi$ would be surjective and so open at $\bar{u}$, which means that the image of $\Phi$ would contain some points of the form $(C(\bar{u}) - \delta, y)$ with $\delta > 0$ small.

Two cases may appear: $\lambda_0 = 1$ or $\lambda_0 = 0$. 

First case : $\lambda_0 = 1$

This is the good case, the Riemannian-like case. The minimizing geodesic can be shown to be solution of a geodesic equation. It is smooth, there is a ”geodesic flow” ...

Second case : $\lambda_0 = 0$

In this case, we have

$$p \cdot D\bar{u}E_{x,1} = 0 \text{ with } p \neq 0,$$

which means that $\bar{u}$ is singular as a critical point of the mapping $E_{x,1}$.
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$$p \cdot D\bar{u}E^x,1 = 0 \text{ with } p \neq 0,$$

which means that $\bar{u}$ is singular as a critical point of the mapping $E^x,1$.

$\rightsquigarrow$ As shown by R. Montgomery, the case $\lambda_0 = 0$ cannot be ruled out.
Singular horizontal paths and Examples

**Definition**

An horizontal path is called **singular** if it is, through the correspondence $\gamma \leftrightarrow u$, a critical point of the End-Point mapping $E^{x,1} : L^2 \to M$.

**Example 1:** Riemannian case

Let $\Delta(x) = T_xM$, any path in $W^{1,2}$ is horizontal. There are no singular curves.

**Example 2:** Heisenberg, fat distributions

In $\mathbb{R}^3$, $\Delta$ given by $X_1 = \partial_x$, $X_2 = \partial_y + x \partial_z$ does not admit nontrivial singular horizontal paths.
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Example 3: Martinet-like distributions

In $\mathbb{R}^3$, let $\Delta = \text{Vect}\{X^1, X^2\}$ with $X^1, X^2$ of the form

\[ X^1 = \partial_{x_1} \quad \text{and} \quad X^2 = (1 + x_1 \phi(x)) \partial_{x_2} + x_1^2 \partial_{x_3}, \]

where $\phi$ is a smooth function and let $g$ be a metric over $\Delta$. 
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where $\phi$ is a smooth function and let $g$ be a metric over $\Delta$.

Theorem (Montgomery)

There exists $\bar{\epsilon} > 0$ such that for every $\epsilon \in (0, \bar{\epsilon})$, the singular horizontal path

$$\gamma(t) = (0, t, 0) \quad \forall t \in [0, \epsilon],$$

is minimizing (w.r.t. $g$) among all horizontal paths joining $0$ to $(0, \epsilon, 0)$. 
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is minimizing (w.r.t. $g$) among all horizontal paths joining $0$ to $(0, \epsilon, 0)$. Moreover, if $\{X^1, X^2\}$ is orthonormal w.r.t. $g$ and $\phi(0) \neq 0$, then $\gamma$ is not the projection of a normal extremal ($\lambda_0 = 1$).
Let \((\Delta, g)\) be a SR structure on \(M\) and \(x \in M\) be fixed.

\[
S_{\Delta, \text{min}}^x = \{\gamma(1) | \gamma : [0, 1] \rightarrow M, \gamma(0) = x, \gamma \text{ hor., sing., min.}\}.
\]

Conjecture (SR or minimizing Sard Conjecture)

The set \(S_{\Delta, \text{min}}^x\) has Lebesgue measure zero.
Let \((\Delta, g)\) be a SR structure on \(M\) and \(x \in M\) be fixed.

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S^x_{\Delta, \text{ming}} = \{ \gamma(1) | \gamma : [0, 1] \to M, \gamma(0) = x, \gamma \text{ hor., sing., min.} \}.
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**Conjecture (SR or minimizing Sard Conjecture)**

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**Conjecture (Sard Conjecture)**

The set \(S^x_{\Delta}\) has Lebesgue measure zero.
The Brown-Morse-Sard Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function of class $C^k$.

**Definition**

- We call **critical point** of $f$ any $x \in \mathbb{R}^n$ such that $d_xf : \mathbb{R}^n \to \mathbb{R}^m$ is not surjective and we denote by $C_f$ the set of critical points of $f$.

- We call **critical value** any element of $f(C_f)$. The elements of $\mathbb{R}^m \setminus f(C_f)$ are called **regular values**.
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H.C. Marston Morse (1892-1977)
Arthur B. Brown (1905-1999)
Anthony P. Morse (1911-1984)
Arthur Sard (1909-1980)
### The Brown-Morse-Sard Theorem

**Theorem (Arthur B. Brown, 1935)**

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be of class \( C^k \). If \( k = \infty \) (or large enough) then \( f(C_f) \) has empty interior.

**Theorem (Anthony P. Morse, 1939)**

Assume that \( m = 1 \) and \( k \geq m \), then \( f(C_f) \) has Lebesgue measure zero.

**Theorem (Arthur Sard, 1942)**

If \( k \geq \max\{1, n - m + 1\} \), \( \mathcal{L}^m(f(C_f)) = 0 \).

**Remark**

Thanks to a construction by Hassler Whitney (1935), the assumption in Sard’s theorem is sharp.
The Sard Theorem is false in infinite dimension. Let \( f : \ell^2 \rightarrow \mathbb{R} \) be defined by

\[
f \left( \sum_{n=1}^{\infty} x_n e_n \right) = \sum_{n=1}^{\infty} \left( 3 \cdot 2^{-n/3} x_n^2 - 2x_n^3 \right).
\]

The function \( f \) is polynomial \( (f^{(4)} \equiv 0) \) with critical set

\[
C(f) = \left\{ \sum_{n=1}^{\infty} x_n e_n \mid x_n \in \{0, 2^{-n/3}\} \right\}.
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The function $f$ is polynomial ($f^{(4)} \equiv 0$) with critical set

$$C(f) = \left\{ \sum_{n=1}^{\infty} x_n e_n \mid x_n \in \{0, 2^{-n/3}\} \right\},$$

and critical values

$$f(C(f)) = \left\{ \sum_{n=1}^{\infty} \delta_n 2^{-n} \mid \delta_n \in \{0, 1\} \right\} = [0, 1].$$
Back to the Sard Conjecture

Let \((\Delta, g)\) be a SR structure on \(M\) and \(x \in M\) be fixed. Set

\[
\Delta^\perp := \left\{ (x, p) \in T^*M \mid p \perp \Delta(x) \right\} \subset T^*M
\]

and (we assume here that \(\Delta\) is generated by \(m\) vector fields \(X^1, \ldots, X^m\)) define

\[
\tilde{\Delta}(x, p) := \text{Span}\left\{ \tilde{h}^1(x, p), \ldots, \tilde{h}^m(x, p) \right\} \quad \forall (x, p) \in T^*M,
\]

where \(h^i(x, p) = p \cdot X^i(x)\) and \(\tilde{h}^i\) is the associated Hamiltonian vector field in \(T^*M\).
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\]

where \(h^i(x, p) = p \cdot X^i(x)\) and \(\vec{h}^i\) is the associated Hamiltonian vector field in \(T^* M\).

**Proposition**

An horizontal path \(\gamma : [0, 1] \to M\) is singular if and only if it is the projection of a path \(\psi : [0, 1] \to \Delta^\perp \setminus \{0\}\) which is horizontal w.r.t. \(\vec{\Delta}\).
The case of Martinet surfaces

Let $M$ be a smooth manifold of dimension 3 and $\Delta$ be a totally nonholonomic distribution of rank 2 on $M$. We define the Martinet surface by

$$\Sigma_\Delta = \{ x \in M | \Delta(x) + [\Delta, \Delta](x) \neq T_x M \}$$

If $\Delta$ is generic, $\Sigma_\Delta$ is a surface in $M$. 

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The Sard Conjecture on Martinet Surfaces
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If $\Delta$ is generic, $\Sigma_\Delta$ is a surface in $M$. If $\Delta$ is analytic then $\Sigma_\Delta$ is analytic of dimension $\leq 2$. 

**Proposition**

The singular horizontal paths are the orbits of the trace of $\Delta$ on $\Sigma_\Delta$. 

Let us fix $x$ on $\Sigma_\Delta$ and see how its orbit look like.
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$\leadsto$ Let us fix $x$ on $\Sigma_\Delta$ and see how its orbit look like.
The Sard Conjecture on Martinet surfaces

Transverse case

$\Sigma_\Delta$
Generic tangent case
(Zelenko-Zhitomirskii, 1995)
The Sard Conjecture on Martinet surfaces

Let $M$ be of dimension 3 and $\Delta$ of rank 2.

$$S^x_\Delta = \{ \gamma(1)|\gamma : [0, 1] \to M, \gamma(0) = x, \gamma \text{ hor.}, \text{ sing.} \}.$$ 

Conjecture (Sard Conjecture)

The set $S^x_\Delta$ has vanishing $\mathcal{H}^2$-measure.

Theorem (Belotto-R, 2016)

The above conjecture holds true under one of the following assumptions:

- The Martinet surface is smooth;
- All datas are analytic and

$$\Delta(x) \cap T_x \text{Sing}(\Sigma_\Delta) = T_x \text{Sing}(\Sigma_\Delta) \quad \forall x \in \text{Sing}(\Sigma_\Delta).$$
Ingredients of the proof

- Control of the divergence of vector fields which generates the trace of $\Delta$ over $\Sigma$ of the form

$$|\text{div } \mathcal{Z}| \leq C |\mathcal{Z}|.$$
Ingredients of the proof

- Control of the divergence of vector fields which generates the trace of $\Delta$ over $\Sigma$ of the form

  $$|\text{div} Z| \leq C |Z|.$$ 

- Resolution of singularities.
An example

In $\mathbb{R}^3$,

$$X = \partial_y \quad \text{and} \quad Y = \partial_x + \left[ \frac{y^3}{3} - x^2y(x + z) \right] \partial_z.$$ 

Martinet Surface: $\Sigma_\Delta = \left\{ y^2 - x^2(x + z) = 0 \right\}$. 

Ludovic Rifford 

The Sard Conjecture on Martinet Surfaces
An example

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Thank you for your attention!!

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