ON THE EXISTENCE OF LOCAL SMOOTH REPULSIVE STABILIZING FEEDBACKS IN DIMENSION THREE

LUDOVIC RIFFORD

Abstract. Given an affine control system in $\mathbb{R}^3$ subject to the Hörmander's condition at the origin, we prove the existence of a local smooth repulsive stabilizing feedback at the origin. Our construction is based on the classical homogeneization procedure, on the existence of a semiconcave control-Lyapunov function, and on the classification of singularities of semiconcave functions in dimension two.

1. Introduction

This paper is concerned with the local stabilization problem for control systems of the form

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x), \tag{1}$$

where $X_1, \cdots, X_m$ are smooth vector fields on $\mathbb{R}^n$ which satisfy the Hörmander's bracket generating condition at the origin, namely,

$$\text{Lie}\{X_1, \cdots, X_m\}(0) = \mathbb{R}^n. \tag{2}$$

According to the classical Chow-Rashevsky theorem (see [6, 12, 33]), under the latter assumption, the control system (1) is locally controllable at the origin. This implies that there exists some neighbourhood of the origin $\mathcal{V}$ such that, for every $x \in \mathcal{V}$, there exists some open-loop control

$$u(\cdot) = (u_1(\cdot), \cdots, u_m(\cdot)) \in L^\infty([0,1]; \mathbb{R}^m)$$

for which the (unique) solution of

$$x(0) = x \text{ and } \dot{x}(t) = \sum_{i=1}^{m} u_i(t) X_i(x(t)) \text{ for almost every } t \geq 0,$$

satisfies $x(1) = 0$. A natural question is to wonder if such a control system is locally asymptotically stabilizable at the origin. In other words, do there exist some neighbourhood of the origin $\mathcal{W}$ and some continuous function

$$k = (k_1, \cdots, k_m) : \mathcal{W} \longrightarrow \mathbb{R}^m,$$

such that for any $\epsilon > 0$ sufficiently small, there is $\delta > 0$ such that all the trajectories of the closed-loop system

$$\dot{x}(t) = \sum_{i=1}^{m} k_i(x(t)) X_i(x(t)), \quad \forall t \geq 0,$$

L. Rifford: Département de Mathématiques d'Orsay, Université de Paris-Sud, Bâtiment 425, 91405 Orsay Cedex, France. Email: Ludovic.Rifford @math.u-psud.fr.

with $|x(0)| < \delta$ satisfy $|x(t)| < \epsilon$ for any $t \geq 0$ (property of Lyapunov stability) , and tend to the origin as t tends to infinity (property of attractivity)? In fact, the so-called stabilization problem can be stated in the much more general case of control systems of the form (1) which are locally asymptotically controllable at the origin.

The control system (1) is said to be locally asymptotically controllable at the origin if the following property is satisfied: There exists some constant $\rho > 0$ such that for every $\epsilon > 0$, there is $\delta > 0$ such that, for each $x \in \mathbb{R}^n$ with $|x| \leq \delta$ there is a control $u(\cdot) \in L^\infty([0,\infty);\mathbb{R}^m)$ such that $\|u(\cdot)\|_\infty \leq \rho$, the unique solution $x(\cdot)$ of

$$x(0) = x \text{ and } \dot{x}(t) = \sum_{i=1}^{m} u_i(t) X_i(x(t)) \text{ for almost every } t \geq 0,$$

tends to 0 as t tends to ∞ and $|x(t)| \leq \epsilon$ for all $t \geq 0$.
In general, such control systems are not locally asymptotically stabilizable at the origin. The Brockett's necessary condition ([10], or [17] for the stronger Coron's necessary condition) asserts that if the control system (1) is locally asymptotically stabilizable at the origin, that is if there exist some neighbourhood of the origin $\mathcal{W}$ and some continuous feedback $k : \mathcal{W} \to \mathbb{R}^m$ which satisfy the property above, then for any $\mu > 0$ sufficiently small, there is $\nu > 0$ (such that $\nu B \subset \mathcal{W}$, where B denotes the open unit ball in $\mathbb{R}^n$) such that

$$\mu B \subset \left\{ \sum_{i=1}^{m} k_i(x) X_i(x) : x \in \nu B \right\} \subset \left\{ \sum_{i=1}^{m} u_i X_i(x) : x \in \nu B, u \in \mathbb{R}^m \right\}.$$

Moreover, many control systems which satisfy the Hörmander's condition (2) do not satisfy the Brockett's necessary condition. For instance the well-known "nonholonomic integrator", given by

$$\dot{x} = u_1 X_1(x) + u_2 X_2(x) = u_1 \begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ -x_1 \end{pmatrix},$$

satisfies the Hörmander's condition (2), but

$$\begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} \notin \left\{ u_1 X_1(x) + u_2 X_2(x) : x \in \mathbb{R}^3, (u_1, u_2) \in \mathbb{R}^2 \right\},$$

for any $\mu \neq 0$. More generally, any control system of the form

$$\dot{x} = u_1 X_1(x) + u_2 X_2(x),$$

which satisfies the Hörmander's condition (2) and such that $X_1(0)$ and $X_2(0)$ are linearly independent, is locally controllable at the origin but not locally asymptotically stabilizable at the origin.

The absence of continuous stabilizing feedbacks motivated several authors to define new types of stabilizing feedbacks; contributions in that direction have been made by Sussmann [49], Artstein [4], Coron [18, 19], Clarke, Ledyaev, Sontag and Subbotin [15], Ancona and Bressan [1], and the author [34, 35, 38, 41, 42][1]. For instance, Ancona and Bressan proved in [1] (see also [38]), that if the control system (1) is locally (resp. globally) asymptotically controllable at the origin, then there exists a feedback law $k : \mathbb{R}^n \to \mathbb{R}^m$ which is locally bounded (and indeed piecewise constant) such that the closed-loop system is locally (resp. globally) asymptotically stable at the origin in the sense of Carathéodory. This means that we can construct some neighbourhood of the origin $\mathcal{W}$ and some function $k : \mathcal{W} \to \mathbb{R}^m$ which is locally bounded, such that, for any $\epsilon > 0$ sufficiently small, there is $\delta > 0$ such that for any $x \in \mathcal{W}$ with $|x| < \delta$, all the absolutely continuous arcs $x(\cdot) : [0, \infty) \to \mathbb{R}^n$ which are solutions (such arcs exist) of

$$x(0) = x \quad \text{and } \dot{x}(t) = \sum_{i=1}^{m} k_i(x(t))X_i(x(t)), \qquad \text{a.e. } t \in [0, \infty),$$

satisfy $|x(t)| < \epsilon$ for any $t \geq 0$ (property of Lyapunov stability), and tend to the origin as t tends to infinity (property of attractivity). These solutions are called the Carathéodory solutions of the closed-loop system

$$\dot{x} = \sum_{i=1}^{m} k_i(x)X_i(x). \tag{3}$$

To be clear, whenever we will say that some feedback $k : \mathcal{W} \to \mathbb{R}^m$ stabilizes (locally or globally) the system (1) to the origin in the sense of Carathéodory, we mean that the Carathéodory solutions of the closed-loop system (3) exist and that all of them satisfy both properties of Lyapunov stability and of attractivity.

In our previous paper [42], we viewed that the Carathéodory stabilizing feedback k can indeed be taken to be smooth outside some stratified closed set $\mathcal{S}$ (called the singular set of the stabilizing feedback) in such a way that the Carathéodory solutions of (3) remain outside the set $\mathcal{S}$ for all time $t \geq 0$ except for t in a locally finite subset of $[0, \infty)$. In other words, for every Carathéodory solution $x(\cdot) : [0, \infty) \to \mathbb{R}^n$ of the closed-loop system (3), the set of times t for which $x(t)$ belongs to $\mathcal{S}$ is a locally finite subset of $[0, \infty)$. In view of this result, it is natural to wonder if we can avoid crossing the singular set $\mathcal{S}$ for positive time. It is proved in [38] that such a property holds for control systems of the form (1) with $m = 1$; and also, whenever the control system (1) admits a certain type of semiconcave control-Lyapunov function. Let us clarify the type of stabilizing feedback we would like to construct.

Definition 1.1. *We say that the control system (1) admits a local smooth repulsive stabilizing feedback at the origin (abreviated $LSRS_0$ feedback in the sequel) if there exist a neighbourhood of the origin $\mathcal{W}$, a set $\mathcal{S} \subset \mathcal{W}$ containing the origin and a feedback $k_{\mathcal{S}} : \mathcal{W} \to \mathbb{R}^m$ such that the following properties are satisfied:*

[1]For more details on the stabilization problem, we recommend to the reader the historical accounts of Coron [20] and Sontag [48]; and for a survey of the contribution of the author, we refer the reader to [40].

(i) The set $\mathcal{S}$ is closed.
(ii) The feedback $k_{\mathcal{S}}$ is locally bounded on $\mathbb{R}^n$ and smooth on $\mathbb{R}^n \setminus \mathcal{S}$.
(iii) The closed-loop system (3) is locally asymptotically stable at the origin in the sense of Carathéodory.
(iv) For any Carathéodory trajectory $x(\cdot)$ of (3),

$$x(t) \notin \mathcal{S}, \quad \forall t > 0.$$

From the point of view of applications, we notice that the smooth repulsive stabilizing feedbacks share the same properties of robustness as the discontinuous stabilizing feedbacks which were constructed in [15, 14, 38] (see also [2, 32]). Moreover, we stress the fact that, whenever a control system is stabilized by means of a smooth repulsive stabilizing feedback, then this feedback depends smoothly on time (for positive times) along any trajectory of the corresponding closed-loop system. We proved in [41] that, if the control system (1) evolves on a smooth surface M, and if the Hörmander's bracket generating condition is satisfied for every $x \in M$, then for any $x_0 \in M$ there exists a smooth repulsive stabilizing feedback which stabilizes the control system globally to the point x_0. In the present paper, our objective is to prove a similar result locally in dimension three. More precisely, we will prove:

Theorem 1. *If $n = 3$ and if the control system (1) satisfies the Hörmander's bracket generating condition at the origin (2), then it admits a local smooth repulsive stabilizing feedback at the origin.*

Our proof is based on the concept of semiconcave control-Lyapunov functions, on the classification of singularities of some stabilizing feedbacks for control systems on surfaces that we proved in [39], and on some classical techniques of homogeneization for control systems satisfying the Hörmander's bracket generating condition. The paper is organized as follows. In Section 2, we provide preliminaries on homogeneous control systems, proving a converse-Lyapunov result for globally asymptotically controllable homogeneous control systems. In Section 3, we prove Theorem 1 in the special case of homogeneous control systems of degree zero with respect to the standard dilation. Then we deduce in Section 4 a proof of Theorem 1. In Section 5, we announce corollaries concerning the stabilization by smooth periodic feedbacks. In Appendix, we present an example of an analytic homogeneous control system in $\mathbb{R}^3$ which is globally asymptotically stable at the origin and which does not admit a smooth repulsive stabilizing feedback at the origin.

Notations:
Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $|\cdot|$ the Euclidean norm of $\mathbb{R}^n$, B the open unit ball $\{x : |x| < 1\}$ in $\mathbb{R}^n$, $\overline{B}$ the closure of B and $B(x,r) = x + rB$ (resp. $\overline{B}(x,r) = x + r\overline{B}$) the ball (resp. the closed ball) centered at x and with radius r. The unit sphere of $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$ and for any $x \in \mathbb{S}^{n-1}$ the tangent space to $\mathbb{S}^{n-1}$ at x is denoted by $T_x\mathbb{S}^{n-1}$. In addition if A is a subset of $\mathbb{R}^n$, then $d_A(\cdot)$ denotes the distance function to the set A in $\mathbb{R}^n$. If m is a positive integer, $|\cdot|_m$ denotes the Euclidean norm of $\mathbb{R}^m$, B_m the open unit ball of $\mathbb{R}^m$, $\overline{B_m}$ the closure of B_m

and $B_m(x,r) = x + rB_m$ (resp. $\overline{B_m}(x,r) = x + r\overline{B_m}$) the ball (resp. the closed ball) centered at x and with radius r. Furthermore, an admissible control for the system (1) is a function $u(\cdot) : [0,\infty) \to \mathbb{R}^m$ which belongs to $\mathcal{U} := L^\infty([0,\infty);\mathbb{R}^m)$; we denote by $\|u(\cdot)\|_\infty$ the supremum norm of $u(\cdot) \in \mathcal{U}$. We recall that if the vector fields $X_1, \cdots X_m$ are assumed to be bounded on $\mathbb{R}^3$ (or on $\mathbb{R}^n$ if we work on $\mathbb{R}^n$), then for every $x \in \mathbb{R}^3$ and for any admissible control $u(\cdot)$, there exists a unique absolutely continuous curve $x(\cdot) : [0,\infty) \to \mathbb{R}^n$ which satisfies

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) X_i(x(t))$$

for almost every $t \in [0,\infty)$ and such that $x(0) = x$. If x is some given state in $\mathbb{R}^3$ and if $u(\cdot)$ is an admissible control, we denote by $x(\cdot; x, u(\cdot))$ the trajectory solution of the system above and such that $x(0; x, u(\cdot)) = x$. Let $\mathcal{K}_\infty$ denote the set of all continuous functions $\rho : [0,\infty) \to [0,\infty)$ for which (i) $\rho(0) = 0$ and (ii) ρ is strictly increasing and unbounded. We let $\mathcal{KL}$ denote the set of all continuous functions $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ for which (1) $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each $t \geq 0$, (2) $\beta(s, \cdot)$ is nonincreasing for each $s \geq 0$, and (3) $\beta(s,t) \to 0$ as $t \to +\infty$ for each $s \geq 0$.

2. Preliminaries on homogeneous control systems

The aim of this section is to develop results about homogeneous control systems which are of interest in the proof of Theorem 1. Until the end of this section, we consider a general control system of the form

$$\dot{x} = Y(x,u) := \sum_{i=1}^{m} u_i Y_i(x) \tag{4}$$

where $Y_1, \cdots, Y_m$ are locally Lipschitz vector fields on $\mathbb{R}^n$ and where the control $u = (u_1, \cdots, u_m)$ belongs to $\mathbb{R}^m$. First we introduce the definitions of dilations and homogeneity, then we define the notion of homogeneous control systems and we prove a homogeneous converse Lyapunov theorem for homogeneous control systems which are globally asymptotically controllable at the origin.

2.1. **Dilations and homogeneity.**

For any $\epsilon > 0$, the dilation δ_ϵ^r associated with a "weight vector" $r = (r_1, \cdots, r_n)$ (where the r_i's are positive integers), is the map $\delta_\epsilon^r : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\delta_\epsilon^r(x_1, \cdots, x_n) := (\epsilon^{r_1} x_1, \cdots, \epsilon^{r_n} x_n).$$

If all $r_i = 1$ we write δ_ϵ^1 and call this the standard dilation.
A continuous function $h : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree $d \geq 0$ (where d is an integer) with respect to δ_ϵ^r, if

$$\forall \epsilon > 0, \forall x \in \mathbb{R}^n, \quad h(\delta_\epsilon^r(x)) = \epsilon^d h(x).$$

A continuous vector field Z on $\mathbb{R}^n$ is said to be homogeneous of degree $k \leq 1$ (where k is an integer) with respect to δ_ϵ^r if for every $j \in \{1, \cdots, n\}$ the j-th component of Z (*i.e.* the function $x \mapsto Z_j(x)$) is homogeneous of degree $r_j - k$ with respect to δ_ϵ^r.

2.2. **Homogeneous control systems.** In this subsection, we will assume that the control system (4) is homogeneous of degree $k \leq 1$ with respect to the dilation δ_ϵ^r, namely, that each vector field Y_i $(i = 1, \cdots, m)$ is homogeneous of degree k with respect to δ_ϵ^r, which means that for every $i = 1, \cdots, m$ and for any $x \in \mathbb{R}^n, \epsilon > 0$, we have

$$Y_i(\delta_\epsilon^r(x)) = \epsilon^{-k} \delta_\epsilon^r(Y_i(x)). \tag{5}$$

Applying suitable coordinate and time transformations, Grüne showed in [22] that we can considerably simplify the class of systems to be considered. We state this idea in the following proposition. We set $\Omega := \mathbb{R}^n \setminus \{0\}$.

Proposition 2.1. *Set $\mu := \min_{i=1,\cdots,n}\{r_i\}$ and $\gamma := \frac{k}{\mu} \in \mathbb{Q}$. There exists a homeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ with $\Phi(0) = 0$ which is an analytic diffeomorphism from Ω into Ω, and such that if we set for every $i = 1, \cdots, m$ and for every $y \in \Omega$, $\tilde{Y}_i(y) := D\Phi(\Phi^{-1}(y)) \cdot Y_i(\Phi^{-1}(y))$, then we have*

$$\forall y \in \Omega, \forall \epsilon > 0, \quad \tilde{Y}_i(\epsilon y) = \epsilon^{1-\gamma} \tilde{Y}_i(y). \tag{6}$$

Instead of refering to Grüne's paper for the proof, we prefer to be complete and give it. So, let us prove Proposition 2.1.

Proof. Following Grüne [22], corresponding to the family of dilations δ_ϵ^r, we define a function $N : \mathbb{R}^n \to [0, \infty)$ which can be interpreted as a dilated norm with respect to δ_ϵ^r. Denoting $l = 2\prod_{i=1}^n r_i > 0$, we define for every $x \in \mathbb{R}^n$, $N(x)$ by

$$N(x) := \left(\sum_{i=1}^n |x_i|^{\frac{l}{r_i}} \right)^{\frac{1}{l}}. \tag{7}$$

We note that N is analytic on Ω. Moreover, we have $N(0) = 0, N(x) > 0$ if $x \in \Omega$, and $N(\delta_\epsilon^r(x)) = \epsilon N(x)$ for any $x \in \mathbb{R}^n, \epsilon \geq 0$. Using the function N, we can define $P : \Omega \to \Omega$ by

$$P(x) = (N(x)^{-r_1} x_1, \cdots, N(x)^{-r_n} x_n), \quad \text{for any } x = (x_1, \cdots, x_n) \in \Omega.$$

The function P defines a projection from Ω into $N^{-1}(1)$ satisfying $P(\delta_\epsilon^r(x)) = P(x)$ for any $x \in \Omega, \epsilon > 0$. Since for any $x \in \Omega$, the function $t \mapsto N(tx)$ is strictly increasing, it is bijective and then the function

$$\begin{array}{rcl} S : N^{-1}(1) & \to & \mathbb{S}^{n-1} \\ x & \mapsto & \dfrac{x}{|x|} \end{array}$$

is an analytic diffeomorphism. Define a coordinate transformation $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi(x) := N(x)^\mu S(P(x)) \text{ if } x \in \Omega, \text{ and } \Phi(0) := 0;$$

it is continuous on $\mathbb{R}^n$ and analytic on Ω. We have

$$\Phi(\delta_\epsilon^r(x)) = \epsilon^\mu \Phi(x), \quad \Phi^{-1}(\epsilon^\mu y) = \delta_\epsilon^r(\Phi^{-1}(y)), \tag{8}$$

and then by differentiation, we obtain that for every $x \in \Omega$ and $v \in \mathbb{R}^n$,

$$D\Phi(\delta_\epsilon^r(x)) \cdot v = \epsilon^\mu D\Phi(x) \cdot [(\delta_\epsilon^r)^{-1}(v)]. \tag{9}$$

Thus defining for every $i = 1, \cdots, m$ and for every $y \in \Omega$,

$$\tilde{Y}_i(y) := D\Phi(\Phi^{-1}(y)).Y_i(\Phi^{-1}(y)),$$

we obtain (with $x = \Phi^{-1}(y)$)

$$\begin{aligned}
\tilde{Y}_i(\epsilon^\mu y) &= D\Phi(\Phi^{-1}(\epsilon^\mu y)).Y_i(\Phi^{-1}(\epsilon^\mu y)) \\
&= D\Phi(\delta_\epsilon^r(x)) \cdot Y_i(\delta_\epsilon^r(x)) \text{ (by (8))} \\
&= \epsilon^\mu D\Phi(x) \cdot ((\delta_\epsilon^r)^{-1}(Y_i(\delta_\epsilon^r(x)))) \text{ (by (9))} \\
&= \epsilon^\mu \epsilon^{-k} D\Phi(x) \cdot (Y_i(x)) \text{ (by homogeneity of } Y_i) \\
&= \epsilon^{\mu-k}\tilde{Y}_i(y).
\end{aligned}$$

We deduce that for for every $i = 1, \cdots, m$ and for any $y \in \Omega$, we have

$$\tilde{Y}_i(\epsilon y) = \epsilon^{1-\frac{k}{\mu}}\tilde{Y}_i(y) = \epsilon^{1-\gamma}\tilde{Y}_i(y).$$

This completes the proof of Proposition 2.1. □

The construction of the $\tilde{Y}_i$'s leads also to the following result.

Proposition 2.2. *Let $i \in \{1, \cdots, m\}$. If we set $\bar{Y}_i(0) := 0$ and*

$$\bar{Y}_i(y) := |y|^\gamma \tilde{Y}_i(y), \quad \forall y \in \Omega, \tag{10}$$

then the vector field $\bar{Y}_i$ is globally Lipschitz on $\mathbb{R}^n$ and homogeneous of degree zero with respect to the standard dilation.

Proof. Let $i \in \{1, \cdots, m\}$. Notice that by construction the vector field $\bar{Y}_i$ is locally Lipschitz on Ω. The fact that $\bar{Y}_i$ is homogeneous of degree zero with respect to the standard dilation is a consequence of (6). This implies that for any $y \in \Omega$,

$$|\bar{Y}_i(y)| = |y| \left| \bar{Y}_i\left(\frac{y}{|y|}\right) \right|.$$

Since the vector field $\bar{Y}_i$ is locally Lipschitz on the compact sphere $\mathbb{S}^{n-1}$, it is bounded on $\mathbb{S}^{n-1}$. Hence the equality above proves that the vector field $\bar{Y}_i$ is continuous at the origin. Furthermore we have for any $y \in \Omega$ and for any $\epsilon > 0$,

$$D\bar{Y}_i(\epsilon y) = D\bar{Y}_i(y).$$

Hence if we denote by $L_{\bar{Y}_i}$ the maximum of $|D\bar{Y}_i(y)|$ for $y \in \mathbb{S}^{n-1}$, we can write for every $y \in \Omega$,

$$|D\bar{Y}_i(y)| \leq L_{\bar{Y}_i}.$$

Let $x, y \in \mathbb{R}^n$. There exist two sequences x_n (resp. y_n) which converge to x (resp. y) such that for each $n \in \mathbb{N}$, the segment $[x_n, y_n]$ belongs to Ω. By the Mean Value theorem, this implies that for each $n \in \mathbb{N}$,

$$|\bar{Y}_i(y_n) - \bar{Y}_i(x_n)| \leq L_{\bar{Y}_i}|x_n - y_n|.$$

By continuity of $\bar{Y}_i$ on $\mathbb{R}^n$, we conclude that the vector field $\bar{Y}_i$ is globally Lipschitz on $\mathbb{R}^n$. □

Finally, if we consider the new control system defined by the vector fields $\bar{Y}_1, \cdots, \bar{Y}_m$, that is the control system

$$\bar{Y}(y, v) := \sum_{i=1}^m v_i \bar{Y}_i(y), \tag{11}$$

where $y \in \mathbb{R}^n$ and where the control $v = (v_1, \cdots, v_m) \in \mathbb{R}^m$, then the following result holds.

Proposition 2.3. *Let $T > 0$ and $x \in \Omega$. If $u(\cdot)$ is some admissible control such that the corresponding trajectory $x(\cdot; x, u(\cdot))$ of (4) remains in Ω, then the arc $y(\cdot)$ on $[0, T]$ defined by*

$$y(t) := \Phi(x(t)), \quad \forall t \in [0, T],$$

is absolutely continuous and is the solution of (11) associated to the control

$$v(\cdot) := |\Phi(x(t))|^{-\gamma} u(\cdot),$$

such that $y(0) = \Phi(x)$.

Proof. The absolutely continuity of $y(\cdot)$ comes from the fact that $x(\cdot)$ is itself absolutely continuous and that Φ is smooth on Ω. Moreover we can write for almost every $t \in [0, T]$,

$$\begin{aligned}
\dot{y}(t) &= D\Phi(x(t)) \cdot \dot{x}(t) \\
&= D\Phi(\Phi^{-1}(y(t))) \cdot \sum_{i=1}^{m} u_i(t) Y_i(x(t)) \\
&= \sum_{i=1}^{m} u_i(t) D\Phi(\Phi^{-1}(y(t))) \cdot Y_i(\Phi^{-1}(y(t))) \\
&= \sum_{i=1}^{m} u_i(t) \tilde{Y}_i(y(t)) \\
&= \sum_{i=1}^{m} v_i(t) |y(t)|^{\gamma} \tilde{Y}_i(y(t)) \\
&= \sum_{i=1}^{m} v_i(t) \bar{Y}_i(y(t));
\end{aligned}$$

which concludes the proof. □

2.3. **Homogeneous control-Lyapunov functions for GAC_0 homogeneous control systems.** The result that we present in this subsection asserts that if the homogeneous control system (4) is globally asymptotically controllable at the origin then it admits a control-Lyapunov function which is semiconcave outside the origin and homogeneous of degree 1 with respect to the same dilation. Before giving the statement of our result, we recall some basic definitions. We first give the definition of GAC_0 control systems, then we present the definition of a semiconcave control-Lyapunov function for some control system. We recall that the concept of nonsmooth control-Lyapunov functions has been initially introduced by Sontag in his seminal paper [47], where he proved the equivalence of global asymptotic controllability and the existence of a continuous control-Lyapunov function. Furthermore we notice that a similar result has been proved by Grüne in [22].

Definition 2.4. [2] *We call the control system (4) globally asymptotically controllable at the origin (abreviated GAC_0 in the sequel) provided there are a nondecreasing function $\sigma : [0,\infty) \to [0,\infty)$ and a function $\beta \in \mathcal{KL}$ satisfying the following properties:*
For each $x \in \mathbb{R}^n$, there exists $u(\cdot) \in \mathcal{U}$ such that
(a) $|x(t;x,u(\cdot))| \leq \beta(|x|,t)$ for all $t \geq 0$.
(b) $\|u(\cdot)\|_\infty \leq \sigma(|x|)$.

Whenever the control system (4) is homogeneous, this definition can be simplified; we have the following result:

Proposition 2.5. *Assume that the control system (4) is homogeneous of degree k with respect to some dilation δ_ϵ^r and fix two constants $R_1, R_2 > 0$ satisfying $R_1 > R_2 > 0$. Then the control system (1) is GAC_0 if and only if there are two constants $M, T > 0$ such that for any $x \in N^{-1}(R_1)$, there exists $u(\cdot) \in \mathcal{U}$ with $\|u(\cdot)\|_\infty \leq M$ such that $N(x(t;x,u(\cdot)) \leq M$ for any $t \in [0,T]$, and such that $N(x(T;x,u(\cdot))) \leq R_2$. Here N denotes the dilated norm with respect to δ_ϵ^r that we defined in the proof of Proposition 2.1.*

Proof. We just have to prove that the property given in the statement of the proposition implies the property given in Definition 2.4. Before beginning the proof, we recall that by construction the dilated norm N is continuous on $\mathbb{R}^n$, analytic outside the origin, positive definite, and satisfies

$$N(\delta_\epsilon^r(x)) = \epsilon N(x), \quad \forall x \in \mathbb{R}^n, \forall \epsilon > 0. \tag{12}$$

Moreover we notice that by homogeneity of (4), for any $x \in \mathbb{R}^n$ and for any $u(\cdot) \in \mathcal{U}$ (such that the corresponding trajectory $x(\cdot;x,u(\cdot))$ of (4) is defined on $[0,\infty)$), we have that for any $\epsilon > 0$,

$$x(t;\delta_\epsilon^r(x), \epsilon^k u(\cdot)) = \delta_\epsilon^r(x(t;x,u(\cdot))), \quad \forall t \geq 0. \tag{13}$$

As before we set $\mu := \min_{i=1,\cdots,n}\{r_i\}$ and $l := 2\pi_{i=1}^n r_i$ (which indeed appears in the definition of N), and in addition we define the constant $\bar{M} > 0$ by,

$$\bar{M} := \max_{x \in N^{-1}(1)} |x|.$$

Since for every $\lambda \in [0,\infty]$ and for any $i = 1,\cdots,m$, we have $\lambda^{2r_i} \leq \max\{\lambda, \lambda^l\}$, and since for any $x \in \mathbb{R}^n \setminus \{0\}$ the point $\delta^r_{N(x)^{-1}}(x)$ belongs to $N^{-1}(1)$ (due to (12)), we deduce that for any $x \in \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned} |x|^2 &= \sum_{i=1}^n N(x)^{2r_i} N(x)^{-2r_i} x_i^2 \\ &\leq \max\left\{N(x)^l, N(x)\right\} |\delta^r_{N(x)^{-1}}(x)|^2 \\ &\leq \bar{M}^2 \max\left\{N(x), N(x)^l\right\}. \end{aligned} \tag{14}$$

[2]A routine argument involving continuity of trajectories with respect to initial states shows that the requirements of the given definition are equivalent to the following apparently weaker pair of conditions used in some references (see [36] and references therein):
1. For each $x \in \mathbb{R}^n$ there is a control $u(\cdot) \in \mathcal{U}$ such that $x(t;x,u(\cdot))$ tends to 0 as $t \to \infty$.
2. There exists $\rho > 0$ such that for each $\epsilon > 0$, there is a $\delta > 0$ such that for each $x \in \mathbb{R}^n$ with $|x| \leq \delta$ there is a control $u(\cdot) \in \mathcal{U}$ such that $\|u(\cdot)\|_\infty \leq \rho$, such that $x(t;x,u(\cdot))$ tends to 0 as $t \to \infty$, and such that $|x(t;x,u(\cdot))| \leq \epsilon$ for all $t \geq 0$.

Furthermore since for every $\lambda \in [0,\infty]$ and for any $i = 1,\cdots,m$, we have $\lambda^{\frac{l}{r_i}} \leq \max\{\lambda,\lambda^l\}$, we have for any $x \in \mathbb{R}^n \setminus \{0\}$,

$$N(x) \leq n^{\frac{1}{l}} \max\left\{|x|^{\frac{1}{l}},|x|\right\}. \tag{15}$$

Define the function $\alpha : [0,\infty) \to \mathbb{R}$ of $\mathcal{K}_\infty$ by,

$$\alpha(s) := n^{\frac{1}{l}} \max\left\{s^{\frac{1}{l}},s\right\}, \quad \forall s \in [0,\infty), \tag{16}$$

and pick some function $\beta \in \mathcal{KL}$ which satisfies

$$\beta(s,t) \geq \bar{M} \max\left\{\sqrt{\frac{M\alpha(s)}{R_1}}\left(\frac{R_2}{R_1}\right)^{\frac{k}{2}}, \left(\frac{M\alpha(s)}{R_1}\right)^{\frac{l}{2}}\left(\frac{R_2}{R_1}\right)^{\frac{kl}{2}}\right\},$$

for any integer $k \geq 0$ and any pair $(s,t) \in (0,\infty)\times(0,\infty)$ such that $t \in [kT,(k+1)T]$, and set for every $x \in \mathbb{R}^n, \sigma(x) := M$. We wish to prove that the control system (4) and the functions β and σ satisfy the properties (a)-(b) of Definition 2.4.
Fix $x \in N^{-1}(R_1)$. By assumption there exists some control $u_0(\cdot) \in \mathcal{U}$ with $\|u_0(\cdot)\|_\infty \leq M$ such that $N(x(t;x,u_0(\cdot))) \leq M$ for any $t \in [0,T]$, and such that $N(x(T;x,u_0(\cdot))) \leq R_2$. In fact since the dynamics vanish for $u = 0$ we can assume without loss of generality that $N(x(t;x,u_0(\cdot))) \geq R_2$ for any $t \geq 0$, and in particular that $N(x(T;x,u_0(\cdot))) = R_2$. Set $y := x(T;x,u(\cdot))$; by (12) the point $\delta^r_{R_1R_2^{-1}}(y)$ belongs to $N^{-1}(R_1)$, hence by the assumption and by (12)-(13), there exists a new control $u_1(\cdot) \in \mathcal{U}$ with $\|u_1(\cdot)\|_\infty \leq M$ such that $R_1^{-1}R_2 \leq N(y(t;y,u_1(\cdot))) \leq R_1^{-1}R_2M$ for any $t \in [0,T]$,and such that $N(y(T;y,u_1(\cdot))) = R_1^{-1}R_2^2$. Continuing this procedure inductively and pasting together the different controls $(u_k)_{k\geq 0}$, we obtain some control $u(\cdot) \in \mathcal{U}$ with $\|u(\cdot)\|_\infty \leq M$ such that for any integer $k \geq 0$, we have

$$0 < N(x(t;x,u(\cdot))) \leq M\left(\frac{R_2}{R_1}\right)^k,$$

whenever $t \in [kT,(k+1)T]$. Using the inequality (14), we deduce that for any integer $k \geq 0$ and any $t \in [kT,(k+1)T]$,

$$\begin{aligned} |x(t;x,u(\cdot))| &\leq \bar{M} \max\left\{\sqrt{N(x(t;x,u(\cdot)))}, N(x(t;x,u(\cdot)))^{\frac{l}{2}}\right\} \\ &\leq \bar{M} \max\left\{\sqrt{M}\left(\frac{R_2}{R_1}\right)^{\frac{k}{2}}, M^{\frac{l}{2}}\left(\frac{R_2}{R_1}\right)^{\frac{kl}{2}}\right\}. \end{aligned}$$

Since by (15)-(16) we have that $N(x) = R_1 \Longrightarrow \alpha(|x|) \geq R_1$, this implies that $|x(t;x,u(\cdot))| \leq \beta(|x|,t)$ for any $t \geq 0$. This proves properties (a)-(b) in the case $x \in N^{-1}(R_1)$. Whenever $x \in \mathbb{R}^n \setminus \{0\}$ (the case $x = 0$ being obvious), by noticing that the point $\delta^r_{R_1N(x)^{-1}}(x)$ belongs to $N^{-1}(R_1)$ and by the same argument as above, we get the existence of some control $u(\cdot) \in \mathcal{U}$ with $\|u(\cdot)\|_\infty \leq M$ such that for any integer $k \geq 0$

$$N(x(t;\delta^r_{R_1N(x)^{-1}}(x),u(\cdot))) \leq M\left(\frac{R_2}{R_1}\right)^k,$$

whenever $t \in [kT,(k+1)T]$. On the other hand by (12)-(13), we have that for any $t \geq 0$,

$$\begin{aligned} N(x(t;x,u(\cdot)) &= R_1^{-1} N(x) N(x(t; \delta^r_{R_1 N(x)^{-1}}(x), u(\cdot))) \\ &\leq R_1^{-1} N(x) M \left(\frac{R_2}{R_1}\right)^k. \end{aligned}$$

Hence by (14)-(15), we deduce that for any integer $k \geq 0$ and any $t \in [kT,(k+1)T]$,

$$\begin{aligned} |x(t;x,u(\cdot))| &\leq \bar{M} \max\left\{ \sqrt{N(x(t;x,u(\cdot)))}, N(x(t;x,u(\cdot)))^{\frac{l}{2}} \right\} \\ &\leq \bar{M} \max\left\{ \sqrt{\frac{M\alpha(|x|)}{R_1}} \left(\frac{R_2}{R_1}\right)^{\frac{k}{2}}, \left(\frac{M\alpha(|x|)}{R_1}\right)^{\frac{l}{2}} \left(\frac{R_2}{R_1}\right)^{\frac{kl}{2}} \right\} \\ &\leq \beta(|x|,t). \end{aligned}$$

This concludes the proof of Proposition 2.5. □

Let us now give the definition of a semiconcave control-Lyapunov function. Before doing that, we need to introduce the notion of semiconcave functions; we refer the reader to the book [11] for an extensive study of semiconcave functions.

Let Ω be an open set in $\mathbb{R}^n$. A function $g : \Omega \to \mathbb{R}$ is said to be semiconcave on Ω provided it is continuous and for any $x_0 \in \Omega$ there are constants $\rho, C > 0$ such that

$$\frac{1}{2}(g(x)+g(y)) - g\left(\frac{x+y}{2}\right) \leq C|x-y|^2, \quad \forall x,y \in x_0 + \rho B.$$

Equivalently, this means that the function g can be written locally as the sum of a concave function and a smooth function. In particular, any semiconcave function is locally Lipschitz on its domain, which by Rademacher's theorem implies that any semiconcave function is differentiable almost everywhere on its domain. We are now ready to define the concept of semiconcave control-Lyapunov function.

Definition 2.6. [3] *A semiconcave control-Lyapunov function for (4) is a function $V : \mathbb{R}^n \to \mathbb{R}$ which is continuous at the origin, semiconcave outside the origin, positive definite, proper and for which there exist a continuous,*

[3] This definition is equivalent to the one that we used in our previous papers. In fact, by classical properties of semiconcave functions (see [11] p. 74), whenever the function V is semiconcave the property (17) is equivalent to the following involving proximal subdifferentials:

$$\forall \zeta \in \partial_P V(x), \quad \min_{|u|_m \leq \alpha(|x|)} \{\langle \zeta, Y(x,u)\rangle\} \leq -W(x),$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. Moreover this property is also equivalent to saying that the function V is a viscosity supersolution to the Hamilton-Jacobi equation

$$\max_{|u|_m \leq \alpha(|x|)} \{-\langle DV(x), Y(x,u)\rangle\} - W(x) \geq 0,$$

on $\mathbb{R}^n \setminus \{0\}$. We refer the reader to [16] and [5] for the definitions of proximal subdifferentials and viscosity solutions.

positive definite function $W : \mathbb{R}^n \to \mathbb{R}$, and a nondecreasing function $\alpha : [0,\infty) \to [0,\infty)$, satisfying

$$\min_{|u|_m \leq \alpha(|x|)} \{\langle \nabla V(x), Y(x,u)\rangle\} \leq -W(x), \tag{17}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$ where V is differentiable.

In [36], we proved that any control system that is globally asymptotically controllable at the origin admits a control-Lyapunov function which is semiconcave outside the origin. We present here the homogeneous version of that result.

Theorem 2. *Let δ_ϵ^r be a dilation. If the control system (4) is homogeneous with respect to δ_ϵ^r and GAC_0, then there exists a semiconcave control-Lyapunov function for (4) which is homogeneous of degree 1 with respect to δ_ϵ^r.*

In fact, for sake of simplicity (and to considerably simplify the proof given in the next section), we prefer to deduce this theorem as a corollary of the corresponding result in the framework of differential inclusions. For that we need to define the concepts of homogeneous differential inclusions, globally asymptotically controllable differential inclusions, and control-Lyapunov functions for differential inclusions.

2.4. **Homogeneous differential inclusions and homogeneous control-Lyapunov functions for $\mathbf{GAC_0}$ homogeneous differential inclusions.** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a multivalued map which satisfies the following assumptions:

(A1) For any $x \in \mathbb{R}^n$, the set $F(x)$ is a compact convex set of $\mathbb{R}^n$ which contains the origin.

(A2) The mapping F is locally Lipschitz on $\mathbb{R}^n$.

Let δ_ϵ^r be a dilation on $\mathbb{R}^n$; we say that the mapping F is homogeneous of degree $k \leq 1$ (where k is an integer) with respect to δ_ϵ^r if for any $x \in \mathbb{R}^n$ and any $\epsilon > 0$ we have,

$$F(\delta_\epsilon^r(x)) = \epsilon^{-k}\delta_\epsilon^r(F(x)). \tag{18}$$

Notice that if we consider m vector fields $Y_1, \cdots, Y_m$ on $\mathbb{R}^n$ which are locally Lipschitz and homogeneous of degree $k \leq 1$ with respect to δ_ϵ^r, then for any $M > 0$ the multivalued map $F : \mathbb{R}^n \to \mathbb{R}^n$ defined by,

$$F(x) := \left\{ \sum_{i=1}^m u_i Y_i(x) : |u|_m \leq M \right\}, \quad \forall x \in \mathbb{R}^n,$$

satisfies the assumptions (A1)-(A2) and is homogeneous of degree k with respect to the dilation δ_ϵ^r.

Assume from now that the mapping F satisfies the assumptions (A1)-(A2); we are interested in the property of global asymptotic controllability of the differential inclusion associated to the mapping F,

$$\dot{x}(t) \in F(x(t)), \quad \text{a.e.} \tag{19}$$

(We refer the reader to [3, 21] for a detailed study of differential inclusions.) Let us present its definition.

Definition 2.7. *We call the differential inclusion (19) globally asymptotically controllable at the origin (abreviated GAC_0) provided that there is a function $\beta \in \mathcal{KL}$ such that for each $x \in \mathbb{R}^n$, there exists a trajectory $x(\cdot) : [0, \infty) \to \mathbb{R}^n$ of (19) with $x(0) = x$ such that $|x(t)| \le \beta(|x|, t)$ for all $t \ge 0$.*

As in the control case (see Proposition 2.5), whenever the multivalued map F is homogeneous with respect to some dilation, this definition is equivalent to another one that is easier to verify.

Proposition 2.8. *Assume that the mapping F is homogeneous of degree k with respect to some dilation δ_ϵ^r and fix two constants $R_1, R_2 > 0$ satisfying $R_1 > R_2 > 0$. Then the differential inclusion (19) is GAC_0 if and only if there are two constants $M, T > 0$ such that for any $x \in N^{-1}(R_1)$, there exists a trajectory $x(\cdot) : [0, T] \to \mathbb{R}^n$ of (19) with $x(0) = x$ such that $N(x(t)) \le M$ for any $t \in [0, T]$, and such that $N(x(T)) \le R_2$.*

The proof of this result being similar to the one we gave for Proposition 2.5, it is left to the reader. Let us now give the definition of semiconcave control-Lyapunov function for (19).

Definition 2.9. *A semiconcave control-Lyapunov function for (19) is defined to be any function $V : \mathbb{R}^n \to \mathbb{R}$ which is continuous at the origin, semiconcave outside the origin, positive definite, proper and for which there exist a continuous, positive definite function $W : \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$\min_{v \in F(x)} \{\langle \nabla V(x), v \rangle\} \le -W(x), \tag{20}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$ where V is differentiable.

We are going to prove the following result.

Theorem 3. *Let δ_ϵ^r be a dilation and F be a multivalued map satisfying the assumptions (A1)-(A2). If the differential inclusion (19)) is homogeneous with respect to δ_ϵ^r and GAC_0, then there exists a semiconcave control-Lyapunov function for (19) that is homogeneous of degree 1 with respect to δ_ϵ^r.*

Theorem 2 is in fact a simple corollary of this result; let us prove it.

Proof of Theorem 2. Assume that the control system (4) is homogeneous of degree k with respect to δ_ϵ^r and GAC_0. By Proposition 2.5, there are two constants $M, T > 0$ such that for any $x \in N^{-1}(1)$, there exists $u(\cdot) \in \mathcal{U}$ with $\|u(\cdot)\|_\infty \le M$ such that $N(x(t; x, u(\cdot)) \le M$ for any $t \in [0, T]$, and such that $N(x(T; x, u(\cdot))) \le 1/2$. Define the multivalued map $F : \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(x) := \left\{ \sum_{i=1}^m u_i Y_i(x) : |u|_m \le M \right\}, \quad \forall x \in \mathbb{R}^n.$$

The mapping F satisfies assumptions (A1)-(A2) and is homogeneous of degree k with respect to δ_ϵ^r. By construction, for any $x \in N^{-1}(1)$, there exists some control $u(\cdot) \in \mathcal{U}$ with $\|u(\cdot)\|_\infty \leq M$ such that the absolutely continuous curve $x(\cdot; x, u(\cdot)) : [0,T] \to \mathbb{R}^n$ satisfies the properties above. In fact, the curve $x(\cdot) := x(\cdot; x, u(\cdot)) : [0,T] \to \mathbb{R}^n$ is a trajectory of the differential inclusion

$$\dot{x}(t) \in F(x(t)), \quad \text{a.e..} \tag{21}$$

Thus this means that for any $x \in N^{-1}(1)$ there exists some trajectory of (21) with $x(0) = x$ such that $N(x(t)) \leq M$ for any $t \in [0,T]$, and such that $N(x(T)) \leq 1/2$. By Proposition 2.8, this proves that the differential inclusion (21) is GAC_0. Hence by Theorem 3, there exists a semiconcave control-Lyapunov function for (21) which is homogeneous of degree 1 with respect to δ^r_ϵ. This function is obviously a semiconcave control-Lyapunov for (4).

Let us now prove Theorem 3.

Proof. We set $\Omega := \mathbb{R}^n \setminus \{0\}$. We are going to prove Theorem 3 in two steps. First we will assume that the multivalued map F is homogeneous of degree zero with respect to the standard dilation, then we will conclude by a change of variables.

Step 1 : Let us first assume that the dilation δ^r_ϵ is the standard dilation δ^1_ϵ and that the mapping F is homogeneous of degree zero with respect to δ^1_ϵ.

Notice that in this case the mapping F is indeed globally Lipschitz on $\mathbb{R}^n$. Moreover by homogeneity of F we have that if $x(\cdot) : [0,\infty) \to \mathbb{R}^n$ is a trajectory of (19) such that $x(0) = x \in \mathbb{R}^n$, then for any $\epsilon > 0$ the absolutely continuous arc $y(\cdot) : [0,\infty) \to \mathbb{R}^n$ defined as,

$$y(t) := \epsilon x(t), \quad \forall t \geq 0,$$

is a trajectory of (19) such that $y(0) = \epsilon x$. In addition we recall that by the assumption of global asymptotic controllability at the origin, there are two constants $M \geq 1, T > 0$ such that for every $x \in \mathbb{S}^{n-1}$, there exists a trajectory $x_x(\cdot) : [0,\infty) \to \mathbb{R}^n$ of (19) with $x(0) = x$ such that

$$|x_x(T)| \leq \frac{1}{2} \quad \text{and } |x_x(t)| \leq M, \quad \forall t \geq 0.$$

Since for any $y \in \mathbb{R}^n$ the set $F(y)$ contains the origin, we can indeed assume that the trajectory $x_x(\cdot)$ satisfies

$$|x_x(T)| = \frac{1}{2} \quad \text{and } \frac{1}{2} \leq |x_x(t)| \leq M, \quad \forall t \geq 0. \tag{22}$$

We claim the following result.

Lemma 1. *For every $x \in \mathbb{R}^n$, there exists a trajectory $\tilde{x}_x(\cdot) : [0,\infty) \to \mathbb{R}^n$ of (19) with $\tilde{x}_x(0) = x$ and such that*

$$\forall l \in \mathbb{N}, \forall t \in [lT, (l+1)T], \quad |\tilde{x}_x(t)| \leq \frac{M}{2^l}|x|. \tag{23}$$

Proof. Let $x \in \mathbb{S}^{n-1}$. By definitions of M and T above, there exists a trajectory $x_x(\cdot)$ of (19) with $x_x(0) = x$ which satisfies the property (22). The point $y := \frac{1}{|x(T)|}x(T) = 2x(T)$ belongs to $\mathbb{S}^{n-1}$, hence we can repeat our argument. There exists a trajectory $x_y(\cdot)$ of (19) with $x_y(0) = y$ which satisfies the property (22). Consequently by homogeneity of F, we deduce that the absolutely continuous arc $\tilde{x}_x(\cdot)$ on $[0, 2T]$ defined by

$$\tilde{x}_x(t) := \begin{cases} x(t) & \text{if } t \in [0, T] \\ \frac{y(t-T)}{2} & \text{if } t \in (T, 2T], \end{cases}$$

is solution of (19) and satisfies

$$|\tilde{x}_x(2T)| = \frac{|y(T)|}{2} = \frac{1}{4},$$

and

$$|\tilde{x}_x(t)| \leq \frac{M}{2}, \quad \forall t \in [T, 2T].$$

Repeating this construction on any interval $[lT, (l+1)T]$, we get the result for $x \in \mathbb{S}^{n-1}$. We conclude easily by homogenity of F. □

Returning to the proof of Theorem 3, we set for any $\rho \in (0, \infty)$,

$$D(\rho) := \frac{2M}{\rho} \left[\frac{e^{\rho T} - 1}{2 - e^{\rho T}} \right].$$

Let L be an integer greater than $4M$. Since $\lim_{\rho \to 0} D(\rho) = 2MT < LT/2$, there exists $\rho > 0$ such that the following inequality holds:

$$D(\rho) < \frac{1 - e^{-\rho LT}}{2\rho}. \tag{24}$$

Define the value function $V_0 : \mathbb{R}^n \to \mathbb{R}$ by,

$$\forall x \in \mathbb{R}^n, \quad V_0(x) := \inf \left\{ \int_0^{LT} e^{\rho t} |x(t)| dt : \dot{x} \in F(x) \text{ a.e. } , x(0) = x \right\}.$$

Notice that since $0 \in F(x)$ for any $x \in \mathbb{R}^n$, the function V_0 is well-defined. We claim the following lemma.

Lemma 2. *For every $x \in \mathbb{R}^n$, the infimum in the definition of $V_0(x)$ is attained. Moreover the function $V_0 : \mathbb{R}^n \to \mathbb{R}$ is positive definite, proper, globally Lipschitz on $\mathbb{R}^n$, and homogeneous of degree 1 with respect to the standard dilation.*

Proof. The mapping F is globally Lipschitz with compact convex values, hence by Gronwall Lemma and Arzéla-Ascoli Theorem any sequence of trajectories $(x_l(\cdot))_{l \in \mathbb{N}}$ of the differential inclusion (19) on the interval $[0, LT]$ which satisfy $x_l(0) = x$ for any $l \in \mathbb{N}$ admits a subsequence which converges uniformly to some trajectory of (19) on $[0, LT]$. This proves that for every $x \in \mathbb{R}^n$ the infimum in the definition of $V_0(x)$ is attained and then that the function V_0 is positive definite. The global Lipschitz regularity of F implies easily, via Gronwall Lemma that V_0 is proper. The homogeneity of V_0 is a consequence of the homogeneity of the norm and of the mapping F. Finally, the regularity of V_0 is a consequence of the fact that if we denote by K the Lipschitz constant of the mapping F on $\mathbb{R}^n$, then we have (see [3, Corollary 1 p. 121]):

For every $x, y \in \mathbb{R}^n$ and for every trajectory $x(\cdot)$ of (19) such that $x(0) = x$, there exists a trajectory $y(\cdot)$ of (19) with $y(0) = y$ and such that

$$|x(t) - y(t)| \leq e^{KLT} |y - x|, \quad \forall t \in [0, LT].$$

□

Let us now prove that there exists some positive definite and continuous function $W_0 : \mathbb{R}^n \to \mathbb{R}$ for which the property (20) is satisfied. To this end we need the following lemma.

Lemma 3. *If $x \in \Omega$ and if $x(\cdot)$ is a trajectory of (19) starting at x such that $V_0(x) = \int_0^{LT} e^{\rho t}|x(t)|dt$, then*

$$|x(LT)| \leq \frac{e^{-\rho LT}}{2}|x|. \tag{25}$$

Proof. Notice that Lemma 1 permits us to bound the quantity $V_0(x)$; we have that for any $x \in \mathbb{R}^n$,

$$\begin{aligned} V_0(x) &\leq \int_0^{LT} e^{\rho t}|\tilde{x}_x(t)|dt \\ &\leq \sum_{l=0}^{L-1} \int_{lT}^{(l+1)T} e^{\rho t}|\tilde{x}_x(t)|dt \\ &\leq \sum_{l=0}^{L-1} \int_{lT}^{(l+1)T} e^{\rho t}\frac{M|x|}{2^l}dt \\ &= \frac{M|x|}{\rho}\sum_{l=0}^{L-1} \frac{e^{\rho(l+1)T} - e^{\rho lT}}{2^l} \\ &= \frac{M|x|}{\rho}(e^{\rho T} - 1)\sum_{l=0}^{L-1}\left(\frac{e^{\rho T}}{2}\right)^l \\ &\leq \frac{M|x|}{\rho}(e^{\rho T} - 1)\sum_{l=0}^{\infty}\left(\frac{e^{\rho T}}{2}\right)^l \\ &= \frac{M|x|}{\rho}\left[\frac{e^{\rho T} - 1}{1 - \frac{e^{\rho T}}{2}}\right]|x| = D(\rho)|x|. \end{aligned} \tag{26}$$

Returning to the proof of Lemma 3, consider $x \in \Omega$ and a trajectory $x(\cdot)$ of (19) starting at x such that $V_0(x) = \int_0^{LT} e^{\rho t}|x(t)|dt$. We claim that there exists $t \in [0, LT]$ such that

$$|x(t)| \leq \frac{e^{-\rho LT}}{2}|x|. \tag{27}$$

As a matter of fact, if it is not the case, this means that the trajectory $x(\cdot)$ remains outside the ball $\overline{B}\left(0, \frac{e^{-\rho LT}}{2}|x|\right)$ on the interval $[0, LT]$, which implies that

$$V_0(x) > \int_0^{LT} e^{\rho t}\frac{e^{-\rho LT}}{2}|x|dt = \frac{1 - e^{-\rho LT}}{2\rho}|x| > D(\rho)|x|,$$

which by (24) and (26) gives a contradiction. In order to conclude, we just notice that since $0 \in F(y)$ for any $y \in \mathbb{R}^n$, then necessarily the quantity $|x(t)|$

is minimal on $[0, LT]$ for $t = LT$. As a matter of fact, denote by $\bar{t} \in [0, LT]$ the maximum time $t \in [0, LT]$ such that

$$|x(t)| = \min_{s \in [0,LT]} \{|x(s)|\}.$$

If $\bar{t} < LT$ then it is clear that the absolutely continuous arc $y(\cdot)$ on $[0, LT]$ defined as,

$$y(t) := \begin{cases} x(t) & \text{if } t \in [0, \bar{t}] \\ x(\bar{t}) & \text{if } t \in (\bar{t}, LT], \end{cases}$$

is a trajectory of (19) on $[0, LT]$ starting at x which satisfies

$$\int_0^{LT} e^{\rho t} |y(t)| dt < V_0(x);$$

which gives a contradiction. This concludes the proof of Lemma 3. □

As we said before, the property (20) is not relevant whenever the function V is not semiconcave. In our case, the function V_0 is not necessary semiconcave on Ω, hence we are going to state this property in terms of proximal subdifferentials[4]. We recall that some vector $\zeta \in \mathbb{R}^n$ belongs to the proximal subdifferential of V_0 at $x \in \mathbb{R}^n$, that we denote by $\zeta \in \partial_P V_0(x)$, if there exists two constants $\eta, \delta > 0$ such that

$$V_0(y) - V_0(x) + \eta |y - x|^2 \geq \langle \zeta, y - x \rangle, \quad \forall y \in x + \delta B. \tag{28}$$

We claim the following result.

Lemma 4. *We have that for any $x \in \Omega$,*

$$\forall \zeta \in \partial_P V_0(x), \quad \min_{v \in F(x)} \{\langle \zeta, v \rangle\} \leq -\rho V_0(x). \tag{29}$$

Proof. Let $x \in \Omega$ and $\zeta \in \partial_P V_0(x)$. By Lemma 2 we know that there exists a trajectory $x(\cdot)$ of (19) on $[0, LT]$ starting at x such that

$$V_0(x) = \int_0^{LT} e^{\rho t} |x(t)| dt.$$

Fix $\bar{t} \in (0, LT)$. The absolutely continuous arc $y(\cdot)$ on $[0, LT]$ defined by

$$y(t) := \begin{cases} x(t + \bar{t}) & \text{if } t \in [0, LT - \bar{t}] \\ x(LT) & \text{if } t \in (LT - \bar{t}, LT], \end{cases}$$

is a trajectory of (19) on $[0, LT]$ which starts at $x(\bar{t})$. Hence by definition of $V_0(x(\bar{t}))$ we have,

$$\begin{aligned} V_0(x(\bar{t})) &\leq \int_0^{LT} e^{\rho t} |y(t)| dt \\ &= \int_0^{LT-\bar{t}} e^{\rho t} |x(t + \bar{t})| dt + \int_{LT-\bar{t}}^{LT} e^{\rho t} |x(LT)| dt \\ &= e^{-\rho \bar{t}} \int_{\bar{t}}^{LT} e^{\rho t} |x(t)| dt + \frac{e^{\rho LT}(1 - e^{-\rho \bar{t}})}{\rho} |x(LT)| \\ &= e^{-\rho \bar{t}} V_0(x) - e^{-\rho \bar{t}} \int_0^{\bar{t}} e^{\rho t} |x(t)| dt + \frac{e^{\rho LT}(1 - e^{-\rho \bar{t}})}{\rho} |x(LT)|, \end{aligned}$$

[4]We recall that we refer the reader to the book [16] for an extensive exposition of proximal calculus.

which implies that

$$\frac{V_0(x(\bar t)) - V_0(x)}{\bar t} \leq \\ \frac{e^{-\rho\bar t}-1}{\bar t}V_0(x) - \frac{e^{-\rho\bar t}}{\bar t}\int_0^{\bar t} e^{\rho t}|x(t)|dt + \frac{e^{\rho LT}(1-e^{-\rho\bar t})}{\rho\bar t}|x(LT)|. \quad (30)$$

Furthermore, there exists a sequence $(\bar t_n)_{n\in\mathbb{N}} \downarrow 0$ and $v \in F(x)$ such that

$$\lim_{n\to\infty} \frac{x(\bar t_n) - x}{\bar t_n} = v.$$

Since $\zeta \in \partial_P V_0(x)$, there exist two constants $\eta, \delta > 0$ such that (28) holds. In consequence, using (30) and passing to the limit we obtain,

$$\begin{aligned} \langle \zeta, v\rangle &\leq -\rho V_0(x) - |x| + e^{\rho LT}|x(LT)| \\ &\leq -\rho V_0(x) \text{ by (25).} \end{aligned}$$

This concludes the proof of the lemma. □

Return to the proof of Theorem 3 (for the standard dilation). The function V_0 satisfies all the properties given in the statement of the theorem but is not semiconcave on Ω. We are going to regularize it by the classical technique of inf-convolution. Before we continue we notice that by classical properties of the proximal subdifferential (see [16]), Lemma 4 implies that

$$\forall x \in \Omega, \forall \zeta \in \partial_P V_0^2(x), \min_{v\in F(x)} \langle \zeta, v\rangle \leq -2\rho V_0(x)^2. \quad (31)$$

Denote by K the Lipschitz constant of the mapping F on $\mathbb{R}^n$ and by K_0 the Lipschitz constant of the function V_0 on $\mathbb{R}^n$, and consider some $\alpha > 2K_0^2$ wich satisfies

$$\frac{8\rho K_0^2}{\alpha} + \frac{4K_0^2 K}{\alpha} + \frac{8K_0^4 K}{\alpha^2} \leq \rho.$$

Define the function $V_1 : \mathbb{R}^n \to \mathbb{R}$ by,

$$\forall x \in \mathbb{R}^n, \quad V_1(x) := \inf_{y\in\mathbb{R}^n} \left\{ V_0(y)^2 + \alpha|y-x|^2 \right\}. \quad (32)$$

We have the following lemma.

Lemma 5. *The function V_1 is semiconcave on $\mathbb{R}^n$, positive definite, proper, and homogeneous of degree 2 with respect to the standard dilation. Furthermore, for every $x \in \mathbb{R}^n$ the infimum in (32) is attained, and moreover if $\bar y \in \mathbb{R}^n$ is such that $V_1(x) = V_0(\bar y)^2 + \alpha|\bar y - x|^2$ then we have,*

$$|\bar y - x| \leq \frac{2K_0}{\alpha} V_0(x). \quad (33)$$

Proof. Let us first prove the second part of the statement. Notice that by definition we have,

$$V_1(x) \leq V_0(x)^2, \quad \forall x \in \mathbb{R}^n. \quad (34)$$

Moreover since V_0 is positive definite we have that for every $y \in \mathbb{R}^n$,

$$V_0(y)^2 + \alpha|y-x|^2 \leq V_0(x)^2 \Longrightarrow V_0(y) \leq V_0(x). \quad (35)$$

This proves that for every $x \in \mathbb{R}^n$, the infimum in (32) can be taken only over the set of points $y \in \mathbb{R}^n$ such that $V_0(y) \leq V_0(x)$. By properness of V_0 this set is compact hence for every $x \in \mathbb{R}^n$ the infimum in (32) is attained.

Consider $\bar{y} \in \mathbb{R}^n$ such that $V_1(x) = V_0(\bar{y})^2 + \alpha|\bar{y} - x|^2$ and let us prove (33). We argue by contradiction and so we assume that

$$|\bar{y} - x| > \frac{2K_0}{\alpha} V_0(x). \tag{36}$$

By (35), we can write

$$\begin{aligned} |V_0(\bar{y})^2 - V_0(x)^2| &= (V_0(\bar{y}) + V_0(x))|V_0(\bar{y}) - V_0(x)| \\ &\leq 2V_0(x)|V_0(\bar{y}) - V_0(x)| \\ &\leq 2K_0 V_0(x)|\bar{y} - x|, \end{aligned} \tag{37}$$

by definition of K_0. Hence we have

$$\begin{aligned} V_1(x) &= V_0(\bar{y})^2 + \alpha|\bar{y} - x|^2 \\ &\geq V_0(x)^2 - 2K_0 V_0(x)|\bar{y} - x| + \alpha|\bar{y} - x|^2 \\ &= V_0(x)^2 + \alpha|\bar{y} - x|\left(|\bar{y} - x| - \frac{2K_0}{\alpha} V_0(x)\right) \\ &> V_0(x)^2 (\text{ by } (36); \end{aligned}$$

which by (34) gives a contradiction. The semiconcavity of V_1 comes from a classical property of inf-convolution; we refer the reader to [11]. The fact that V_1 is positive definite, proper, and homogeneous of degree 2 with respect to δ^1_ϵ being straightforward to show, it is left to the reader. □

We are going to prove the property of type (29)-(31) for the new function V_1. For that we need the following result. (We refer the reader to [16, Theorem 5.1, p. 44] for its proof.)

Lemma 6. *Suppose that $x \in \mathbb{R}^n$ is such that $\partial_P V_1(x)$ is nonempty. Then there exists a point $\bar{y} \in \mathbb{R}^n$ satisfying the following properties:*
(a) The infimum in (32) is attained uniquely at $\bar{y}$.
(b) The proximal subgradient $\partial_P V_1(x)$ is the singleton $\{2\alpha(x - \bar{y})\}$.
(c) $2\alpha(x - \bar{y}) \in \partial_P V_0^2(\bar{y})$.

We are going to prove the following.

Lemma 7. *We have that for any $x \in \Omega$,*

$$\forall \zeta \in \partial_P V_1(x), \quad \min_{v \in F(x)} \{\langle \zeta, v \rangle\} \leq -\rho V_1(x). \tag{38}$$

Proof. Let $x \in \Omega$ and $\zeta \in \partial_P V_1(x)$. By the lemmae above, we know that there exists $\bar{y} \in \mathbb{R}^n$ such that $V_1(x) = V_0(\bar{y})^2 + \alpha|\bar{y} - x|^2$ and such that $|\bar{y} - x| \leq \frac{2K_0}{\alpha} V_0(x)$. Notice that $\bar{y}$ cannot be zero. As a matter of fact, if $\bar{y} = 0$ then we deduce that

$$|x| \leq \frac{2K_0}{\alpha} V_0(x) \leq \frac{2K_0^2}{\alpha} |x|,$$

which implies that $\alpha \leq 2K_0^2$ and then gives a contradiction. In consequence $\bar{y} \in \Omega$ and thus from Lemma 6 (b)-(c) and (31), we deduce that $\zeta \in \partial_P V_0^2(\bar{y})$ and that there exists $v \in F(\bar{y})$ such that

$$\langle \zeta, v \rangle \leq -2\rho V_0(\bar{y})^2.$$

Furthermore, there exists $w \in F(x)$ such that $|w - v| \leq K|x - \bar{y}|$. Hence using the fact that $\partial_P V_0^2(\bar{y}) = 2V_0(\bar{y})\partial_P V_0(\bar{y})$ and that $|\zeta| \leq 2K_0 V_0(\bar{y})$, and the inequalities (33) and (37) we obtain,

$$\begin{array}{rcl} \langle \zeta, w \rangle & \leq & \langle \zeta, v \rangle + |\zeta||w - v| \\ & \leq & -2\rho V_0(\bar{y})^2 + 2K_0 V_0(\bar{y}) K |x - \bar{y}| \\ & \leq & -2\rho \left(V_0(x)^2 - 2K_0 V_0(x)|\bar{y} - x| \right) \\ & & +2K_0 K |\bar{y} - x| \left(V_0(x) + K_0 |\bar{y} - x| \right) \\ & = & -2\rho V_0(x)^2 + (4\rho K_0 + 2K_0 K) V_0(x)|\bar{y} - x| + 2K_0^2 K |\bar{y} - x|^2 \\ & \leq & -2\rho V_0(x)^2 + \left(\frac{8\rho K_0^2}{\alpha} + \frac{4K_0^2 K}{\alpha} + \frac{8K_0^4 K}{\alpha^2} \right) V_0(x)^2 \\ & \leq & -\rho V_0(x)^2 \leq -\rho V_1(x), \end{array}$$

by construction of the constant α. This concludes the proof of Lemma 7. □

Finally we define $V : \mathbb{R}^n \to \mathbb{R}$ by $V(x) := \sqrt{V_1(x)}$, for any $x \in \mathbb{R}^n$. This function is continuous at the origin, semiconcave on Ω, positive definite, proper, homogeneous of degree 1 with respect to δ_ϵ^1, and satisfies,

$$\forall x \in \Omega, \forall \zeta \in \partial_P V(x), \quad \min_{v \in F(x)} \langle \zeta, v \rangle \leq -\frac{\rho}{2} V(x).$$

Recall that by semiconcavity, this property implies

$$\min_{v \in F(x)} \{ \langle \nabla V(x), v \rangle \} \leq -\frac{\rho}{2} V(x),$$

for all $x \in \Omega$ where V is differentiable. This proves that the function V is a control-Lyapunov function for the differential inclusion (19); which concludes the Step 1.

Step 2 : We prove Theorem 3 in the general case.

The Proposition 2.1 can be adapted in the case of differential inclusions; the proof is left to the reader.

Proposition 2.10. *Set $\mu := \min_{i=1,\cdots,n}\{r_i\}$ and $\gamma := \frac{k}{\mu} \in \mathbb{Q}$. There exists a homeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ with $\Phi(0) = 0$ which is an analytic diffeomorphism from Ω into Ω, and such that if we set for every $y \in \Omega$,*

$$\tilde{F}(y) := \left\{ D\Phi(\Phi^{-1}(y)) \cdot v \text{ for } v \in F(\Phi^{-1}(y)) \right\},$$

then the mapping $\tilde{F}$ satisfies,

$$\forall y \in \Omega, \forall \epsilon > 0, \quad \tilde{F}(\epsilon y) = \epsilon^{1-\gamma} \tilde{F}(y).$$

We notice that the homeomorphism Φ is indeed exactly the same as the one we constructed in the proof of Proposition 2.1. We recall that it satisfies for any $x \in \mathbb{R}^n$,

$$|\Phi(x)| = N(x)^\mu \quad \text{and } \Phi(\delta_\epsilon^r(x)) = \epsilon^\mu \Phi(x). \tag{39}$$

Define the multivalued map $\bar{F} : \mathbb{R}^n \to \mathbb{R}^n$ by $\bar{F}(0) = 0$ and,

$$\bar{F}(y) := |y|^\gamma \tilde{F}(y), \quad \forall y \in \Omega.$$

The mapping $\bar{F}$ satisfies the assumptions (A1)-(A2) and is homogeneous of degree zero with respect to δ_ϵ^1. Let us prove that the differential inclusion

$$\dot{y}(t) \in \bar{F}(y(t)), \quad \text{a.e.} \tag{40}$$

is GAC$_0$. In fact since the differential inclusion (19) is GAC$_0$, there are two constants $M, T > 0$ such that for any $x \in N^{-1}(1)$, there exists a trajectory $x_x(\cdot) : [0,T] \to \mathbb{R}^n$ of (19) with $x_x(0) = x$ such that $1/2 \leq N(x_x(t)) \leq M$ for any $t \in [0,T]$, and such that $N(x_x(T)) = 1/2$. Fix $y \in \mathbb{S}^{n-1}$; by definition of the function Φ (see (39)), the point $x := \Phi^{-1}(y)$ belongs to $N^{-1}(1)$. Define the function $\theta_y : [0,T] \to \mathbb{R}$ by

$$\theta_y(t) := \int_0^t |\Phi(x_x(s))|^{-\gamma} ds, \quad \forall t \in [0,T].$$

Since we know that $N(x(s)) \geq 1/2$ for any $s \in [0,T]$, the function θ is increasing and hence it is a bijection from $[0,T]$ into $[0,\bar{T}_y]$, where $\bar{T}_y$ is defined by

$$\bar{T}_y := \int_0^T |\Phi(x_x(t))|^{-\gamma} dt. \tag{41}$$

Define the absolutely continuous arc $y_y(\cdot) : [0,\bar{T}_x] \to \mathbb{R}^n$ by,

$$y_y(\bar{t}) := \Phi\left(x_x\left(\theta_y^{-1}(\bar{t})\right)\right), \quad \forall \bar{t} \in [0,\bar{T}_y].$$

We have for almost every $\bar{t} \in [0,\bar{T}_y]$,

$$\begin{aligned}
\dot{y}(\bar{t}) &= D\Phi\left(x_x\left(\theta_y^{-1}(\bar{t})\right)\right) \cdot \left[\frac{d}{d\bar{t}}\left(x_x\left(\theta_y^{-1}(\bar{t})\right)\right)\right] \\
&= D\Phi\left(x_x\left(\theta_y^{-1}(\bar{t})\right)\right) \cdot \left[\frac{1}{\theta'(\theta_y^{-1}(\bar{t})}\dot{x}_x\left(\theta_y^{-1}(\bar{t})\right)\right] \\
&= |y_y(\bar{t})|^\gamma D\Phi\left(\Phi^{-1}(y_y(\bar{t}))\right) \cdot \dot{x}_x\left(\theta_y^{-1}(\bar{t})\right) \\
&\in \bar{F}(y_y(\bar{t})).
\end{aligned}$$

Thus we deduce that the arc $y_y(\cdot)$ is a trajectory of (40) which satisfies

$$|y_y(\bar{t})| = N(x_x(\theta_y^{-1}(\bar{t})))^\mu, \quad \forall \bar{t} \in [0,\bar{T}_y].$$

This gives

$$|y_y(\bar{T}_y)| = \frac{1}{2^\mu}, \quad \text{and } \frac{1}{2^\mu} \leq |y_y(\bar{t})| \leq M^\mu, \quad \forall \bar{t} \in [0,\bar{T}_y].$$

If we denote by $\bar{M}$ the maximum of the function $x \mapsto |\Phi(x)|^{-\gamma}$ over the compact set $N^{-1}([1/2,M])$ then we have that $\bar{T}_y \leq T\bar{M}$ for any $y \in \mathbb{S}^{n-1}$. Since for any $z \in R^n$ the set $F(z)$ contains the origin, this proves that for any $y \in \mathbb{S}^{n-1}$ there exists some trajectory $y_y(\cdot) : [0,T\bar{M}] \to \mathbb{R}^n$ of (40) with $y_y(0) = y$ such that $2^{-\mu} \leq |y_y(\bar{t})| \leq M^\mu$ for any $\bar{t} \in [0,T\bar{M}]$, and such that $|y_y(T\bar{M})| = 2^{-\mu}$. By Proposition 2.8, this proves that the differential inclusion (40) is GAC$_0$.

As a consequence we can apply the result of Step 1 to the differential inclusion (40). Hence we obtain the existence of some constant $\rho > 0$ and some function $\bar{V}:\mathbb{R}^n\to\mathbb{R}$ which is positive definite, proper, globally Lipschitz, semiconcave on Ω, homogeneous of degree 1 with respect to δ_ϵ^1, and which satisfies,

$$\min_{w\in\bar{F}(y)}\left\{\langle\nabla\bar{V}(y),w\rangle\right\}\leq-\frac{\rho}{2}\bar{V}(y), \tag{42}$$

at all points of differentiation. Define the function $V:\mathbb{R}^n\to\mathbb{R}$ by

$$V(x):=\bar{V}(\Phi(x))^{\frac{1}{\mu}},\quad\forall x\in\mathbb{R}^n.$$

The function V is obviously continuous on $\mathbb{R}^n$, positive definite and proper, and in addition by (39) it is homogeneous of degree 1 with respect to δ_ϵ^r. Moreover since Φ is smooth from Ω onto Ω, the function V is, like $\bar{V}$, semiconcave outside the origin. Besides $\bar{V}$ is differentiable at $x\in\Omega$ if and only if $\bar{V}$ is differentiable at $\Phi(x)$, and we have

$$\nabla V(x)=\frac{1}{\mu}\bar{V}(\Phi(x))^{\frac{1}{\mu}-1}D\Phi(x)^*\cdot\nabla\bar{V}(\Phi(x)).$$

Let $x\in\Omega$ such that V is differentiable at x. By properties of the function $\bar{V}$, there exists $w\in\bar{F}(\Phi(x))$ which satisfies

$$\langle\nabla\bar{V}(\Phi(x)),w\rangle\leq-\frac{\rho}{2}\bar{V}(\Phi(x)).$$

By construction of the mapping $\bar{F}$, we have that $|\Phi(x)|^{-\gamma}w\in\tilde{F}(\Phi(x))$, which implies that the vector $v\in\mathbb{R}^n$ defined by

$$v:=D\Phi^{-1}(\Phi(x))\cdot(|\Phi(x)|^{-\gamma}w)$$

belongs to $F(x)$. On the other hand, we have

$$\begin{aligned}\langle\nabla V(x),v\rangle &= \frac{1}{\mu}\bar{V}(\Phi(x))^{\frac{1}{\mu}-1}\langle\nabla\bar{V}(\Phi(x)),D\Phi(x)\cdot v\rangle\\ &= \frac{1}{\mu}\bar{V}(\Phi(x))^{\frac{1}{\mu}-1}\langle\nabla\bar{V}(\Phi(x)),|\Phi(x)|^{-\gamma}w\rangle\\ &\leq -\frac{\rho}{2\mu}\bar{V}(\Phi(x))^{\frac{1}{\mu}-1}|\Phi(x)|^{-\gamma}\bar{V}(\Phi(x))\\ &\leq -\frac{\rho}{2\mu}|\Phi(x)|^{-\gamma}\bar{V}(\Phi(x))^{\frac{1}{\mu}}.\end{aligned}$$

Define the function $W:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ by

$$W(x):=\frac{\rho}{2\mu}|\Phi(x)|^{-\gamma}\bar{V}(\Phi(x))^{\frac{1}{\mu}},\quad\forall x\in\mathbb{R}^n.$$

This function is positive and continuous on $\mathbb{R}^n\setminus\{0\}$. Moreover if we denote by $\bar{K}$ the Lipschitz constant of the function $\bar{V}$, then we can write for any $x\in\mathbb{R}^n$,

$$0\leq W(x)\leq\frac{\rho}{2\mu}K^{\frac{1}{\mu}}|\Phi(x)|^{\frac{1}{\mu}-\gamma}=\frac{\rho}{2\mu}K^{\frac{1}{\mu}}|\Phi(x)|^{\frac{1-k}{\mu}}.$$

If $k<1$, then the function W can be extended continuously at the origin by setting $W(0)=0$. If $k=1$ then this means that the function W is bounded by a constant on $\mathbb{R}^n\setminus\{0\}$. But there exists clearly another function $W':\mathbb{R}^n\to\mathbb{R}$ continuous and positive definite such that $W'\leq W$. In any case this concludes the proof of Theorem 3. □

3. Proof of Theorem 1 in the standard homogeneous case

The purpose of this section is to provide a proof of Theorem 1 in the case of control systems which are homogeneous of degree zero with respect to the standard dilation, and analytic outside the origin. Therefore, until the end of this section we consider a control system of the form

$$\dot{x} = Y(x,u) := \sum_{i=1}^{m} u_i Y_i(x), \tag{43}$$

where $Y_1, \cdots, Y_m$ are vector fields on $\mathbb{R}^3$ which are globally Lipschitz on $\mathbb{R}^3$, analytic on $\mathbb{R}^3 \setminus \{0\}$, homogeneous of degree 0 with respect to the standard dilation, and which satisfy

$$\mathrm{Lie}\{Y_1, \cdots, Y_m\}(x) = \mathbb{R}^3, \tag{44}$$

for any $x \in \mathbb{R}^3 \setminus \{0\}$[5]. We will in fact prove something much more precise than Theorem 1; we postpone the statement of our result to the end of this section. We first need to develop preliminaries on semianalytic sets.

3.1. Preliminaries on semianalytic sets.

Here we recall some basic facts about semianalytic sets; we refer the reader to [29, 46] for more details.

A set $A \subset \mathbb{R}^n$ is called semianalytic if and only if for every $x \in \mathbb{R}^n$, we can find a neighbourhood U of x in $\mathbb{R}^n$ and $2pq$ real analytic functions $g_{i,j}$ and $h_{i,j}$ ($1 \leq i \leq p$ and $1 \leq j \leq q$) such that

$$A \cap U = \bigcup_{i=1}^{p} \{y \in U : g_{i,j}(y) = 0 \text{ and } h_{i,j}(y) > 0 \text{ for } j = 1, \cdots, q\}.$$

The property "semianalytic" is preserved by the following operations: finite union, finite intersection, and difference of any two. Moreover we have the following theorem of stratification of semianalytic sets.

Theorem 4. *If $A \subset \mathbb{R}^n$ is semianalytic then it admits a stratification, that is a locally finite decomposition*

$$A = \bigcup_{\alpha \in I} \Gamma_\alpha,$$

into a disjoint union of connected real analytic submanifolds such that if $\overline{\Gamma_\alpha} \cap \Gamma_\beta \neq \emptyset$, then $\Gamma_\beta \subset \overline{\Gamma_\alpha}$, and $\mathrm{dim}\Gamma_\beta \leq \mathrm{dim}\Gamma_\alpha - 1$ whenever $\alpha \neq \beta$.

In the sequel, we will express Theorem 4 by saying that a semianalytic set in $\mathbb{R}^n$ can be stratified into a disjoint union of strata of dimension d with $d \in \{0, \cdots, n\}$, where each stratum of dimension d is a connnected real analytic submanifold of dimension d. Furthermore we will say that a semianalytic set has dimension D if each stratum of its stratification has dimension less or equal than D. Notice that by the stratification theorem, any semianalytic

[5]Notice that we do not assume that the control system (43) satisfies the Hörmander's condition at the origin. This is due to the fact that the change of variables given in Proposition 2.1 transforms the initial control system into a control system which is no longer smooth at the origin. However whenever the initial control system satisfies the Hörmander condition at the origin then by homogeneity this condition is satisfied everywhere in $\mathbb{R}^3$. That is why we are allowed to assume that (43) satisfies the Hörmander's condition outside the origin.

set which is compact or even relatively compact (that is such that its closure is compact) has a finite number of connected components. Here is a lemma that will be very useful in the sequel.

Lemma 8. *Let $\mathcal{A} \subset \mathbb{R}^n$ be a semianalytic set which is open, connected and relatively compact. There exist two constants $\mu_{\mathcal{A}}, l_{\mathcal{A}}, > 0$ such for any $0 \leq \mu \leq \mu_{\mathcal{A}}$, the set $\mathcal{A}^{\mu}$ defined as,*

$$\mathcal{A}_{\mu} := \{x \in \mathcal{A} \text{ s.t. } d(x, \mathbb{R}^n \setminus \mathcal{A}) \geq \mu\},$$

is nonempty, connected and such that for any pair $x, y \in \mathcal{A}_{\mu}$ there exists some absolutely continuous path

$$\gamma_{x,y} : [0, l_{\mathcal{A}}] \longrightarrow \mathcal{A}_{\mu}$$

such that $\gamma_{x,y}(0) = x, \gamma_{x,y}(l_{\mathcal{A}}) = y$ and which satisfies

$$|\dot{\gamma}_{x,y}(t)| \leq 1, \quad a.e.\ t \in [0, l_{\mathcal{A}}].$$

The proof of Lemma 8 relies on the concept of subanalytic sets. Since we do not want to enlarge too much on that subject here, we refer the reader to [26, 27, 46] for basic facts about subanalyticity and we just sketch the proof of the lemma.

Proof. Set $\mathcal{A}^c := \mathbb{R}^n \setminus \mathcal{A}$ and denote by $d_{\mathcal{A}^c} : \mathbb{R}^n \to \mathbb{R}$ the distance function to the set $\mathbb{R}^n \setminus \mathcal{A}$. It is well known (see for instance [5]) that the function $d_{\mathcal{A}^c}$ is globally Lipschitz on $\mathbb{R}^n$ and semiconcave on $\mathcal{A}$. In fact it is not difficult to show that this function is also subanalytic on $\mathbb{R}^n$, which means that its graph in $\mathbb{R}^n \times \mathbb{R}$ is a subanalytic set. Denote for any $x \in \mathbb{R}^n$, by $\partial d_{\mathcal{A}^c}(x)$, the Clarke generalized gradient of the function $d_{\mathcal{A}^c}$ at the point x, and define the singular set of $d_{\mathcal{A}^c}$ by,

$$\Sigma\left(d_{\mathcal{A}^c}\right) := \{x \in \mathcal{A} \text{ s.t. } \partial d_{\mathcal{A}^c}(x) \text{ is not a singleton}\}.$$

By semiconcavity, this set coincides with the set of points of $\mathcal{A}$ where $d_{\mathcal{A}^c}$ is not differentiable and so has measure zero, and it is subanalytic. Define also the critical set of $d_{\mathcal{A}^c}$ by,

$$\mathcal{C}\left(d_{\mathcal{A}^c}\right) := \{x \in \mathcal{A} \text{ s.t. } 0 \in \partial d_{\mathcal{A}^c}(x)\} \subset \mathbb{R}^n.$$

This set is included in the singular set and is subanalytic. In consequence, by the stratification theorem for subanalytic sets, it admits a stratification into a disjoint union of connected real analytic submanifolds of dimension zero and one. Since by semiconcavity, the function $d_{\mathcal{A}^c}$ is constant on every stratum of the set $\mathcal{C}\left(d_{\mathcal{A}^c}\right)$, we deduce that there exists some constant $\bar{\mu}$ such that the set $\mathbb{R}^n \setminus \mathcal{A}_{\bar{\mu}}$ does not intersect $\mathcal{C}\left(d_{\mathcal{A}^c}\right)$. Fix μ such that $0 \leq \mu \leq \bar{\mu}$ and denote by M_1 the maximum of the function $d_{\mathcal{A}^c}$ on the set $\mathcal{A}$. By semiconcavity, there exists for any $x \in \mathcal{A}_{\mu}$ a unique solution $x(\cdot)$ of the differential inclusion

$$\dot{x} \in \partial d(x(t)),$$

such that $x(0) = x$. As long as $x(t) \notin \Sigma(d_{\mathcal{A}^c})$, we have

$$d_{\mathcal{A}^c}(x(t)) = d_{\mathcal{A}^c}(x) + t,$$

and there exists $t \leq M_1$ such that $x(t)$ belongs to the singular set $\Sigma\left(d_{\mathcal{A}^c}\right)$. Moreover, if $x(\bar{t}) \in \Sigma(d_{\mathcal{A}^c})$ for some $\bar{t} \in [0, M_1]$ then $x(t) \in \Sigma(d_{\mathcal{A}^c})$ for any $t \geq \bar{t}$, and $d_{\mathcal{A}^c}(x(t)) \geq d_{\mathcal{A}^c}(x(\bar{t}))$ for any $t \geq \bar{t}$. In fact, by semiconcavity of the distance function, by connectedness of $\mathcal{A}$, and by subanalyticity of the singular set, it can be proven that there exists a constant M_2 such that for any pair $x, y \in \Sigma(d_{\mathcal{A}^c}) \cap \mathcal{A}_\mu$ there exists some absolutely continuous path

$$\gamma_{x,y} : [0, M_2] \longrightarrow \Sigma(d_{\mathcal{A}^c})$$

such that $\gamma_{x,y}(0) = x, \gamma_{x,y}(M_2) = y$ and which satisfies

$$|\dot{\gamma}_{x,y}(t)| \leq 1, \quad \text{a.e. } t \in [0, M_2].$$

We conclude easily. □

3.2. **A useful lemma.** Here we prove a lemma which will be very useful for the proof of Theorem 1 in the case of control systems which are homogeneous of degree zero with respect to the standard dilation. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a multivalued map which satisfies the assumptions (A1)-(A2) and which is homogeneous of degree zero with respect to the standard dilation. Set $\Omega := \mathbb{R}^3 \setminus \{0\}$, and define the mapping $\tilde{F} : \mathbb{S}^2 \to \mathbb{R}^3$ by,

$$\forall x \in \mathbb{S}^2, \quad \tilde{F}(x) := \mathrm{Proj}_{T_x\mathbb{S}^2}(F(x)) = \left\{ \mathrm{Proj}_{T_x\mathbb{S}^2}(v) : v \in F(x) \right\},$$

where $\mathrm{Proj}_{T_x\mathbb{S}^2}$ denotes the projection onto the vector space $T_x\mathbb{S}^2$. We notice that the multivalued map $\tilde{F}$ satisfies the assumptions (A1)-(A2) on the sphere $\mathbb{S}^2$ and that for any $x \in \mathbb{S}^2$, the set $\tilde{F}(x)$ is included in the tangent space $T_x\mathbb{S}^2$. This means that for any $x \in \mathbb{S}^2$, any trajectory of the differential inclusion

$$\dot{x}(t) \in \tilde{F}(x(t)), \quad \text{a.e.} \tag{45}$$

starting at x, remains on the sphere $\mathbb{S}^2$ and can be extended on $[0, \infty)$. These trajectories can be associated to those of the differential inclusion

$$\dot{x}(t) \in F(x(t)), \quad \text{a.e.} \tag{46}$$

as follows.

Lemma 9. *If $x(\cdot) : [a, b] \to \Omega$ is a trajectory of (46) then the absolutely continuous arc $y(\cdot) : [a, b] \to \mathbb{S}^2$ defined by,*

$$y(t) := \frac{x(t)}{|x(t)|}, \quad \forall t \in [a, b],$$

is a trajectory of (45). In addition, if $y(\cdot) : [a, b] \to \mathbb{R}^3$ is a trajectory of (45), then there exists some trajectory $x(\cdot) : [a, b] \to \Omega$ with $x(a) \in \mathbb{S}^2$ such that

$$y(t) = \frac{x(t)}{|x(t)|}, \quad \forall t \in [a, b].$$

Proof. Let us prove the first part of the lemma. If $x(\cdot) : [a, b] \to \Omega$ is a trajectory of (46), then we notice that for almost all $t \in [a, b]$, we have

$$\frac{d}{dt}\left(\frac{x(t)}{|x(t)|}\right) = \frac{\dot{x}(t)}{|x(t)|} - \frac{\langle x(t), \dot{x}(t)\rangle}{|x(t)|^3} x(t).$$

By homogeneity of F, we have obviously that for almost all $t \in [a, b]$,

$$\frac{\dot{x}(t)}{|x(t)|} \in F\left(\frac{x(t)}{|x(t)|}\right);$$

moreover the projection of the vector $\frac{\dot{x}(t)}{|x(t)|}$ on the tangent space $T_{\frac{x(t)}{|x(t)|}}\mathbb{S}^2$ reads

$$\frac{\dot{x}(t)}{|x(t)|} - \left\langle \frac{\dot{x}(t)}{|x(t)|}, \frac{x(t)}{|x(t)|} \right\rangle \frac{x(t)}{|x(t)|}.$$

This proves the first part of the lemma. Let us now consider some trajectory $y(\cdot) : [a,b] \to \mathbb{R}^3$ of (45). Since $y(\cdot)$ is absolutely continuous on $[a,b]$, the multivalued map $G : [a,b] \to \mathbb{R}^3$ defined by,

$$\forall t \in [a,b], \quad G(t) := \left\{ v \in F(y(t)) \text{ s.t. } \mathrm{Proj}_{T_y\mathbb{S}^2}(v) = \dot{y}(t) \right\},$$

is measurable with nonempty compact convex values and is bounded on $[a,b]$. Thus by the Measurable Selection Theorem (see for instance [21]), there are two L^1 functions $v(\cdot) : [a,b] \to \mathbb{R}^3$ and $\alpha(\cdot) : [a,b] \to \mathbb{R}$ such that $v(t) \in F(y(t))$ for any $t \in [a,b]$, and such that

$$\dot{y}(t) = v(t) - \alpha(t)y(t), \qquad \text{a.e } t \in [a,b]. \tag{47}$$

Set for any $t \in [a,b]$,

$$\mathcal{K}(t) := \exp\left(\int_0^t \alpha(s)ds \right),$$

and define the absolutely continuous arc $x(\cdot) : [a,b] \to \Omega$ by,

$$\forall t \in [a,b], \quad x(t) := \mathcal{K}(t)y(t).$$

Obviously $x(t)/|x(t)| = y(t)$ for any $t \in [a,b]$. Moreover by (47) we have that for almost every $t \in [a,b]$,

$$\begin{aligned} \dot{x}(t) &= \dot{\mathcal{K}}(t)y(t) + \mathcal{K}(t)\dot{y}(t) \\ &= \alpha(t)\mathcal{K}(t)y(t) + \mathcal{K}(t)(v(t) - \alpha(t)y(t)) \\ &= \mathcal{K}(t)(v(t) \in \mathcal{K}(t)F(y(t)) = F(x(t)), \end{aligned}$$

by homogeneity of F. This concludes the proof of Lemma 9. □

Let us now start the proof of Theorem 1.

3.3. **A relevant $\mathbf{GAC}_0$ differential inclusion.** As before we set $\Omega := \mathbb{R}^3 \setminus \{0\}$. We define for any $x \in \mathbb{R}^3$, the set of velocities of the control system (43) at x by

$$F_0(x) := \left\{ \sum_{i=1}^m u_i Y_i(x) : u \in \mathbb{R}^m \right\}.$$

The mapping $F_0(x)$ is lower semicontinuous on $\mathbb{R}^3$, and for every $x \in \mathbb{R}^3$ the set $F_0(x)$ is a vector subspace of $\mathbb{R}^3$ of dimension $1, 2$ or 3. We set for every $l = 1, 2, 3$,

$$\mathcal{D}_l := \left\{ x \in \mathbb{S}^2 \text{ s.t. } \dim F_0(x) = l \right\},$$

and we define $\mathcal{R} \subset \mathbb{S}^2$ as,

$$\mathcal{R} := \left\{ x \in \mathbb{S}^2 \text{ s.t. } x \in F_0(x) \right\}.$$

We claim the following result.

Lemma 10. *The sets $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ and $\mathcal{R}$ are semianalytic and satisfy the following properties:*

(i) *The set* $\mathcal{D}_1$ *is compact, of dimension* ≤ 1*, and satisfies*

$$\mathcal{D}_1 \cap \mathcal{R} = \emptyset.$$

(ii) *The set* $\mathcal{D}_3$ *is either open and dense in* $\mathbb{S}^2$*, or empty; moreover it satisfies* $\mathcal{D}_3 \subset \mathcal{R}$*.*

(iii) *The set* $\mathcal{D}_2 \cap \mathcal{R}$ *has dimension* ≤ 1 *and satisfies*

$$\overline{\mathcal{D}_2 \cap \mathcal{R}} \subset (\mathcal{D}_2 \cap \mathcal{R}) \cup \mathcal{D}_1.$$

Proof. The semianalyticity of $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ and $\mathcal{R}$ is an easy consequence of the analyticity of the vector fields $Y_1, \cdots, Y_m$. Let us prove the three properties (i)-(iii).
(i) The compactness of $\mathcal{D}_1$ comes from the lower semicontinuity of the mapping F_0; on the other hand the fact that it is nowhere dense in the sphere and that $\mathcal{D}_1 \cap \mathcal{R} = \emptyset$ are consequences of (44).
(ii) If we denote by Σ the set of injective maps $\sigma : \{1,2,3\} \to \{1, \cdots, m\}$ and if we define the map $\phi : \mathbb{S}^2 \to \mathbb{R}$ by

$$\varphi(x) := \sum_{\sigma \in \Sigma} \det(Y_{\sigma(1)}(x), Y_{\sigma(2)}(x), Y_{\sigma(3)}(x))^2, \quad \forall x \in \mathbb{R}^3,$$

then the set $\mathcal{D}_3$ satisfies

$$\mathcal{D}_3 = \left\{ x \in \mathbb{S}^2 \text{ s.t.} \varphi(x) \neq 0 \right\}.$$

Since the map φ is analytic and homogeneous of degree 2 with respect to the standard dilation, the set $\mathcal{D}_3$ is obviously either open and dense in $\mathbb{S}^2$, or empty. The fact that $\mathcal{D}_3 \subset \mathcal{R}$ is a direct consequence of the definition of $\mathcal{R}$.
(iii) Let us prove that $\mathcal{D}_2 \cap \mathcal{R}$ has empty interior; we argue by contradiction. Assume that the exists $\bar{x} \in \mathbb{S}^2$ and $\mu > 0$ such that

$$\bar{x} + \mu B \subset \mathcal{D}_2 \cap \mathcal{R}.$$

Since $\bar{x}$ belongs to $\mathcal{D}_2$ there exists $i \in \{1, \cdots, m\}$ such that the vectors $Y_i(\bar{x})$ and $\bar{x}$ are independent; without loss of generality we can assume that $i = 1$. Moreover there exists also $j \in \{1, \cdots, m\}$ such that the vectors $Y_1(\bar{x})$ and $Y_j(\bar{x})$ are independent; as before we can assume that $j = 2$. Now since $\bar{x}$ belongs to $\mathcal{R}$ there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\bar{x} = \alpha_1 Y_1(\bar{x}) + \alpha_2 Y_2(\bar{x});$$

notice that by construction we have $\alpha_2 \neq 0$. In fact since the open ball $B(\bar{x}, \mu)$ is contained in $\mathcal{D}_2 \cap \mathcal{R}$, reducing μ if necessary there exist two smooth[6] functions $\alpha_1 : B(\bar{x}, \mu) \to \mathbb{R}^*$ and $\alpha_2 : B(\bar{x}, \mu) \to \mathbb{R}$ which satisfy

$$x = \alpha_1(x) Y_1(x) + \alpha_2(x) Y_2(x),$$

for any $x \in \bar{x} + \mu B$. Define the vector field Z on $B(\bar{x}, \mu)$ by $Z(x) := x$ for any $x \in \bar{x} + \mu B$. Setting $\beta_1(x) := 1/\alpha_2(x)$ and $\beta_2(x) := -\alpha_1(x)/\alpha_2(x)$ on $B(\bar{x}, \mu)$, we obtain

$$Y_2(x) = \beta_1(x) Z(x) + \beta_2(x) Y_1(x), \quad \forall x \in \bar{x} + \mu B.$$

[6] In fact up to reduce the constant μ, we can assume that the vector fields Y_1 and Y_2 are independent on $B(\bar{x}, \mu)$. Thus the functions α_1 and α_2 are solutions of a Cramer system, hence they are smooth.

Hence we deduce that for any $x \in \bar{x} + \mu B$,

$$[Y_1, Y_2](x) = \beta_1(x)[Y_1, Z](x) + \langle \nabla \beta_1(x), Y_1(x) \rangle Z(x) + \langle \nabla \beta_2(x), Y_1(x) \rangle Y_1(x). \quad (48)$$

But since the vector field Y_1 is homogeneous of degree zero with respect to the standard dilation, we have that $Y_1(\epsilon x) = \epsilon Y_1(x)$, for any $\epsilon > 0$ and any $x \in \mathbb{R}^3$. This implies that

$$DY_1(x) \cdot x = Y_1(x), \quad \forall x \in \Omega.$$

Hence from (48) we deduce that the bracket $[Y_1, Y_2](x)$ belongs to the vector space $\mathrm{span}\{Y_1(x), Y_2(x)\}$ for any $x \in \bar{x} + \mu B$. Furthermore since all the vector fields $Y_3, \cdots, Y_m$ can be written as a combination of Y_1 and Y_2 on $B(\bar{x}, \mu)$, we obtain that all the brackets of the form $[Y_i, Y_j](x)$ with $i, j \in \{1, \cdots, m\}$ and $x \in \bar{x} + \mu B$ belong to the vector space spanned by the Y_i's (on $B(\bar{x}, \mu)$); this contradicts (44).
Let us now prove the second property. Consider $x \in \overline{\mathcal{D}_2 \cap \mathcal{R}}$. Since $\overline{\mathcal{D}_2} \subset \mathcal{D}_2 \cup \mathcal{D}_1$, we have that $x \in \mathcal{D}_2 \cup \mathcal{D}_1$. Consequently, we just have to prove that if x belongs to $\mathcal{D}_2$ then it belongs to $\mathcal{R}$ as well; we argue by contradiction. Let us assume that x belongs to $\mathcal{D}_2$ and not to $\mathcal{R}$. This means that there are two independent vectors v and v' in $F_0(x)$ which do not belong to the vector line $\mathrm{span}\{x\}$. Hence by lower semicontinuity of the mapping F_0, for all y close enough to x, there exist two independent vectors v_y and v'_y in $F_0(y)$ which do not belong to the vector line $\mathrm{span}\{y\}$. But as $x \in \overline{\mathcal{D}_2 \cap \mathcal{R}}$, there exist such y in $\mathcal{D}_2 \cap \mathcal{R}$ that is such that y belongs to $F_0(y)$. This implies that $F_0(y)$ has dimension three and then contradicts the fact that $y \in \mathcal{D}_2$. □

Three cases appear.

Case A: $\mathcal{R} = \mathbb{S}^2$.

In that case for every $x \in \mathbb{S}^2$, there exists $u^x \in \mathbb{R}^m$ such that

$$\sum_{i=1}^{m} u_i^x Y_i(x) = -x.$$

In addition, there exists $\mu_x > 0$ such that

$$\langle \sum_{i=1}^{m} u_i^x Y_i(y), y \rangle \leq -\frac{|y|^2}{2}$$

for any $y \in x + \mu_x B$. By compactness of the sphere $\mathbb{S}^2$, there exists $p \in \mathbb{N}$ and p points $x_1, \cdots, x_p$ on the sphere such that

$$\mathbb{S}^2 \subset \bigcup_{j=1,\cdots,p} B(x_j, \mu_{x_j}).$$

Let $\{\Psi_j\}_{j=1,\cdots,p}$ be a smooth partition of unity subordinate to the covering $\{B(x_j, \mu_{x_j})\}_{j=1,\cdots,p}$, that is, a family of smooth maps $\{\Psi_j\}_{j=1,\cdots,p} : \mathbb{S}^2 \to \mathbb{R}$

such that $\mathrm{Supp}(\Psi_j) \subset B(x_j, \mu_{x_j})$ for every $j = 1, \cdots, p$, and such that

$$\sum_{j=1}^{p} \Psi_j(x) = 1, \quad \forall x \in \mathbb{S}^2.$$

We define the feedback $k^A : \mathbb{S}^2 \to \mathbb{R}^m$ by

$$k^A(x) := \sum_{j=1}^{p} \Psi_j(x) u_{x_j}.$$

We verify easily that for every $x \in \mathbb{S}^2$,

$$\langle \sum_{i=1}^{m} k_i^A(x) Y_i(x), x \rangle \leq -\frac{|x|^2}{2}.$$

Finally we extend k to $\mathbb{R}^3$ by setting for every $x \in \mathbb{R}^3$,

$$k^A(x) := |x| \sum_{j=1}^{p} \Psi_j(x) u_{x_j}.$$

By construction, the feedback k^A is globally Lipschitz on $\mathbb{R}^3$, smooth on Ω, homogeneous of degree zero with respect to the standard dilation, and the homogeneous function $V : \mathbb{R}^3 \to \mathbb{R}$ defined by $V(x) = |x|^2/2$ satisfies

$$\forall x \in \mathbb{R}^3, \quad \langle \nabla V(x), \sum_{i=1}^{m} k_i^A(x) Y_i(x) \rangle \leq -V(x).$$

Hence we obtain the following result.

Proposition 3.1. *If $\mathcal{R} = \mathbb{S}^2$, then there exists some feedback $k_A : \mathbb{R}^3 \to \mathbb{R}^m$ which is globally Lipschitz on $\mathbb{R}^3$, smooth outside the origin, homogeneous of degree zero with respect to the standard dilation and such that the closed-loop system*

$$\dot{x} = \sum_{i=1}^{m} k_i^A(x) Y_i(x), \tag{49}$$

is globally asymptotically stable at the origin (abreviated GAS_0 in the sequel).

Case B: $\overline{\mathcal{R}} = \mathbb{S}^2$.

Since $\mathcal{R}$ is a semianalytic, its interior is dense in $\mathbb{S}^2$. Hence from Lemma 10 (i)-(iii), the set $\mathcal{D}_3$ is dense in $\mathbb{S}^2$ and the set $\mathcal{D}_2$ has empty interior. Moreover since $\mathcal{D}_3$ is semianalytic, it has a finite number of connected components; we denote them by $\mathcal{D}_3^1, \cdots, \mathcal{D}_3^C$. By Lemma 8 there exist two constants $\bar{\mu}, l > 0$ such that for any $0 < \mu \leq \bar{\mu}$ and for any $c \in \{1, \cdots, C\}$ the set

$$\mathcal{D}_3^{c,\mu} := \{x \in \mathcal{D}_3^c \text{ s.t. } d(x, \mathcal{D}_1 \cup \mathcal{D}_2) \geq \mu\}$$

is nonempty, connected and such that for any pair $x, y \in \mathcal{D}_3^{c,\mu}$ there exists some absolutely continuous path

$$\gamma_{x,y} : [0, l] \longrightarrow \mathcal{D}_3^{c,\mu}$$

such that $\gamma_{x,y}(0) = x, \gamma_{x,y}(l) = y$ and $|\dot{\gamma}_{x,y}(t)| \leq 1$ for almost every $t \in [0, l]$. Furthermore there exists some constant $\bar{\rho} > 0$ and C points $\bar{x}_1, \cdots, \bar{x}_C \in \mathbb{S}^2$ which satisfy

$$\forall c = 1, \cdots, C, \quad B(\bar{x}_c, \bar{\rho}) \cap \mathbb{S}^2 \subset \mathcal{D}_3^{c,\bar{\mu}}.$$

We are going to prove the following.

Proposition 3.2. *If $\overline{\mathcal{R}} = \mathbb{S}^2$ then there exists a multivalued map $F_B : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies the assumptions (A1)-(A2), which is homogeneous of degree zero with respect to the standard dilation, such that $F_B(x) \subset F_0(x)$ for any $x \in \mathbb{R}^3$, such that the differential inclusion*

$$\dot{x}(t) \in F_B(x(t)), \quad a.e. \tag{50}$$

is GAC_0, and such that the two following properties are satisfied:

(i) *For any $c = 1, \cdots, C$ and for any $x \in \bar{x}_c + (\bar{\rho}/2)B$,*

$$F_B(x) = \left\{ (|x|\bar{x}_c - x) + \frac{2}{\bar{\rho}} \left(|x - |x|\bar{x}_c| - (\bar{\rho}/2)|x| \right) \bar{x}_c \right\}.$$

(ii) *For any $x \in \mathbb{R}^n \setminus \{\bar{x}_1, \cdots, \bar{x}_C\}$,*

$$-\lambda x \notin F_B(x), \quad \forall \lambda > 0.$$

Proof. We are first going to define the mapping F_B on a neighbourhood of the set $\mathcal{D}_1 \cup \mathcal{D}_2$, then on the set $\mathbb{S}^2$, and finally extend it by homogeneity to the whole space $\mathbb{R}^3$; we will do it in five steps. Before beginning, we need to define the three sets $\mathcal{D}_2^a, \mathcal{D}_2^b, \mathcal{D}_2^c$; we set

$$\mathcal{D}_2^a := \{x \in \mathcal{D}_2 \text{ s.t. } x \in F_0(x)\},$$
$$\mathcal{D}_2^b := \{x \in \mathcal{D}_2 \text{ s.t. } x \perp F_0(x)\},$$
$$\text{and} \quad \mathcal{D}_2^c := \mathcal{D}_2 \setminus \left(\mathcal{D}_2^a \cup \mathcal{D}_2^b\right).$$

We notice that these three sets are semianalytic sets of dimension at most one, hence from the Stratification Theorem 4 we will be able to stratify them by disjoint unions of strata of dimension zero and one. Furthermore we define a family of functions $(\phi_i^r)_{i=1,2,r>0} : \mathbb{S}^2 \to \mathbb{R}$ as,

$$\phi_i^r(x, v) := \max \left\{ 0, \min \left\{ \frac{d(x, \mathcal{D}_i)}{r}, 1 \right\} - |v|^2 \right\}, \quad \forall x \in \mathbb{S}^2.$$

We notice that for $i = 1, 2$ and for any $r > 0$, the function ϕ_i^r is globally Lipschitz on the sphere $\mathbb{S}^2$ and that it satisfies

$$x \in \mathcal{D}_i, v \in \overline{B} \implies \phi_i^r(x, v) = 0, \tag{51}$$
$$d(x, \mathcal{D}_i) \geq r, v \in \overline{B} \implies \phi_i^r(x, v) = 1 - |v|^2, \tag{52}$$
$$d(x, \mathcal{D}_i) \geq r/2, v \in \overline{B} \implies \phi_i^r(x, v) \geq \max\left\{0, 1/2 - |v|^2\right\}. \tag{53}$$

Let us construct the mapping F_B on a neighbourhood of the set $\mathcal{D}_1 \cup \mathcal{D}_2$.

Step 1: Let us first show how we could define the mapping F_B on a neighbourhood of the set $\mathcal{D}_1$. We define $F_1 : \mathcal{D}_1 \to \mathbb{R}^3$ by,

$$F_1(x) := F_0(x) \cap \overline{B}, \quad \forall x \in \mathcal{D}_1.$$

This mapping is globally Lipschitz, and we assert that there exists some neighbourhood $\mathcal{U}$ of $\mathcal{D}_1$ such that it can be extended into a globally Lipschitz mapping $F_1 : \mathcal{U} \to \mathbb{R}^3$ in such a way that the following properties are satisfied [7]:

(i) For any $x \in \mathcal{U}$, the set $\mathrm{span}(F_1(x))$ is a vector line which does not contain the vector x, $F_1(x) \subset F_0(x)$, and $F_1(x) = \mathrm{span}(F_1(x)) \cap \overline{B}$.
(ii) For any $x \in \mathcal{D}_1, F_1(x) = F_0(x) \cap \overline{B}$.

Define the mapping $\tilde{F}_1 : \mathcal{U} \to \mathbb{R}^3$ by ,

$$\forall x \in \mathbb{S}^2, \quad \tilde{F}_1(x) := \mathrm{Proj}_{T_x\mathbb{S}^2}\left(F_1(x)\right);$$

the following result holds.

Lemma 11. *There is a positive constant μ_1 such that $(\mathcal{D}_1 + \mu_1 B) \cap \mathbb{S}^2 \subset \mathcal{U}$, and such that for any $0 < \mu \leq \mu_1$ and for any $x \in (\mathcal{D}_1 + \mu B) \cap \mathbb{S}^2$, there is a trajectory $y(\cdot) : [0,1] \to \mathcal{U}$ of the differential inclusion*

$$\dot{y}(t) \in \tilde{F}_1(y(t))) \quad a.e.\ t \in [0,1], \tag{54}$$

which starts at y and which satisfies

$$d(y(t), \mathcal{D}_1) \leq \mu, \quad \forall t \in [0,1], \tag{55}$$

$$and \quad d(y(1), \mathcal{D}_1) = \mu. \tag{56}$$

Proof. First extend by homogeneity the mapping F_1 on the set Ω. For that we set

$$\hat{\mathcal{U}} := \left\{ x \in \Omega \text{ s.t. } \frac{x}{|x|} \in \mathcal{U} \right\}$$

and we define $\hat{F}_1 : \hat{\mathcal{U}} \to \mathbb{R}^3$ as ,

$$\hat{F}_1(x) := |x| F_1(x), \quad \forall x \in \hat{\mathcal{U}}.$$

The mapping $\hat{F}_1$ satisfies the assumption (A1)-(A2) on the set $\hat{\mathcal{U}}$ and is homogeneous of degree zero with respect to the standard dilation on this set. Recall that the set $\mathcal{D}_1$ is semianalytic and nowhere dense, hence by Theorem 4, it admits a stratification with strata of dimension zero and one. By homogeneity this implies that the set $\hat{\mathcal{D}}_1$ defined as ,

$$\hat{\mathcal{D}}_1 := \left\{ x \in \Omega \text{ s.t. } \frac{x}{|x|} \in \mathcal{D}_1 \right\},$$

---

[7] As a matter of fact, we first notice that for any $\bar{x} \in \mathcal{D}_1$ there exists $i_{\bar{x}} \in \{1, \cdots, m\}$ such that $F_0(\bar{x}) = \mathrm{span}\{Y_{i_{\bar{x}}}(\bar{x})\}$. Hence there is a small ball $\mathcal{B}_{\bar{x}}$ centered at $\bar{x}$ such that for any $x \in \mathcal{B}_{\bar{x}}$ the vector space $\mathrm{span}\{Y_{i_{\bar{x}}}(x)\}$ has dimension one; we set for any $x \in \mathcal{B}, G_{\bar{x}}(x) := \mathrm{span}\{Y_{i_{\bar{x}}}(x)\} \cap \overline{B}$. Now by compactness, the set $\mathcal{D}_1$ can be covered by a finite union of balls $\{\mathcal{B}_{\bar{x}_i}\}_{i \in I}$. Moreover, since the set $\mathcal{D}_1$ is semianalytic, it can be stratified (by Theorem 4), so this covering can be constructed in such a way that for any triple $i, i', i" \in I$ the intersection $\mathcal{B}_{\bar{x}_i} \cap \mathcal{B}_{\bar{x}_{i'}} \cap \mathcal{B}_{\bar{x}_{i"}}$ is empty. In addition, we can also assume that if x belongs to the intersection of two balls $\mathcal{B}_{\bar{x}_i} \cap \mathcal{B}_{\bar{x}_{i'}}$ then the vector lines $G_{\bar{x_i}}(x), G_{\bar{x_{i'}}}(x)$ are not orthogonal. Consider a smooth partition of unity $\{\psi_i\}_{i \in I}$ subordinate to this covering, and set for any $x \in \cup_{i \in I} \mathcal{B}_{\bar{x}_i} =: \mathcal{U}$

$$F_1(x) := \mathrm{span}\left(\tilde{\sum}_{i \in I} \psi_i(x) G_{\bar{x}_i}(x) \right) \cap \overline{B}.$$

(Here whenever Δ_1 and Δ_2 are two vector lines which are not orthogonal, and whenever ψ_1 and ψ_2 are two nonnegative constants such that $\psi_1 + \psi_2 = 1$, the notation $\tilde{\sum}_{i=1,2} \psi_i \Delta_i$ denotes the set $\mathrm{span}\{\psi_1 u_1 + \psi_2 u_2\}$ where $u_1 \in \Delta_1$, and $u_2 \in \Delta_2$ are taken such that $|u_1| = 1, |u_2| = 1$, and $\langle u_1, u_2 \rangle > 0$.) We leave the reader to prove that up to reducing the set $\mathcal{U}$, the mapping F satisfies the desired properties.

admits a stratification with homogeneous strata of dimension one and two. Fix $x \in \mathcal{D}_1$; two cases appear.
Case 1: x belongs to some stratum S of $\hat{\mathcal{D}}_1$ of dimension two.
If all the trajectories of the differential inclusion

$$\dot{x}(t) \in \hat{F}_1(x(t)) \quad \text{a.e.} \tag{57}$$

starting from x stay in the stratum S for small time, then this means that $\hat{F}_1(y) \subset T_yS$ whenever y belongs to a small neighbourhood of x. Since S is a real analytic submanifold and since $\mathrm{span}(\hat{F}_1)$ coincides with F_0 on S, this implies also that

$$\mathrm{Lie}\left\{Y_1, \cdots, Y_m\right\}(y) \subset T_yS \subsetneq \mathbb{R}^3,$$

for any y in a small neighbourhood of x.This fact contradicts the Hörmander's condition (44), which means that there are two constants $\epsilon_x, \mu_x \in (0,1)$ and a trajectory $x(\cdot) : [0, \epsilon_x] \rightarrow \hat{\mathcal{U}}$ of (57) starting at x such that

$$d(x(\epsilon_x), \hat{\mathcal{D}}_1) \geq \mu_x. \tag{58}$$

Case 2: x belongs to some stratum of dimension one.
Since $\mathrm{span}(\hat{F}_1(x))$ has dimension one and does not equal $\mathrm{span}\{x\}$, there is necessarly a trajectory of (57) which leaves the vector line $\mathrm{span}\{x\}$. By the previous case, we deduce easily the existence of a pair $\epsilon_x, \mu_x \in (0,1)$ and of a trajectory $x(\cdot) : [0, \epsilon_x] \rightarrow \mathbb{R}^3$ of (57) starting at x which satisfies (58).
We notice now that if for some point $x \in \mathcal{D}_1$ there are two constants $\epsilon_x, \mu_x > 0$ and some trajectory $x(\cdot) : [0, \epsilon_x] \rightarrow \mathbb{R}^3$ of (57) starting at x which satisfies (58), then by Gronwall Lemma there exists $\rho_x > 0$ such that for any $y \in B(x, \rho_x)$ there exists a trajectory $x^y(\cdot) : [0, \epsilon_x] \rightarrow \mathbb{R}^3$ of (57) starting at y such that

$$d(x^y(\epsilon_x), \hat{\mathcal{D}}_1) \geq \frac{\mu_x}{2}.$$

We conclude easily by compactness of the set $\mathcal{D}_1$, by the fact that 0 belongs to $\hat{F}_1(x)$ for any $x \in \hat{\mathcal{U}}$, and by Lemma 9. $\square$

Fix $r \in (0, \mu_1)$ and pick some function $\psi_1^r : \mathcal{U} \rightarrow [0,1]$ which is globally Lipschitz and which satisfies the following properties:

$$\forall x \in \mathcal{U}, \quad d(x, \mathcal{D}_1) \leq \frac{r}{2} \Longrightarrow \psi_1^r(x) = 1, \tag{59}$$

$$\forall x \in \mathcal{U}, \quad d(x, \mathcal{D}_1) \geq r \Longrightarrow \psi_1^r(x) = 0. \tag{60}$$

Define the mapping $G_1^r : \mathcal{U} \rightarrow \mathbb{R}^3$ as,

$$G_1^r(x) := \mathrm{span}\left(\psi_1^r(x) F_1(x) + (1 - \psi_1^r(x)) \tilde{F}_1(x)\right) \cap \overline{B},$$

for any $x \in \mathcal{U}$. This mapping is globally Lipschitz on $\mathcal{U}$ and satisfies from (59)-(60) the following properties:

(iii) For any $x \in \mathcal{U}$, the set $\mathrm{span}(G_1^r(x))$ is a vector line which does not contain the vector x and $G_1^r(x) = \mathrm{span}(G_1^r(x)) \cap \overline{B}$.
(iv) For any $x \in \mathcal{D}_1 + (r/2)\overline{B}, G_1^r(x) = F_1(x) \subset F_0(x)$.
(v) For any $x \in \mathcal{U} \setminus (\mathcal{D}_1 + rB), G_1^r \subset T_x\mathbb{S}^2$.

For any $x \in \mathcal{U}$, we denote by $w_1^r(x)$ the unique vector $w \in \mathbb{S}^2$ that is orthogonal to $\mathrm{span}(G_1^r(x))$ and which maximizes the quantity $\langle w, x\rangle$; moreover we denote by $L_1^r(x)$ the vector line which is orthogonal to the vector plane $\mathrm{span}(G_1^r(x), w_1^r(x))$. It is not difficult to prove[8] that the vector field $w_1^r : \mathcal{U} \to \mathbb{R}^3$ and the multivalued map $x \in \mathcal{U} \mapsto L_1^r(x) \cap \overline{B}$ are globally Lipschitz on $\mathcal{U}$. Define the mapping $F_1^r : \mathcal{U} \to \mathbb{R}^3$ as,

$$\forall x \in \mathcal{U}, \quad F_1^r(x) := \{v + w + \alpha w_1^r(x) : (v, w, \alpha) \in H_1^r(x)\},$$

where the set $H_1^r(x)$ is defined by the set of triple (v, w, α) such that $v \in G_1^r(x), \alpha \geq 0, w \in L_1^r(x)$ such that $0 \leq |w|^2 + \alpha^2 \leq \phi_1^r(x, v)$. The mapping F_1^r is globally Lipschitz on $\mathcal{U}$; moreover from (iii)-(v) and (51)-(52), it satisfies the following properties:

(A) For any $x \in \mathcal{U}$, the set $F_1^r(x)$ is a compact convex set which contains the origin, which is included in $F_0(x)$, and which intersects the cone $\{\lambda x : \lambda \leq 0\}$ only at the origin.
(B) For any $x \in \mathcal{D}_1 + (r/2)\overline{B}, F_1^r(x) = F_1(x)$.
(C) For any $x \in \mathcal{U}$ such that $d(x, \mathcal{D}_1) \geq r$,

$$F_1^r(x) = \overline{B} \cap \{v \in \mathbb{R}^3 \text{ s.t. } \langle x, v\rangle \geq 0\}.$$

(D) There exists $\rho_1 > 0$ such that for any $x \in \mathcal{U}$ with $d(x, \mathcal{D}_1) \geq r/2$,

$$\rho_1 \overline{B} \cap T_x\mathbb{S}^2 \subset \mathrm{Proj}_{T_x\mathbb{S}^2}\left(F_1^r(x)\right).$$

Step 2: Let K^a be a compact subset of $\mathcal{D}_2^a \cup \mathcal{D}_2^c$; let us show how to construct some mapping F_2 on a neighbourhood of the set K^a. Since K^a is a compact set, since $K^a \cap \mathcal{D}_1 = \emptyset$ and since the set $\mathcal{D}_2^b$ is closed in $\mathbb{S}^2 \setminus \mathcal{D}_1$, there exist some neighbourhood $\mathcal{V}$ of K^a in $\mathbb{S}^2$ which does not intersect the set $\mathcal{D}_1 \cup \mathcal{D}_2^b$. Define the mapping $L_2 : \mathcal{D}_2 \cap \mathcal{V} \to \mathbb{R}^3$ by

$$L_2(x) := \left(F_0(x) \cap T_x\mathbb{S}^2\right) \cap \overline{B}, \quad \forall x \in \mathcal{D}_2 \cap \mathcal{V}.$$

This mapping is globally Lipschitz, besides it can be extended into a mapping $L_2 : \mathcal{V} \to \mathbb{R}^3$ in such a way that for any $x \in \mathcal{V}$ the following property is satisfied[9]:

(vi) The set $\mathrm{span}\{L_2(x)\}$ is a vector line, $L_2(x) = \mathrm{span}(L(x)) \cap \overline{B}$, and $L_2(x) \subset T_x\mathbb{S}^2$.

For any $x \in \mathcal{D}_2 \cap \mathcal{V}$, we denote by $w_2(x)$ the unique vector $w \in F_0(x) \cap \overline{B}$ which maximizes the quantity $\langle x, w\rangle$. We notice that since the vector field

[8]By symmetry, for any $x \in \mathcal{U}$ the vector $w_1^r(x)$ belongs to the vector space $\mathrm{span}(G_1^r(x), x)$. Hence if the vector line $G_1^r(x)$ reads locally $G_1^r(x) = \mathrm{span}\{g(x)\}$ where $|g(x)| = 1$, then the vector $w_1^r(x)$ writes $w_1^r(x) = \lambda_1(x)g(x) + \lambda_2 x$ where $\lambda_1(x), \lambda_2(x) \in \mathbb{R}$ satisfy $\lambda_1(x) + \lambda_2(x)\langle x, g(x)\rangle = 0, \lambda_1(x)^2 + \lambda_2(x)^2 + 2\lambda_1(x)\lambda_2(x)\langle x, g(x)\rangle = 1$ and $\lambda_2(x)^2 > 0$. We deduce that $\lambda_2(x) = (1 - \langle x, g(x)\rangle)^{\frac{1}{2}}$ which is well defined since $g(x) \neq x$; we conclude easily.

[9]We apply the same construction as the one we did in Step 1 to extend F on a neighbourhood of $\mathcal{D}_1$.

$w_2 : \mathcal{D}_2 \cap \mathcal{V} \to \mathbb{S}^2$ is Lipschitz on $\mathcal{D}_2 \cap \mathcal{V}$[10], it can be extended into a Lipschitz vector field $w_2 : \mathcal{V} \to \mathbb{R}^3$. In fact up to reduce the neighbourhood $\mathcal{V}$ and to set $w_2 := w_2/|w_2|$, we can assume that the Lipschitz vector field $w_2 : \mathcal{V} \to \mathbb{R}^3$ satisfies the following property:

(vii) For any $x \in \mathcal{V}, |w_2(x)| = 1$ and $\langle x, w_2(x)\rangle \in [1/2, 1]$.

Fix $r > 0$ such that $K^a + r\overline{B} \subset \mathcal{V}$, and pick some function $\psi_2^r : \mathcal{V} \to [0, 1]$ which is globally Lipschitz and which satisfies the following properties:

$$\forall x \in \mathcal{D}_2 \cap \mathcal{V}, \quad \psi_2^r(x) = 1, \tag{61}$$

$$\forall x \in \mathcal{V}, \quad d(x, \mathcal{D}_2) \geq r \Longrightarrow \psi_2^r(x) = 0. \tag{62}$$

Set for any $x \in \mathcal{V}$,

$$w_2^r(x) := \psi_2^r(x)w_2(x) + (1 - \psi_2^r(x))x,$$

and define the mapping $G_{2,a}^r : \mathcal{V} \to \mathbb{R}^3$ by,

$$G_{2,a}^r(x) := \{v \in \text{span}\{L_2(x), w_2^r(x)\} \text{ s.t. } |v| \leq 1 \text{ and } \langle x, v\rangle \geq 0\},$$

for any $x \in \mathcal{V}$. As in Step 1, we define $\tilde{G}_{2,a}^r : \mathcal{V} \to \mathbb{R}^3$ as,

$$\tilde{G}_{2,a}^r(x) := \text{Proj}_{T_x\mathbb{S}^2}\left(G_{2,a}^r(x)\right), \quad \forall x \in \mathcal{V};$$

the following result holds (the proof of this result being similar to the proof of Lemma 11, it is left to the reader).

Lemma 12. *There exists some constant $\mu_2 > 0$ such that $(K^a + \mu_2 B) \cap \mathbb{S}^2 \subset \mathcal{V}$, and such that for any $0 < \mu \leq \mu_2$ and for any $y \in (K^a + \mu B) \cap \mathbb{S}^2$, there is a trajectory $y(\cdot) : [0, 1] \to \mathcal{V}$ of the differential inclusion*

$$\dot{y}(t) \in \tilde{G}_{2,a}^r(y(t))) \quad a.e.\ t \in [0, 1], \tag{63}$$

which starts at y and which satisfies

$$d(y(t), \mathcal{D}_2) \leq \mu, \quad \forall t \in [0, 1], \tag{64}$$

$$\textit{and} \quad d(y(1), \mathcal{D}_2) = \mu. \tag{65}$$

For any $x \in \mathcal{V}$, we denote by $L_2'(x)$ the vector line which is orthogonal to the vector plane $\text{span}(G_{2,a}^r(x))$, and we define the mapping $F_{2,a}^r : \mathcal{V} \to \mathbb{R}^3$ as,

$$F_{2,a}^r(x) := \left\{v + w' : v \in G_{2,a}^r(x), w' \in L_2'(x) \text{ s.t. } 0 \leq |w'|^2 \leq \phi_2^r(x, v)\right\}, \tag{66}$$

for any $x \in \mathcal{V}$. From (vi)-(vii) and (51)-(53), it is not difficult to show that the mapping $F_{2,a}^r$ is globally Lipschitz on $\mathcal{V}$ and that it satisfies the following properties:

---

[10] As a matter of fact, for any $\bar{x} \in \mathcal{D}_2 \cap \mathcal{V}$ there exist $i, j \in \{1, \cdots, m\}$ such that $F_0(x) = \text{span}\{Y_i(\bar{x}), Y_j(\bar{x})\}$. Hence there is a small ball $\mathcal{B}$ centered at $\bar{x}$ such that for any $x \in \mathcal{B}$ the vector space $\text{span}\{Y_i(x), Y_j(x)\}$ has dimension two. In addition, up to orthonormalize the basis Y_i, Y_j on $\mathcal{B}$ (that is up to set $Z_1(x) := Y_i(x)/|Y_i(x)|$ and $Z_2(x) := \alpha(x)Y_i(x) + \beta(x)Y_j(x)$ where $\alpha(x), \beta(x)$ satisfy $|Z_2(x)| = 1$ and $\langle Z_1(x), Z_2(x)\rangle$ for any $x \in \mathcal{B}$), we can assume that for any $x \in \mathcal{B}$, both vectors $Z_1(x) \equiv Y_I(x), Z_2(x) := Y_j(x)$ define an orthonomal basis of $F_0(x)$. This means that for any $x \in \mathcal{B}$, the vector $w_2(x)$ writes $w_2(x) = \alpha_1(x)Z_1(x) + \alpha_2 Z_2(x)$, where

$$\alpha_1(x)^2 + \alpha_2(x)^2 = 1 \quad \text{and} \quad -\alpha_2(x)\langle Z_1(x), x\rangle + \alpha_1(x)\langle Z_2(x), x\rangle = 0.$$

Since we have necessarily $\langle Z_1(x), x\rangle \neq 0$ or $\langle Z_2(x), x\rangle \neq 0$ (because $x \notin \mathcal{D}_2^b$), we leave the reader to deduce that the vector field w_2 is Lipschitz on the ball $\mathcal{B}$.

(E) For any $x \in \mathcal{V}$, the set $F^r_{2,a}(x)$ is a compact convex set which contains the origin and the set $G^r_{2,a}(x)$, which is included in $F_0(x)$, and which intersects the cone $\{\lambda x : \lambda \leq 0\}$ only at the origin.

(F) For any $x \in \mathcal{D}_2 \cap \mathcal{V}, F^r_{2,a}(x) = F_0(x) \cap \{v \in \overline{B} \text{ s.t. } \langle x, v\rangle \geq 0\}$.

(G) For any $x \in \mathcal{V}$ such that $d(x, \mathcal{D}_2) \geq r$,

$$F^r_{2,a}(x) = \overline{B} \cap \{v \in \mathbb{R}^3 \text{ s.t. } \langle x, v\rangle \geq 0\}.$$

Step 3: Let K^b be a compact subset of $\mathcal{D}^b_2 \cup \mathcal{D}^c_2$ and $r > 0$; let us show how to construct some mapping F_2 on a neighbourhood of the set K^b. Since K^b is a compact set, since $K^b \cap \mathcal{D}_1 = \emptyset$ and since the set $\mathcal{D}^a_2$ is closed in $\mathbb{S}^2 \setminus \mathcal{D}_1$, there exist some neighbourhood $\mathcal{W}$ of K^b in $\mathbb{S}^2$ which does not intersect the set $\mathcal{D}_1 \cup \mathcal{D}^b_2$. Since for any $x \in \mathcal{D}_2 \cap \mathcal{W}$ the vector plane $F_0(x)$ does not contain the vector line span$\{x\}$, it is clear that there exists some mapping $G^r_{2,b} : \mathcal{W} \to \mathbb{R}^3$ which is globally Lipschitz and which satisfies the following properties:

(viii) For any $x \in \mathcal{W}$, the set $\text{span}(G^r_{2,b}(x))$ is a vector plane which intersects the vector line span$\{x\}$ only at the origin, $G^r_{2,b}(x) = \text{span}(G^r_{2,b}(x)) \cap \overline{B}$, and $G^r_{2,b}(x) \subset F_0(x)$.

(ix) For any $x \in \mathcal{D}_2 \cap \mathcal{W}, G^r_{2,b}(x) = F_0(x) \cap \overline{B}$.

(x) For any $x \in \mathcal{W}$ such that $d(x, \mathcal{D}_2) \geq r, G^r_{2,b} = T_x\mathbb{S}^2 \cap \overline{B}$.

For any $x \in \mathcal{W}$, we denote by $w'(x)$ the unique vector of $\mathbb{S}^2$ which is orthogonal to $\text{span}(G^r_{2,b}(x))$ and such that $\langle x, w'(x)\rangle > 0$, and we define for any $r > 0$ the mapping $F^r_{2,b} : \mathcal{W} \to \mathbb{R}^3$ by,

$$F^r_{2,b}(x) := \left\{v + \alpha w'(x) : v \in G^r_{2,b}(x), 0 \leq \alpha \leq \phi^r_2(x, v)\right\}, \quad \forall x \in \mathcal{W}. \quad (67)$$

From (v)-(vii) and (51)-(52) it is not difficult to show that the mapping $F^r_{2,b}$ is globally Lipschitz on $\mathcal{W}$ and that it satisfies the following properties:

(H) For any $x \in \mathcal{W}$, the set $F^r_{2,b}(x)$ is a compact convex set which contains the origin and the set $G^r_{2,b}(x)$, which is included in $F_0(x)$, and which intersects the cone $\{\lambda x : \lambda \leq 0\}$ only at the origin.

(I) For any $x \in \mathcal{D}_2 \cap \mathcal{W}, F^r_{2,b}(x) = F_0(x) \cap \overline{B}$.

(J) For any $x \in \mathcal{W}$ such that $d(x, \mathcal{D}_2) \geq r$,

$$F^r_{2,b}(x) = \overline{B} \cap \{v \in \mathbb{R}^3 \text{ s.t. } \langle x, v\rangle \geq 0\}.$$

(K) There exists $\rho_3 > 0$ such that for any $x \in \mathcal{W}$,

$$\rho_3\overline{B} \cap T_x\mathbb{S}^2 \subset \text{Proj}_{T_x\mathbb{S}^2}\left(F^r_{2,b}(x)\right).$$

Step 4: We glue together the constructions given in Steps 1-3. For that we first notice that since the set $\mathcal{D}_1$ is semianalytic, we can write the following result.

Lemma 13. *There exist two semianalytic open sets $\mathcal{V}_1, \mathcal{V}_2$ such that $\mathcal{V}_1 \subset \mathcal{U}$ and $\mathcal{V}_1 \subset \mathcal{V}_2 \subset (\mathcal{D}_1 + \bar{\mu}B) \cap \mathbb{S}^2$, a positive integer N, and N sets $S_1, \cdots, S_N$*

such that

$$\mathcal{D}_2 \cap \mathcal{V}_2 = \bigcup_{k=1}^{N} S_k,$$

and such that for each $k \in \{1, \cdots, N\}$, the following properties are satisfied:

(i) *The stratum S_k is an open connected real analytic submanifold of $\mathcal{V}_2$ of dimension one.*
(ii) *The stratum S_k is either totally included in $\mathcal{D}_2^a$, either totally included in $\mathcal{D}_2^b$, either totally included in $\mathcal{D}_2^b$.*
(iii) *The set $S_k \cap \mathcal{V}_1$ is an open connected real analytic manifold of dimension one and the set $S_k \cap \partial\mathcal{V}_1$ is a singleton.*
(iv) *The set $\overline{S_k}$ is analytically diffeomorphic to the interval $[0,1]$ and intersects the set $\mathcal{D}_1$ (respectively the set $\partial\mathcal{V}_2$) at a unique point.*
(v) *If there exists $k' \neq k$ such that $\overline{S_k} \cap \overline{S_{k'}} \neq \emptyset$ then there is $x \in \mathcal{D}_1$ such that $\overline{S_k} \cap \overline{S_{k'}} = \{x\}$.*

Let $r > 0$ be such that $r < \bar{\mu}, \mu_1$ and $\mathcal{D}_1 + r\overline{B} \subset \mathcal{V}_1$. From Step 1 we know that there exists some Lipschitz mapping $F_1^r : \mathcal{U} \rightarrow \mathbb{R}^3$ which satisfies properties (A)-(D).
Set $\mathcal{U}' := \mathbb{S}^2 \setminus \overline{\mathcal{V}_1}$ and write the stratification of the set $\mathcal{D}_2^c \cap \mathcal{U}'$. There are two positive integers p, p', p distinct points $x_1, \cdots, x_p$ in $\mathbb{S}^2$ and p' disjoint open connected real analytic submanifolds of $\mathbb{S}^2$ of dimension one $M_1, \cdots, M_{p'}$ such that

$$\mathcal{D}_2^c \cap \mathcal{U}' = \{x_1, \cdots, x_p\} \cup \left(\bigcup_{j=1}^{p'} M_j \right).$$

Since both sets $\mathcal{D}_2^a \cap (\mathbb{S}^2 \setminus \mathcal{V}_2)$ and $\mathcal{D}_2^b \cap (\mathbb{S}^2 \setminus \mathcal{V}_2)$ are closed, there exists a compact subset K^a of $\mathcal{D}_2^a \cup \mathcal{D}_2^c$ which is included in $\mathbb{S}^2 \setminus \overline{\mathcal{V}_1}$, which contains the set $\mathcal{D}_2^a \cap (\mathbb{S}^2 \setminus \mathcal{V}_2)$, and which contains the points $x_1, \cdots, x_p$. From Step 2, we deduce that there exists some neighbourhood $\mathcal{V}$ of K^a which is included in $\mathbb{S}^2 \setminus \overline{\mathcal{V}_1}$, which does not intersect the set $\mathcal{D}_2^b$, and such that for any $r > 0$ which satisfies $K^a + r\overline{B} \subset \mathcal{V}$, there is some Lipschitz mapping $F_{2,a}^r : \mathcal{V} \rightarrow \mathbb{R}^3$ which satisfies properties (E)-(G).
Fix now $r > 0$ such that $r < \mu_1, \mu_2$, $\mathcal{D}_1 + rB \subset \mathcal{V}_1$ and $K^a + r\overline{B} \subset \mathcal{V}$; and set $K^b := \mathcal{D}_2^b \cap (\mathbb{S}^2 \setminus \mathcal{V}_2)$. From Step 3, we deduce that there exist some neighbourhood $\mathcal{W}$ of K^b which is included in $\mathbb{S}^2 \setminus \overline{\mathcal{V}_1}$ and which does not intersect $\mathcal{V}$, and some Lipschitz mapping $F_{2,b}^r : \mathcal{W} \rightarrow \mathbb{R}^3$ which satisfies properties (H)-(K).
Let us now explain how to glue together the three mappings $F_1^r, F_{2,a}^r, F_{2,b}^r$ constructed above on a neighbourhood of the set $\mathcal{D}_1 \cup \mathcal{D}_2$. In fact, we notice that without loss of generality on the construction of the neighbourhoods $\mathcal{V}$ and $\mathcal{W}$, we can assume that there is an integer $\bar{i} \in \{1, \cdots, p'\}$ and $\bar{i}$ distinct integers $j_1, \cdots, j_{\bar{i}}$ in $\{1, \cdots, p'\}$ such that the strata $M_{j_1}, \cdots, M_{\bar{i}}$ intersects both sets $\mathcal{V}$ and $\mathcal{W}$ as follows:
For any $i = 1, \cdots, \bar{j}$, the manifold M_{j_i} (which is diffeomorphic to the open interval $(0,1)$) can be partitionned into the union of two connected and open submanifolds $M_{j_i}^{\mathcal{V}}$ and $M_{j_i}^{\mathcal{W}}$ which correspond to both ends of S_{j_i}, and of

one closed and connected submanifold M'_{j_i} such that

$$M_{j_i} \cap \mathcal{V} = M_{j_i}^{\mathcal{V}}, \quad M_{j_i} \cap \mathcal{W} = M_{j_i}^{\mathcal{W}},$$
$$\text{and} \quad M_{j_i} \cap \big(\mathcal{U}' \setminus (\mathcal{V} \cup \mathcal{W})\big) = M'_{j_i}.$$

From this observation and from Lemma 13 above, it becomes easy to glue together the three mappings $F_1^r, F_{2,a}^r$ and $F_{2,b}^r$ along the strata $S_1, \cdots, S_N$ and $M_{j_1}, \cdots, M_{j_{\bar{i}}}$ in such a way to obtain an open set $\mathcal{X}$ of $\mathbb{S}^2$ which contains the three neighbourhoods $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ and the set $\mathcal{D}_1 \cup \mathcal{D}_2 + r\overline{B}$, and a Lipschitz mapping $F_B : \mathcal{X} \to \mathbb{R}^3$ which satisfy the following properties[11]:

(L) For any $x \in \mathcal{X}$, the set $F_B(x)$ is a compact convex set which contains the origin, which is included in $F_0(x)$, and which intersects the cone $\{\lambda x : \lambda \leq 0\}$ only at the origin.

(M) For any $x \in \mathcal{D}_1 + (r/2)\overline{B}, F_B(x) = F_1^r(x)$.

(N) For any $x \in \mathcal{D}_2 \cap \mathcal{V}, F_B(x) = F_{2,a}^r(x)$.

(O) For any $x \in \mathcal{D}_2 \cap \mathcal{W}, F_B(x) = F_{2,b}^r(x)$.

(P) For any $x \in \mathcal{X}$ such that $d(x, \mathcal{D}_1 \cup \mathcal{D}_2) \geq r$,
$$F_B(x) = \overline{B} \cap \{v \in \mathbb{R}^3 \text{ s.t. } \langle x, v\rangle \geq 0\}.$$

(Q) [12] If we set for every $x \in \mathcal{X}, \tilde{F}_B(x) := \mathrm{Proj}_{T_x\mathbb{S}^2}(F_B(x))$, then there exists $T > 0$ such that for every $y \in \mathcal{X}$, there is a trajectory $y(\cdot) : [0, T] \to \mathcal{X}$ of the differential inclusion
$$\dot{y}(t) \in \tilde{F}_B(y(t)) \quad \text{a.e. } t \in [0, T],$$
which starts at y, which stays in $\mathcal{X}$, and which satisfies
$$d(y(T), \mathcal{D}_1 \cup \mathcal{D}_2) = r.$$

[11]Fix $i \in \{1, \cdots, \bar{i}\}$; we notice that from the construction that we made in Steps 2 and 3 we have that
$$F_{2,a}^r(x) = F_0(x) \cap \left\{v \in \overline{B} \text{ s.t. } \langle x, v\rangle \geq 0\right\},$$
for any $x \in M_{j_i}^{\mathcal{V}}$, and that
$$F_{2,b}^r(x) = F_0(x) \cap \overline{B},$$
for any $x \in M_{j_i}^{\mathcal{W}}$. For any $x \in M'_{j_i}$, denote by $w_2(x)$ the unique vector $w \in F_0(x) \cap \overline{B}$ which maximizes the quantity $\langle x, w\rangle$ and set $L_2(x) := (F_0(x) \cap T_x\mathbb{S}^2) \cap \overline{B}$; both mappings w_2, L_2 are globally Lipschitz. Pick some function $\psi : M_{j_i} \to [0, 1]$ which is globally Lipschitz and such that $\psi(x) = 0$ for every $x \in M_{j_i}^{\mathcal{V}}$ and $\psi(x) = 1$ for every $x \in M_{j_i}^{\mathcal{W}}$. Then set
$$F_B(x) := \left\{v + \alpha w_2(x) : v \in L_2(x), |v|^2 + \alpha^2 \leq 1 \text{ and } \alpha \geq -\psi(x)\right\},$$
for every $x \in M_{j_i}$. In this way, we glue together both mappings $F_{2,a}^r$ and $F_{2,b}^r$ along the stratum M_{j_i}. In fact, by using the definitions of these mappings on $\mathcal{V}$ and $\mathcal{W}$, we are able to glue them together on a neighbourhood of each stratum M_{j_i} in such a way that properties (L)-(P) are satisfied. Moreover, from (i)-(v) in Lemma 13 and the constructions of the mappings $F_1^r, F_{2,a}^r, F_{2,b}^r$, we can also glue these mappings together along each stratum S_k for $k = 1, \cdots, N$ in such a way that properties (L)-(P) are satisfied.

[12]The property (Q) is a consequence of Lemma 11 together with property (D), of Lemma 12, and of property (K). For instance, if y belongs to $\mathcal{D}_1$, then we know by Lemma 11 together with (B) that there exists a trajectory $y(\cdot)$ of the differential inclusion
$$\dot{y}(t) \in \mathrm{Proj}_{T_y\mathbb{S}^2} F_1^r(y(t)) \quad \text{a.e.}$$
which starts at y, which remains in $\mathcal{U}$, and such that $d(y(1), \mathcal{D}_1) = r/2$. Besides from (D), we can lead $y(1)$ in time at most $r/(2\rho_1)$ to some point $z \in \mathcal{U}$ such that $d(z, \mathcal{D}_1)$.

In order to complete the construction of the mapping F_B on the sphere, it just remains to define it outside the set $\mathcal{X}$; we proceed as follows:
Let $x \in \mathbb{S}^2 \setminus \mathcal{X}$, we set

$$F_B(x) := \overline{B} \cap \left\{ v \in \mathbb{R}^3 \text{ s.t. } \langle x, v\rangle \geq 0 \right\}$$

if $x \notin \bigcup_{c=1}^C B(\bar{x}_c, \bar{\rho})$, by

$$F_B(x) := \left\{ (\bar{x}_c - x) + \frac{2}{\bar{\rho}} \left(|x - \bar{x}_c| - \bar{\rho}/2 \right) \bar{x}_c \right\}.$$

if $x \in B(\bar{x}_c, \bar{\rho}/2) \cap \mathbb{S}^2$ and $c \in \{1, \cdots, C\}$, by

$$F_B(x) := \left\{ \bar{x}_c - x - \frac{6}{\bar{\rho}} \left(|x - \bar{x}_c| - \bar{\rho}/2 \right) \langle \bar{x}_c - x, x\rangle x \right\} \subset T_x\mathbb{S}^2,$$

if $x \in (B(\bar{x}_c, 2\bar{\rho}/3) \setminus B(\bar{x}_c, \bar{\rho}/2)) \cap \mathbb{S}^2$ and $c \in \{1, \cdots, C\}$, and if $x \in (B(\bar{x}_c, \bar{\rho}) \setminus B(\bar{x}_c, 2\bar{\rho}/3)) \cap \mathbb{S}^2$ for some $c \in \{1, \cdots, C\}$ by,

$$F_B(x) := \left\{ v_1(x) + v_2 : |v_2| \leq \frac{3}{\bar{\rho}} \left(|x - \bar{x}_c| - 2\bar{\rho}/3 \right) \text{ and } \langle v_2, x\rangle \geq 0 \right\}.$$

where the vector $v_1(x)$ is defined by,

$$v_1(x) := \frac{3}{\bar{\rho}} \left(\bar{\rho} - |x - \bar{x}_c| \right) \left(x - \bar{x}_c - \langle \bar{x}_c - x, x\rangle x \right) \in T_x\mathbb{S}^2.$$

In conclusion, we obtain a mapping $F_B : \mathbb{S}^2 \to \mathbb{R}^3$ which is globally Lipschitz, with values which are compact convex subsets of $\mathbb{R}^n$ which contain the origin, and which satisfies properties (i)-(ii) of Proposition 3.2.

Step 5: Finally we define $F_B : \mathbb{R}^3 \to \mathbb{R}^3$ by $F_B(0) = 0$ and,

$$F_B(x) := |x| F_B\left(\frac{x}{|x|} \right), \quad \forall x \in \mathbb{R}^3 \setminus \{0\}.$$

By conctruction the mapping F_B satisfies the assumptions (A1)-(A2), is homogeneous of degree zero with respect to the standard dilation, is contained in F_0, and satisfies properties (i)-(ii). Let us show that the differential inclusion (50) is GAC$_0$.
If we set for any $x \in \mathbb{S}^2$, $\tilde{F}_B(x) := \mathrm{Proj}_{T_x\mathbb{S}^2}(F_B(x))$, then from (Q) and the fact that $0 \in F_B(x)$ for any $x \in \mathbb{S}^2$, we know that for every $y \in \mathbb{S}^2$ there is a trajectory $y(\cdot) : [0, T] \to \mathbb{S}^2$ of the differential inclusion

$$\dot{y}(t) \in \tilde{F}_B(y(t)) \quad \text{a.e. } t \in [0, T], \tag{68}$$

which starts at y and such that

$$y(T) \in \bigcup_{c=1,\cdots,C} \mathcal{D}_3^{c,r}.$$

Therefore since $r < \bar{\mu}$, and since we have that

$$\forall x \in \bigcup_{c=1,\cdots,C} \left(\mathcal{D}_3^{c,r} \setminus B(\bar{x}_c, \bar{\rho}) \right), \quad \tilde{F}_B(x) = T_x\mathbb{S}^2 \overline{B},$$

we deduce that the trajectory $y(\cdot)$ can be extended into a trajectory $y(\cdot)$: $[0, T+l] \to \mathbb{S}^2$ of (68) which satisfies

$$y(T+l) \in \bigcup_{c=1,\cdots,C} B(\bar{x}_c, \bar{\rho}).$$

We conclude easily by the construction of the mapping F_B inside the balls $B(\bar{x}_c, \bar{\rho})$, Lemma 9 and Proposition 2.8. □

Case C: $\overline{\mathcal{R}} \subsetneq \mathbb{S}^2$.

From Lemma 10, the set $\mathcal{D}_3$ is empty, the set $\mathcal{D}_2$ is dense in $\mathbb{S}^2$, and the set $\mathcal{D}_2 \cap \mathcal{R}$ has dimension ≤ 1. We claim the following result.

Proposition 3.3. *If $\overline{\mathcal{R}} \subsetneq \mathbb{S}^2$ then there exists a multivalued map $F_C : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies assumptions (A1)-(A2), which is homogeneous of degree zero with respect to the standard dilation, such that $F_C(x) \subset F_0(x)$ for any $x \in \mathbb{R}^3$, such that the differential inclusion*

$$\dot{x}(t) \in F_C(x(t)), \quad a.e. \tag{69}$$

is GAC_0, and such that the following property is satisfied:

$$\forall x \in \Omega, \forall \lambda > 0, \quad -\lambda x \notin F_C(x). \tag{70}$$

Proof. As in the previous case, we define the set $\mathcal{D}_2^a$ as,

$$\mathcal{D}_2^a := \{x \in \mathcal{D}_2 \text{ s.t. } x \in F_0(x)\}.$$

From Lemma 10 (iii), this set is a closed semianalytic set of dimension at most one. We are first going to define the mapping F_C on a neighbourhood of the set $\mathcal{D}_1 \cup \mathcal{D}_2^a$, then on the set $\mathbb{S}^2$, and finally extend it by homogeneity to the whole space $\mathbb{R}^3$; we will do it in four steps.

Step 1: We set for any $x \in \mathcal{D}_1$,

$$F_1(x) := \{v \in F_0(x) \text{ s.t. } |v| \leq 1\}.$$

As in Step 1 of the previous case, it can be easily shown that there exists some neighbourhood $\mathcal{U}$ of $\mathcal{D}_1$ such that the mapping F_1 can be extended into a globally Lipschitz mapping $F_1 : \mathcal{U} \to \mathbb{R}^3$ in such a way that the following properties are satisfied:

(i) For any $x \in \mathcal{U}$, the set $\mathrm{span}(F_1(x))$ is a vector line which does not contain the vector x, $F_1(x) \subset F_0(x)$, and $F_1(x) = \mathrm{span}(F_1(x)) \cap \overline{B}$.
(ii) For any $x \in \mathcal{D}_1, F_1(x) = F_0(x) \cap \overline{B}$.

Moreover as before, if we define the mapping $\tilde{F}_1 : \mathcal{U} \to \mathbb{R}^3$ by ,

$$\forall x \in \mathbb{S}^2, \quad \tilde{F}_1(x) := \mathrm{Proj}_{T_x\mathbb{S}^2}(F_1(x)),$$

then Lemma 11 holds.
Fix $r \in (0, \mu_1)$; for any $x \in \mathcal{U} \setminus (\mathcal{D}_1 + (r/2)\overline{B})$ we denote by $w_1^r(x)$ the unique vector $w \in \mathbb{S}^2$ which is orthogonal to $\mathrm{span}(F_1(x))$ and wich belongs to $F_0(x)$; it is not difficult to prove that the vector field $w_1^r : \mathcal{U} \setminus (\mathcal{D}_1 + (r/2)\overline{B}) \to \mathbb{R}^3$

is globally Lipschitz. We pick some function $\psi_1^r : \to [0,1]$ which is globally Lipschitz, nondecreasing, and which satisfies the following properties:

$$\forall x \in \mathcal{U}, \quad d(x, \mathcal{D}_1) \leq \frac{r}{2} \Longrightarrow \psi_1^r(x) = 0, \tag{71}$$

$$\forall x \in \mathcal{U}, \quad d(x, \mathcal{D}_1) \geq r \Longrightarrow \psi_1^r(x) = 1. \tag{72}$$

Then we define the mapping $F_1^r : \mathcal{U} \to \mathbb{R}^3$ as,

$$F_1^r(x) := \left\{ v + \alpha s w_1^r(x) : v \in F_1(x), |v|^2 + s^2 \leq 1, 0 \leq \alpha \leq \psi_1^r(x) \right\},$$

for any $x \in \mathcal{U}$. The mapping F_1^r is globally Lipschitz on $\mathcal{U}$; moreover from (i)-(ii) and (71)-(72), it satisfies the following properties:

(A) For any $x \in \mathcal{U}$, the set $F_1^r(x)$ is a compact convex set which contains the origin and which intersects the cone $\{\lambda x : \lambda \leq 0\}$ only at the origin.

(B) For any $x \in \mathcal{D}_1 + (r/2)\overline{B}, F_1^r(x) = F_1(x) \subset F_0(x)$.

(C) For any $x \in \mathcal{U}$ such that $d(x, \mathcal{D}_1) \geq r, F_1^r(x) = F_0(x) \cap \overline{B}$.

Step 2: Define two sets $\mathcal{U}^r, \mathcal{D}^r \subset \mathbb{S}^2$ as,

$$\mathcal{U}^r := \mathbb{S}^2 \setminus (\mathcal{D}_1 + r\overline{B}) \quad \text{and } \mathcal{D}^r := \mathcal{U}^r \cap \mathcal{D}_2^a.$$

We are going to show how to construct the mapping F_C on a neighbourhood of the set $\mathcal{D}^r$. Since the closure of $\mathcal{D}^r$ does not intesect $\mathcal{D}_1$, there exists some neighbourhood $\mathcal{V}^r \subset \mathcal{U}^r$ of $\mathcal{D}^r$ such that x is not orthogonal to $F_0(x)$ for any $x \in \mathcal{V}^r$. Define the mapping $L_2 : \mathcal{V}^r \to \mathbb{R}^3$ by

$$L_2(x) := \left(F_0(x) \cap T_x\mathbb{S}^2 \right) \cap \overline{B}, \quad \forall x \in \mathcal{V}^r,$$

and denote for any $x \in \mathcal{V}^r$, by $w_2(x)$ the unique vector $w \in F_0(x) \cap \overline{B}$ which maximizes the quantity $\langle x, w \rangle$. It is easy to prove that the mapping L_2 and the vector field $w_2 : \mathcal{V}^r \to \mathbb{R}^3$ are globally Lipschitz on $\mathcal{V}^r$. We define the mapping $G_2^r : \mathcal{V} \to \mathbb{R}^3$ as,

$$G_2^r(x) := \left\{ v + \alpha w : v \in L_2(x), \alpha \geq 0, |v|^2 + \alpha^2 \leq 1 \right\},$$

for any $x \in \mathcal{V}^r$. The mapping G_2^r is globally Lipschitz on $\mathcal{V}^r$ and satisfies the following properties:

(iii) For any $x \in \mathcal{V}^r$, the set $G_2^r(x)$ is a compact convex set which contains the origin, which is included in $F_0(x)$, and which intersects the cone $\{\lambda x : \lambda \leq 0\}$ only at the origin.

(iv) For any $x \in \mathcal{D}^r \cap \mathcal{V}^r, G_2^r(x) = F_0(x) \cap \{x \in \overline{B} \text{ s.t. } \langle x, v \rangle \geq 0\}$.

Moreover if we define $\tilde{G}_2^r : \mathcal{V}^r \to \mathbb{R}^3$ as,

$$\tilde{G}_2^r := \mathrm{Proj}_{T_x\mathbb{S}^2} \left(G_2^r(x) \right), \quad \forall x \in \mathcal{V}^r,$$

then the following result holds (the proof of this result being similar to the proof of Lemma 11, it is left to the reader).

Lemma 14. *There exists some constant $\mu_2 > 0$ such that $(\mathcal{D}^r + \mu_2 B) \cap \mathcal{U}^r \subset \mathcal{V}^r$, and such that for any $0 < \mu \leq \mu_2$ and for any $y \in (\mathcal{D}^r + \mu B) \cap \mathbb{S}^2$, there is a trajectory $y(\cdot) : [0,1] \to \mathcal{V}^r$ of the differential inclusion*

$$\dot{y}(t) \in \tilde{G}_2^r(y(t))) \quad a.e. \ t \in [0,1], \tag{73}$$

which starts at y *and which satisfies*

$$d(y(t), \mathcal{D}_2^a) \leq \mu, \quad \forall t \in [0,1], \tag{74}$$

$$\textit{and} \quad d(y(1), \mathcal{D}_2^a) = \mu. \tag{75}$$

For any $r' > 0$, we define the function $\beta^{r'} : \mathcal{V}^r \to \mathbb{R}$ by

$$\beta^{r'}(x) := \min\left\{\frac{d(x, \mathcal{D}_2^a)}{r'}, 1\right\}, \forall x \in \mathbb{S}^2.$$

We notice that for any $r' > 0$, the function $\beta^{r'}$ is globally Lipschitz on $\mathbb{S}^2$ and satisfies

$$x \in \mathcal{D}_2^a, \implies \beta^{r'}(x) = 0, \tag{76}$$

$$d(x, \mathcal{D}_2^a) \geq r, \implies \beta^{r'} = 1. \tag{77}$$

Fix now $r' \in (0, \mu_2)$ and define the mapping $F_2^{r,r'} : \mathcal{V}^r \to \mathbb{R}^3$ as,

$$F_2^{r,r'} := \left\{v + \alpha w : v \in L_2(x), \alpha \geq -\beta^{r'}(x), |v|^2 + \alpha^2 \leq 1\right\},$$

for any $x \in \mathcal{V}^r$. The mapping $F_2^{r,r'}$ is globally Lipschitz on $\mathcal{V}^r$, moreover from (iii)-(iv) and (76)-(77) it satisfies the following properties:

(D) For any $x \in \mathcal{V}^r$, the set $F_2^{r,r'}(x)$ is a compact convex set which contains the origin, which is included in $F_0(x)$, and which intersects the cone $\{\lambda x : \lambda \leq 0\}$ only at the origin.

(E) For any $x \in \mathcal{D}_2^r \cap \mathcal{V}^r, F_2^{r,r'}(x) \subset G_2^r(x)$.

(F) For any $x \in \mathcal{V}^r$ such that $d(x, \mathcal{D}_2^a) \geq r', F_2^{r,r'}(x) = F_0(x) \cap \overline{B}$.

Step 3: As in Step 4 of the previous case, it is easy from Lemma 13 to glue together both mappings that we constructed in Steps 1 and 2 above. Setting $F_C(x) := F_0 \cap \overline{B}$, for any x outside $\mathcal{U} \cup \mathcal{V}^r$, we obtain F_C on the sphere.

Step 4: Finally we define $F_C : \mathbb{R}^3 \to \mathbb{R}^3$ by $F_B(0) = 0$ and,

$$F_C(x) := |x| F_C\left(\frac{x}{|x|}\right), \quad \forall x \in \mathbb{R}^3 \setminus \{0\}.$$

By conctruction the mapping F_C satisfies the assumptions (A1)-(A2), is homogeneous of degree zero with respect to the standard dilation, is contained in the mapping F_0, and satisfies property (70). Furthermore (E) together with Lemma 11, Lemma 14 and Lemma 9 prove that the differential inclusion (69) is GAC_0. □

3.4. **A stabilizing feedback with bifurcation singularities.** As before, we set $\Omega := \mathbb{R}^3 \setminus \{0\}$. Assume that $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a multivalued map which satisfies assumptions (A1)-(A2) and which is homogeneous of degree zero with respect tot the standard dilation. By Theorem 3, we know that if the associated differential inclusion

$$\dot{x}(t) \in F(x(t)) \quad \text{a.e.} \tag{78}$$

is globally asymptotically controllable at the origin, then there exists a semiconcave control-Lyapunov function

$$V : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

for (78) which is homogeneous of degree 1 with respect to the standard dilation. Let us denote by $\Sigma(V)$ the set of $x \in \Omega$ where the function V is not differentiable; this set is called the singular set of the function V in the set Ω. We recall that if we denote for every $x \in \Omega$ by $\partial V(x)$ the Clarke's generalized gradient of V at x (we refer the reader to [13, 16] for an extensive study of the Clarke's generalized gradient of locally Lipschitz functions), then since V is semiconcave on Ω the singular set can also be defined by,

$$\Sigma(V) = \{x \in \Omega \text{ s.t. } \partial V(x) \text{ is not a singleton}\}.$$

Furthermore we notice that by homogeneity of V,the singular set $\Sigma(V)$ is homogeneous with respect to the standard dilation, that is

$$\forall x \in \Omega, \forall \lambda > 0, \quad x \in \Sigma(V) \Longrightarrow \lambda x \in \Sigma(V);$$

in addition we have also that

$$\nabla V(\lambda x) = \nabla V(x), \quad \forall x \in \Omega \setminus \Sigma(V), \forall \lambda > 0,$$

and more generally that

$$\partial V(\lambda x) = \partial V(x) \quad \forall x \in \Omega, \forall \lambda > 0.$$

Actually, since the function V is homogeneous, there exists some constant $\bar{\delta} > 0$ such that

$$\min_{v \in F(x)} \{\langle \nabla V(x), v \rangle\} \leq -\bar{\delta}|x|, \quad \forall x \in \Omega \setminus \Sigma(V). \tag{79}$$

According to the method that we applied in [39, 42] in order to construct stabilizing feedbacks, we define the function $\Psi_V : \Omega \to \mathbb{R}$ by

$$\forall x \in \Omega, \quad \Psi_V(x) := \max_{\zeta \in \partial V(x)} \min_{v \in F(x)} \{\langle \zeta, v \rangle\} = \min_{v \in F(x)} \max_{\zeta \in \partial V(x)} \{\langle \zeta, v \rangle\}.$$

The function Ψ_V is upper semicontinuous on Ω, besides by homogeneity of F and V it is homogeneous of degree 1 with respect to the standard dilation. Fix $\delta \in (0, \bar{\delta})$ and define $\Sigma_\delta \subset \Sigma(V)$ as,

$$\Sigma_\delta(V) := \{x \in \Omega \text{ s.t. } \Psi_V(x) > -\delta|x|\}.$$

Up to modifying[13] slighty the function V and the parameter δ, we can indeed assume that the sets $\Sigma(V)$ and $\Sigma_\delta(V)$ are homogeneous Whitney stratifications. Moreover, since we work with homogeneous objects in dimension 3, we can fit the two-dimensional results to our context. In this way, as described in [39, 42], we are able to construct a selection $v^* : \mathbb{R}^3 \to \mathbb{R}^3$ of

[13] We proved in [42] that generically any control-Lyapunov function of a given control system is stratified semiconcave on Ω. As we explained in [42] (see also [39]), this property implies that the singular set of V is a Whitney stratification, that is roughly speaking the singular set is stratified by a locally finite union of strata of dimension zero, one and two. Moreover, since we work with homogeneous dynamics and since the control-Lyapunov function V is homogeneous, we can indeed modify V homogeneously. Therefore we can assume without loss of generality that each stratum of the $\Sigma(V)$ is homogeneous with respect to the standard dilation.

the mapping F (that is such that $v^*(x) \in F(x)$ for any $x \in \mathbb{R}^3$) which is smooth outside Σ_δ, which stabilizes in the sense of Carathéodory and such that the discontinuities of the vector field $\tilde{v}^* : \mathbb{S}^2 \to T\mathbb{S}^2$ defined as,

$$\forall x \in \mathbb{S}^2, \quad \tilde{v}^* := \mathrm{Proj}_{T_x\mathbb{S}^2}(v^*(x)),$$

correspond to the classification that we gave in [39]. Let us state this result precisely.

Theorem 5. *If the differential inclusion (78) is GAC_0, then there exists a selection $v^* : \mathbb{R}^3 \to \mathbb{R}^3$ of F and a set $\mathcal{S} \subset \mathbb{R}^3$ such that the following properties are satisfied:*

(i) *The set $\mathcal{S}$ is closed, homogeneous with respect to the standard dilation and admits a Whitney stratification with homogeneous strata of dimension one and two.*
(ii) *The vector field v^* is homogeneous with respect to the standard dilation of degree zero and smooth on $\mathbb{R}^3 \setminus \mathcal{S}$.*
(iii) *The different types of discontinuities of the vector field $\tilde{v}^*$ on the sphere $\mathbb{S}^2$ are those described in Figure 1.*
(iv) *The system $\dot{x} = v^*(x)$ is GAS_0 in the sense of Carathéodory.*
(v) *For every bifurcation point $\bar{x}$, the Cauchy problem $\dot{x} = -v^*(x), x(0) = \bar{x}$ admits locally a unique solution.*

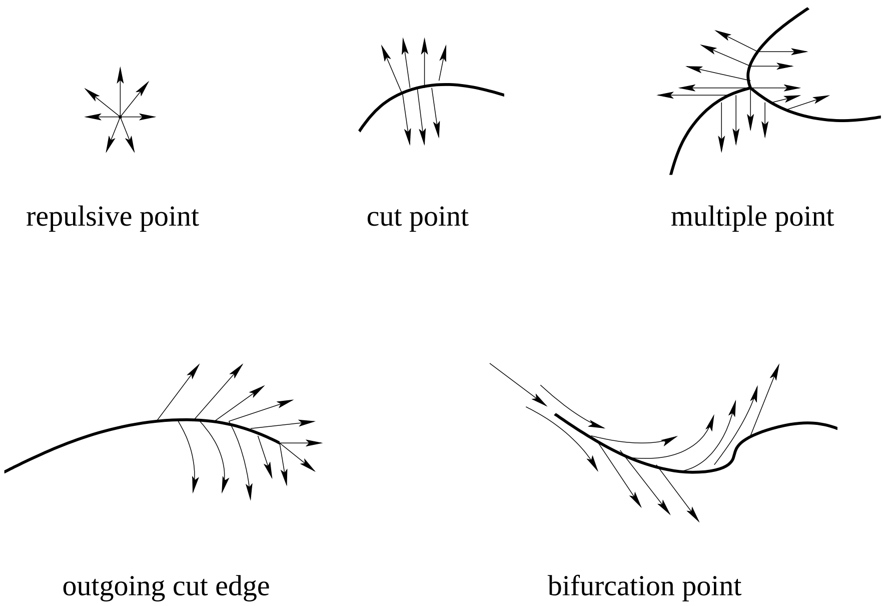

Figure 1. Different types of singularities.

Let us apply Theorem 5 to the cases B and C of the previous section.

Case B:

Since the multivalued map F_B satisfies the assumptions (A1)-(A2), is homogeneous of degree zero with respect to the standard dilation, and since the differential inclusion (50) is GAC_0, Theorem 5 gives.

Proposition 3.4. *If $\overline{\mathcal{R}} = \mathbb{S}^2$ then there exists a selection $v_B^* : \mathbb{R}^3 \to \overline{B}$ of $F_B \subset F_0$ and a set $\mathcal{S}_B \subset \mathbb{R}^3$ such that the properties (i)-(v) of Theorem 5 are satisfied. In particular, the two following properties are satisfied:*

$$\forall x \in \mathbb{S}^2 \setminus \{\bar{x}_1, \cdots, \bar{x}_C\}, \quad \tilde{v}_B^*(x) \neq 0, \tag{80}$$

and for every $c = 1, \cdots, C$ and every $x \in B\left(\bar{x}_c, \frac{\bar{\rho}}{2}\right)$,

$$\tilde{v}_B^*(x) = (\bar{x}_c - x) + \frac{2}{\bar{\rho}}\left(|x - \bar{x}_c| - (\bar{\rho}/2)\right)\bar{x}_c. \tag{81}$$

Case C:

Since the multivalued map F_C satisfies the assumptions (A1)-(A2), is homogeneous of degree zero with respect to the standard dilation, and since the differential inclusion (69) is GAC_0, Theorem 5 gives.

Proposition 3.5. *If $\overline{\mathcal{R}} \subsetneq \mathbb{S}^2$ then there exists a selection $v_C^* : \mathbb{R}^3 \to \overline{B}$ of $F_C \subset F_0$ and a set $\mathcal{S}_C \subset \mathbb{R}^3$ such that the properties (i)-(v) of Theorem 5 are satisfied. In particular, for every $x \in \mathbb{S}^2$, the vector $v_C^*(x)$ does not vanish.*

3.5. **Cancellation of bifurcation singularities.** In this section, our aim is to "eliminate" the singularities of bifurcation. Let us first explain how to do that in the case B.

Case B:

By Proposition 3.4, we have v_B^* and $\mathcal{S}_B$ for which properties (i)-(v) of Theorem 5 are satisfied. In particular, we know that the vector field $\tilde{v}_B^*$ is smooth on $\mathbb{S}^2 \setminus \mathcal{S}_B$ and that its singularities are those described in Figure 1. By compactness, there is only a finite number of bifurcation points in $\mathbb{S}^2$. Let us denote them by $x_1, \cdots, x_p$ and show how by modifying v_B^* we can eliminate these singularities. Fix $i \in \{1, \cdots, p\}$.

From assertion (v) of Theorem 5, there are $\epsilon > 0$ and a C^1 curve $x_i(\cdot) : [0, \epsilon] \to \mathbb{S}^2$ which satisfies

$$\dot{x}_i(t) = -\tilde{v}_B^*(x_i(t)), \quad \forall t \in [0, \epsilon], \quad x_i(0) = x_i, \tag{82}$$

and such that for every $t \in (0, \epsilon], x_i(t) \notin \mathcal{S}_B$. In fact, since the vector field $\tilde{v}_B^*$ is smooth outside the trace of the singular set on the sphere $\mathcal{S}_B \cap \mathbb{S}^2$, there exists $\bar{t} > 0$ such that the curve $x_i(\cdot)$ can be extended into a maximal solution to the Cauchy problem (82) in the open set $\mathbb{S}^2 \setminus \mathcal{S}_B$ on the interval $[0, \bar{t})$. Moreover we notice that from (81), the trajectory $x_i(\cdot)$ cannot enter the balls $B(\bar{x}_c, \bar{\rho}/2)$ for $c = 1, \cdots, C$. In consequence two different cases appear.

First case: $\bar{t} < \infty$.

From the description of singularities of v_B^* and then of $\tilde{v}_B^*$ given in Theorem 5 (iii), we deduce that the curve $x_i(\cdot)$ can be necessarily extended into a C^1 curve on the closed interval $[0,\bar{t}]$ and that $x_i(\bar{t}) \in \mathbb{S}^2 \cap \mathcal{S}_B$. Whence the point $x_i(\bar{t})$ is either a repulsive point, either a cut point, either a multiple point, either an outgoing cut edge or a bifurcation point. We describe below how to modify the vector field v_B^* and the control-Lyapunov function V in each of these situations.

First subcase: The point $x_i(\bar{t})$ is a repulsive point.
We first need the following result which will be illustrated in Figure 2.

Lemma 15. *There are two curves $y_i^1(\cdot), y_i^2(\cdot) : [0,\bar{t}] \to \mathbb{S}^2$ such that for all $t \in [0,\bar{t}]$,*

$$\langle x_i(t), \dot{y_i^1}(t) \rangle > 0, \quad \langle x_i(t), \dot{y_i^1}(t) \rangle > 0, \tag{83}$$

and such that the curve $x_i(\cdot) : [0,\bar{t}] \to \mathbb{S}^2$ is contained in the small open region $\mathcal{R}_i$ which is delimited by $y_i^1(\cdot)$ and $y_i^2(\cdot)$ in $\mathbb{S}^2$ (that is the region which we coloured grey in Figure 2).

Proof. Since we are dimension two, we can define define ξ_1 and ξ_2 two continuous sections of the unit normal bundle of the curve $x_i(\cdot) : [0,\bar{t}] \to \mathbb{S}^2$ in $\mathbb{S}^2$. Set for $j = 1,2$ and for every $t \in [0,\bar{t}]$,

$$y_i^j(t) := x_i(t) + \mu \int_0^t \xi_j(s)ds.$$

By construction, we have for almost every $t \in [0,\bar{t}]$,

$$\begin{aligned} \langle \dot{x}_i(t), \dot{y}_i^j(t) \rangle &= \mu \|\dot{x}_i(t)\|^2 + \mu \int_0^t \langle \dot{x}_i(s), \xi_j(s) \rangle ds \\ &= \|\dot{x}_i(t)\|^2 > 0. \end{aligned}$$

Moreover we have for any $t \in [0,\bar{t}]$,

$$\|y_i^j(t) - x_i(t)\| \le \|\mu \int_0^t \xi_j(s)ds\| \le \mu\bar{t}.$$

This conclude the proof of the lemma. □

Since the point $x_i(\bar{t})$ is repulsive, we have necessarily,

$$\mathrm{Proj}_{T_x\mathbb{S}^2}\left(F_0(x)\right) = T_x\mathbb{S}^2.$$

Define the set $\mathcal{S}_B' \subset \mathbb{R}^3$ as,

$$\mathcal{S}_B' := \mathcal{S}_B \cup \left\{ \lambda y_i^j(t) : \lambda \ge 0 t \in [0,\bar{t}], j = 1,2 \right\},$$

and the new vector field $(v_B^*)' : \mathbb{R}^3 \to \mathbb{R}^3$ by $(v_B^*)'(0) = 0$ and

$$\forall x \in \Omega, (v_B^*)'(x) := \begin{cases} v_B^*(x) & \text{if} \quad x \notin \mathcal{R} \\ -v_B^*(x) & \text{if} \quad x \in \mathcal{R}. \end{cases}$$

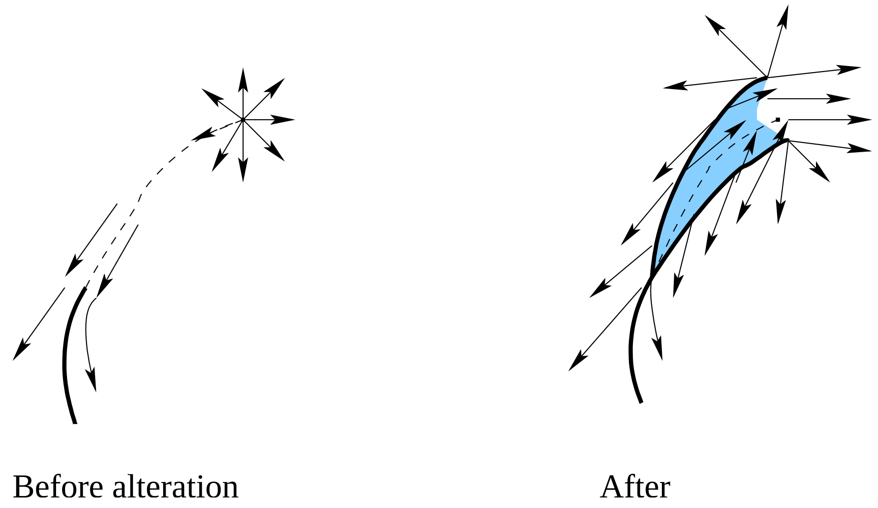

Figure 2.

We notice that the set $\mathcal{S}'_B$ is homogeneous with respect to the standard dilation and that $(v_B^*)'$ is homogeneous of degree zero with respect to δ^1. Moreover we observe that by (83), for every $x \in \mathcal{S}'_B$ the vectors $(v_B^*)'(y)$ are always pointing outward the set $\mathcal{S}'_B$ for y in a small neighbrouhood of x. Which means that there exists a neighbourhood $\mathcal{V}$ of the set $\mathcal{S}'_B$ in the sphere and a constant $\Delta > 0$ such that the following property is satisfied[14]:

$$\forall x \in \mathcal{V} \setminus \mathcal{S}'_B, \forall \xi \in \partial d_{\mathcal{S}'_B}(x), \quad \left\langle (v_B^*)'(x), \xi \right\rangle \geq \Delta.$$

Unfortunately, we notice that the vector field (v_B^*) is not smooth outside $\mathcal{S}'_B$; hence we have to refine the construction of $(v_B^*)'$. In fact we claim that we can glue together both vector fields v_B^* and $-v_B^*$ in such a way that the new vector field $(v_B^*)'$ is smooth outside the set $\mathcal{S}'$. For that, it suffices to recall that the set $F_0(x_i)$ is convex and symmetric with respect to the origin. As a matter of fact, since for every $x \in \mathcal{R}_i$, the vectors $v_B^*(x)$ and $(v_B^*)'$ belong to $F_0(x)$, the latter set contains necessarily a convex disc passing through these two vectors, thus it becomes easy to glue the vector fields $(v_B^*)'$ and $-v_B^*$ together as shown in Figure 3.

[14]Here, $\partial d_{\mathcal{S}'_B}(x)$ denotes the Clarke's generalized gradient of the distance function $d_{\mathcal{S}'_B}$ at the point x. We recall to the reader that we refer to [13, 16] for an extensive study of nonsmooth calculus.

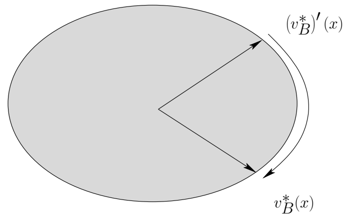

Figure 3.

Second subcase: The point $x_i(\bar{t})$ is a cut point.
In this case, a slightly different version of Lemma 15 provides two curves $y_i^1(\cdot), y_i^2(\cdot) : [0, \bar{t}] \to \mathbb{S}^2$ which satisfy (83) and such that $y_i^1(\bar{t})$ and $y_i^2(\bar{t})$ belong to the set $\mathcal{S}_B$. As before we add these curves to the singular set $\mathcal{S}_B$ and we modify the vector field v_B^* as shown in Figure 4.

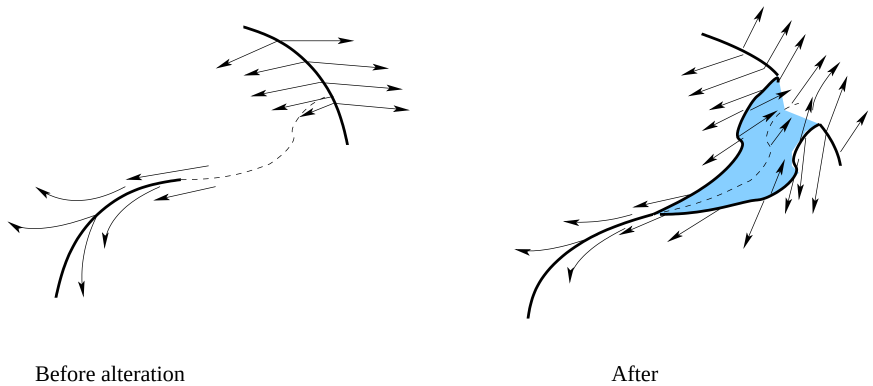

Figure 4.

Other subcases: The point $x_i(\bar{t})$ is a multiple point or an outgoing cut edge or a bifurcation point.
All these subcases are very similar to the previous ones, so we leave the reader to treat them. For instance we show in Figure 5 what happens in the case of an outgoing cut edge.

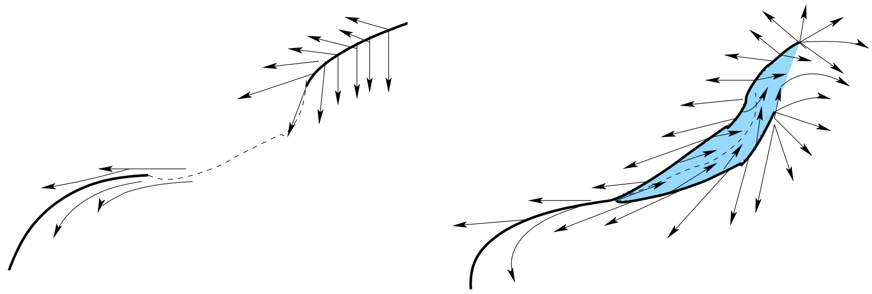

Figure 5.

Second case : $\bar{t} = \infty$.

Since $\mathbb{S}^2$ is compact, the ω-limit set of the curve $x_i(\cdot)$ defined as,

$$\mathcal{K} := \left\{ \lim_{n\to\infty} x_i(t_n) : (t_n)_n \uparrow \infty \right\},$$

is compact. Moreover by repulsivity of the set $\mathcal{S}_B$ with respect to the vector field v_B^* and by the fact that $\bar{t} = \infty$, we have necessarily that

$$\mathcal{K} \cap \mathcal{S}_B = \emptyset.$$

Furthermore, we notice that since the trajectory $x_i(\cdot)$ can not enter the balls $B(\bar{x}_c, \bar{\rho}/2)$ for $c = 1, \cdots, C$, the set $\mathcal{K}$ contains no equilibrium point of $\tilde{v}_B^*$ (we recall that $\tilde{v}_B^*$ does not vanish outside the points $\bar{x}_1, \cdots, \bar{x}_C$). Thus from Poincaré-Bendixon Theorem (see for instance the book [28]) we deduce that the set $\mathcal{K}$ is a closed orbit of the vector field $\tilde{v}_B^*$ in $\mathbb{S}^2 \setminus \mathcal{S}_B$. This means that there exist $\tau > 0$ and some trajectory $x(\cdot) : [0, \tau] \to \mathbb{S}^2$ of the dynamical system,

$$\dot{x}(t) = v_B^*(x(t)), \forall t \in [0, \tau],$$

which verifies $x(0) = x(\tau)$ and such that,

$$\mathcal{K} := \{x(t) : t \in [0, \tau]\}.$$

We need the following result:

Lemma 16. *There exists $x \in \mathcal{K}$ and $\tilde{v} \in \mathrm{Proj}_{T_x\mathbb{S}^2}(F_0(x))$ such that*

$$\tilde{v} \notin T_x\mathcal{K}.$$

Proof. We argue by contradiction. If we have that,

$$\forall x \in \mathcal{K}, \quad \mathrm{Proj}_{T_x\mathbb{S}^2}(F_0(x)) \subset T_x\mathcal{K},$$

then this implies easily that setting

$$\hat{\mathcal{K}} := \left\{ x \in \mathbb{R}^3 \setminus \{0\} \text{ s.t. } \frac{x}{|x|} \in \mathcal{K} \right\},$$

gives

$$\forall x \in \hat{\mathcal{K}}, \quad F_0(x) \subset T_x\hat{\mathcal{K}}.$$

The latter property contradicts assumption (44). □

This result allows us to assume without loss of generality that the closed orbit $\mathcal{K}$ is isolated in the set of closed orbits of the dynamical $\dot{x} = v_B^*(x)$ and that the set $\mathcal{K}$ is repulsive with respect to the trajectories of $\dot{x} = v_B^*(x)$. As a matter of fact, since the set $\mathcal{K}$ is the ω-limit set of a trajectory of the dynamical system $\dot{x} = -v_B^*(x)$, it cannot be attractive with respect to the trajectories of $\dot{x} = v_B^*(x)$ on both sides of it in $\mathbb{S}^2$. In addition, if $\mathcal{K}$ is repulsive on one side and attractive on the other, then up to modify the vector field v_B^* in a neighbourhood of some x given by Lemma 16 as shown in Figure 6, then we can eliminate the closed orbit $\mathcal{K}$.

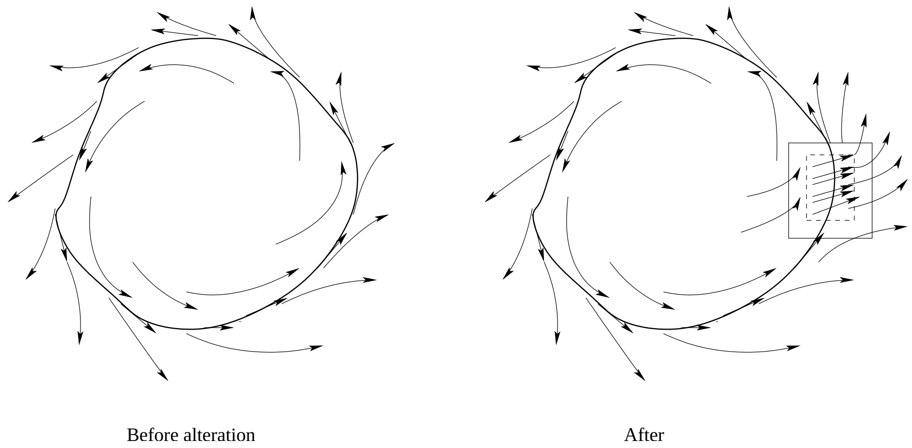

Figure 6.

Now as before, we construct two curves which encompass the closed orbit $\mathcal{K}$ and we we modify the vector fields v_B^* as shown in Figure 5.

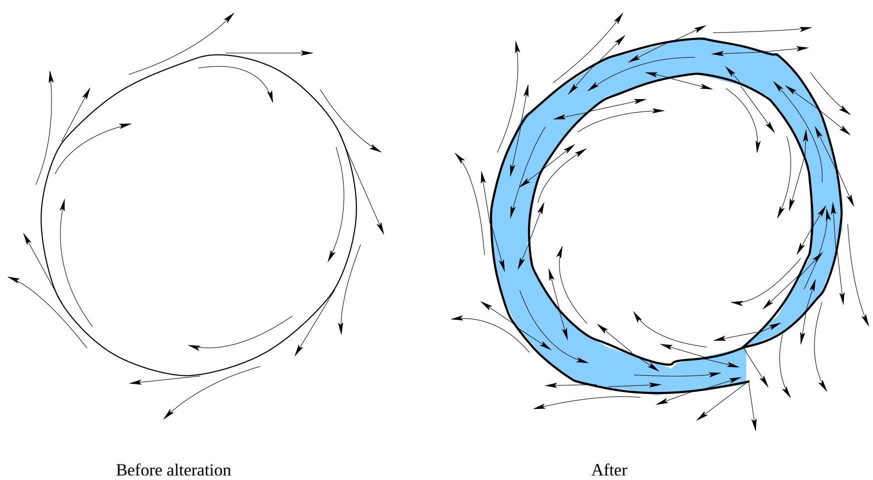

Figure 7.

This concludes the elimination of singularities in case B. If we are in case C, then we notice that from Proposition 3.5 since the vector field v_C^* does not vanish on the sphere, all the work we did in case B works as well. The cancellation of bifurcation singularities that we managed in this section applies to the vector fields v_B^* and v_C^*. However, since the vector field v_B^* (resp. v_C^*) is a selection of the multivalued map F_B (resp. F_C) which satisfies $F_B(x) \subset F_0(x)$ for every $x \in \mathbb{R}^3$ and since the control system (43) is affine in the control, using Michael's selection theorem (see [30] and [38]) we can construct a feedback $k_B^* : \mathbb{R}^3 \to \mathbb{R}^m$ (resp. $k_C^* : \mathbb{R}^3 \to \mathbb{R}^m$) such that the corresponding vector field

$$x \longmapsto \sum_{i=1}^m \left(k_B^*(x)\right)_i Y_i(x) \quad \left(\text{resp. } x \longmapsto \sum_{i=1}^m \left(k_C^*(x)\right)_i Y_i(x)\right),$$

satisfies the same properties as v_B^* (resp. v_C^*). Furthermore, we specify that in each transformation that described above, we are indeed able to change the initial control-Lyapunov function and to construct some selection of $\mathbb{R}^m$ in such a way that we obtain the following result:

Theorem 6. *If the control system (43) satisfies (44) then there exist a semiconcave control-Lyapunov function $V : \mathbb{R}^3 \to \mathbb{R}$ which is homogeneous of degree 1 with respect to the standard dilation, two sets $\mathcal{S}, \mathcal{V}_{\mathcal{S}} \subset \mathbb{R}^3$, a feedback $k_{\mathcal{S}} : \mathbb{R}^3 \to \mathbb{R}^m$, and a constant $\Delta > 0$ such that the following properties are satisfied:*

(i) *The set $\mathcal{S}$ is closed, homogeneous with respect to the standard dilation and stratified with strata of dimension one and two.*
(ii) *The feedback $k_{\mathcal{S}}$ is locally bounded on $\mathbb{R}^3$, smooth on $\mathbb{R}^3 \setminus \mathcal{S}$ and for every $i = 1, \cdots, m$ the i-th component $(k_{\mathcal{S}})_i$ of the feedback $k_{\mathcal{S}}$ is homogeneous of degree 1 with respect to the standard dilation.*
(iii) *For every $x \in \mathcal{S}$, there exists some sequence $(x_n)_n \in \mathbb{R}^3 \setminus \mathcal{S}$ which converges to x and such that $k_{\mathcal{S}}(x_n)$ tends to $k_{\mathcal{S}}(x)$.*
(iv) *For every $x \in \mathbb{R}^3 \setminus \mathcal{S}$ and for any $\zeta \in \partial V(x)$,*

$$\left\langle \sum_{i=1}^m \left(k_{\mathcal{S}}(x)\right)_i Y_i(x), \zeta \right\rangle \leq -\Delta |x|^2.$$

(v) *The set $\mathcal{V}_{\mathcal{S}}$ is homogeneous with respect to the standard dilation and open in $\mathbb{R}^3 \setminus \{0\}$.*
(vi) *For every $x \in \mathcal{V}_{\mathcal{S}} \setminus \mathcal{S}$ and every $\xi \in \partial d_{\mathcal{S}}(x)$,*

$$\left\langle \sum_{i=1}^m \left(k_{\mathcal{S}}(x)\right)_i Y_i(x), \xi \right\rangle \geq \Delta |x|^2.$$

4. Proof of Theorem 1

We are now ready to prove Theorem 1. So we assume from now that the control system (1) satisfies the Hörmander's condition (2). Before giving the proof of our main result, we need to recall a classical technique of homogeneization of control systems and to prove a preliminary lemma related to the perturbation of smooth repulsive stable control systems.

4.1. **Local approximation by homogeneous control systems.** We present below some classical results about the approximation of a given family of smooth vector fields by vector fields with nilpotent Lie algebra. Actually various kinds of nilpotent approximations have been used in the study of hypoelliptic operators and in nonlinear control theory; see for instance the works of Rothschild ans Stein [45], Bressan [8], Hermes [23], or Bellaiche [6]. Here we follow the presentation given in Bellaiche's monograph.

If $X_1, \cdots, X_m$ is a given family of smooth vector fields in $\mathbb{R}^3$, then for each positive integer s and each s-tuple of numbers $\pi := (i_1, \cdots, i_s) \in \{1, \cdots, m\}^s$, the commutator X_π of $X_1, \cdots, X_m$ of length s is defined by

$$X_\pi := \left[X_{i_1}\left[\cdots\left[X_{i_{s-1}}, X_{i_s}\right]\right]\cdots\right].$$

Denote by $\mathcal{L}$ the Lie algebra generated by the vector fields $X_1, \cdots, X_m$ and construct an increasing filtration of $\mathcal{L}$ at zero.
We set $F_0 := \emptyset$, $F_1 := \mathrm{span}\{X_1, \cdots, X_m\}$, and we define by induction the family $\{F_j\}_{j\in\mathbb{N}}$ by,

$$F_{j+1} := \{[X_i, X] : X \in F_j, i = 1, \cdots, m\}.$$

The sequence of vector spaces $\{F_j(0)\}_{j\in\mathbb{N}}$ is nondecreasing; moreover since the family $X_1, \cdots, X_m$ satisfies the Hörmander's condition at the origin, there exists an integer N such that $F_N(0) = \mathbb{R}^3$. Set for any $j \in \mathbb{N}$, $n_j := \dim F_j(0)$ and let us show how to contruct the dilation adapted to the filtration $\mathcal{F} := \{F_j\}_{j\in\mathbb{N}}$. Four different cases appear.

First case: $n_1 = 3$.
There exist $i_1, i_2, i_3 \in \{1, \cdots, m\}$ such that

$$\mathbb{R}^3 = F_1(0) = \mathrm{span}\left\{X_{i_1}(0), X_{i_2}(0), X_{i_3}(0)\right\}.$$

We set $\pi_1 := (i_1), \pi_2 := (i_2), \pi_3 := (i_3)$ and $r := (1, 1, 1)$.

Second case: $n_1 = 2$.
There exist $i_1, i_2 \in \{1, \cdots, m\}$ such that

$$F_1(0) = \mathrm{span}\left\{X_{i_1}(0), X_{i_2}(0)\right\}.$$

Denote by j_2 the smallest $j \in \mathbb{N}$ such that $n_j = 3$. Thus there exists some j_2-tuple $I \in \{1, \cdots, m\}^{j_2}$ such that

$$\mathbb{R}^3 = F_{n_2}(0) = \mathrm{span}\left\{X_{i_1}(0), X_{i_2}(0), X_I(0)\right\}.$$

We set $\pi_1 := (i_1), \pi_2 := (i_2), \pi_3 := I$ and $r := (1, 1, j_2)$.

Third case: $n_1 = 1$.
There exists $i \in \{1, \cdots, m\}$ such that

$$F_1(0) = \mathrm{span}\left\{X_i(0)\right\}.$$

Denote by j_2 the smallest $j \in \mathbb{N}$ such that $n_j > 1$. Two subcases appear.

Subcase 1: $n_{j_2} = 3$.
In this subcase, there exist two j_2-tuples in $I_1, I_2 \in \{1, \cdots, m\}^{j_2}$ such that

$$\mathbb{R}^3 = F_{n_{j_2}}(0) = \mathrm{span}\left\{X_i(0), X_{I_1}(0), X_{I_2}(0)\right\}.$$

We set $\pi_1 := (i), \pi_2 := I_1, \pi_3 := I_2$ and $r := (1, j_2, j_2)$.

Subcase 2: $n_{j_2} = 2$,.
There exists a j_2-tuple $I \in \{1, \cdots, m\}^{j_2}$ such that

$$F_{n_{j_2}} = \text{span}\{X_i(0), X_I(0)\}.$$

Morever if we denote by j_3 the smallest j such that $n_j = 3$, there exists a j_3-tuple $J \in \{1, \cdots, m\}^{j_3}$ such that

$$\mathbb{R}^3 = F_{n_{j_3}}(0) = \text{span}\{X_i(0), X_I(0), X_J(0)\}.$$

We set $\pi_1 := (i), \pi_2 := I, \pi_3 := J$ and $r := (1, j_2, j_3)$.

In each case, we have constructed tuples π_1, π_2, π_3 and a triple r such that

$$\mathbb{R}^3 = \text{span}\{X_{\pi_1}(0), X_{\pi_2}(0), X_{\pi_3}(0)\}. \tag{84}$$

We call δ_ϵ^r the dilation adapted to the filtration $\mathcal{F}$. We have the following result; we refer the reader to Proposition 5.17 and Theorem 5.19 in [6][15].

Theorem 7. *There exists a smooth change of coordinates in the space $\mathbb{R}^3$ in which each vector field X_i $(i = 1, \cdots, m)$ takes the form*

$$X_i = \hat{X}_i + R_i,$$

where $\hat{X}_i$ is homogeneous of order 1 with respect to the dilation δ_ϵ^r, and R_i is "of order ≤ 0" with respect to δ_ϵ^r at the origin. In fact, for each $j = 1, 2, 3$, the j-th coordinate of the vector field $\hat{X}_i$ is an homogeneous polynomial of degree 1 and the j-th coordinate of R_i satisfies:

$$(R_i(x))_j = O\left(\left(|x_1|^{\frac{1}{r_1}} + |x_2|^{\frac{1}{r_2}} + |x_3|^{\frac{1}{r_3}}\right)^{r_j}\right). \tag{85}$$

In addition, the vector fields $\hat{X}_1, \cdots, \hat{X}_m$ satisfy the Hörmander's condition at the origin.

4.2. **Perturbations of smooth repulsive stable control system.** A standard result in the asymptotic stability of approximations of homogeneous vector fields is given by the following result; we refer the reader to [44, Theorem 3], also to the papers [23, 24].

Theorem 8. *Let F be a continuous vector field on $\mathbb{R}^n$ with $F(0) = 0$, which is homogeneous of degree $k \leq 1$ with respect to some dilation $\delta_\epsilon^r(x)$. Let G be a continuous vector field such that for each $i = 1, \cdots, n$, its i-th coordinate satisfies the following property:*

$$(G(x))_i = O\left(\left(|x_1 l^{\frac{1}{r_1}} + |x_2|^{\frac{1}{r_2}} + |x_3|^{\frac{1}{r_3}}\right)^{r_j-(k-1)}\right).$$

Then if the dynamical system $\dot{x} = F(x)$ is locally asymptotically stable at the origin, then the system $\dot{x} = F(x) + G(x)$ is locally asymptotically stable at the origin too.

Our main result is indeed based on the nonsmooth version on the theorem above which follows from the following lemma:

[15]We warn the reader that in [6], Bellaiche does not use the same definition of the degree of a homogeneous vector field.

Lemma 17. *Let $Z : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, associated with a semiconcave control-Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ and a set $\mathcal{S} \subset \mathbb{R}^n$ such that the following properties are satisfied:*

(i) *The set $\mathcal{S}$ is closed and stratified with strata of dimension less or equal than $n-1$.*

(ii) *The vector field Z is locally bounded on $\mathbb{R}^n$, smooth on $\mathbb{R}^n \setminus \mathcal{S}$ and satisfies $Z(0) = 0$.*

(iii) *For every $x \in \mathcal{S}$, there exists some sequence $(x_n)_n \in \mathbb{R}^3 \setminus \mathcal{S}$ which converges to x and such that $Z(x_n)$ tends to $Z(x)$.*

(iv) *There exists a continuous, positive definite function $W : \mathbb{R}^n \to \mathbb{R}$ such that,*

$$\forall x \in \mathbb{R}^n \setminus \mathcal{S}, \forall \zeta \in \partial V(x), \quad \langle Z(x), \zeta \rangle \leq -W(x).$$

(v) *There exists a neighbourhood $\mathcal{V}_{\mathcal{S}}$ of the set $\mathcal{S} \setminus \{0\}$ in $\mathbb{R}^n \setminus \{0\}$ and a continuous, positive definite function $W : \mathbb{R}^n \to \mathbb{R}$ such that,*

$$\forall x \in \mathcal{V}_{\mathcal{S}} \setminus \mathcal{S}, \forall \xi \in \partial d_{\mathcal{S}}(x), \quad \langle Z(x), \xi \rangle \geq w(x).$$

Then the dynamical system

$$\dot{x} = Z(x) \tag{86}$$

is globally asymptotically asymptotically stable at the origin in the sense of Carathéodory and any trajectory of it satisfies $x(t) \notin \mathcal{S}$ for any $t \geq 0$.

Proof. Assumption (iii) together with (ii) implies that for every $x \in \mathcal{S}$, the vector $Z(y)$ does not belong to the Bouligand tangent cone $T_{\mathcal{S}}^B(x)$[16] whenever y is closed enough to x. Hence if $x(\cdot) : [0, \epsilon] \to \mathbb{R}^n$ is some Carathéodory solution of (86) on the interval $[0, \epsilon]$ such that $x(0) \in \mathcal{S}$, then there exists $\mu \in (0, \epsilon)$ such that $x(t) \notin \mathcal{S}$ for any $t \in (0, \mu)$. Furthermore we notice that if $x(\cdot) : [a, b] \to \mathbb{R}^n$ is some Carathéodory solution of (86) which remains inside the set $\mathcal{V} \setminus \mathcal{S}$, then from Lebourg's Theorem (see [13, Theorem 2.3.7 p. 41]), for almost every $t \in [a, b]$ there exists $\xi \in \partial d_{\mathcal{S}}(x(t))$ such that

$$\frac{d}{dt} d_{\mathcal{S}}(x(t)) = \langle \xi, Z(x(t)) \rangle.$$

Thus we deduce that the function $t \in [a, b] \mapsto d_{\mathcal{S}}(x(t)) - \int_0^t w(s) ds$ is nondecreasing, which implies that the function $t \in [a, b] \mapsto d_{\mathcal{S}}(x(t))$ is increasing. Since assumptions (i), (iii) and (v) imply that for every $x \in \mathcal{S}$, there exists $\delta > 0$ such that for any $t_1 \leq \delta$, the point x_1 defined as $x_1 := x + t_1 Z(x)$ does not belong to $\mathcal{S}$ and since the vector field Z is smooth on $\mathbb{R}^n \setminus \mathcal{S}$, it is clear that for every $x_0 \in \mathbb{R}^n$ the Cauchy problem

$$\dot{x} = Z(x(t)), \quad \text{a.e. and} \quad x(0) = x_0,$$

admits a Carathéodory solution. Now since we know that Carathéodory solutions of (86) always exist (that is for every initial state) and satisfy $x(t) \notin \mathcal{S}$ for every positive time, it remains to prove that the system (86) is globally

[16] The Bouligand tangent cone to the set $\mathcal{S}$ at x is defined by

$$T_{\mathcal{S}}^B(x) := \left\{ v \in \mathbb{R}^3 \text{ s.t. } \liminf_{t \downarrow 0} \frac{d_{\mathcal{S}}(x + tv)}{t} = 0 \right\}.$$

asymptotically stable at the origin in the sense of Carathéodory. This result is a simple consequence of assumption (iv) together with Lebourg's Theorem which give that any Carathéodory solution of (86) verifies,

$$\frac{d}{dt}V(x(t)) \leq W(x(t)), \quad \forall t > 0.$$

We conclude easily; we refer the reader to [37] for the details of the proof. □

4.3. **Proof of Theorem 1.** As before we set $\Omega := \mathbb{R}^3 \setminus \{0\}$. From Theorem 7, up to make a change of variables, we can assume that for every $i = 1, \cdots, m$, the vector field X_i writes

$$X_i = \hat{X}_i + R_i,$$

where $\hat{X}_i$ is homogeneous of degree 1 with respect to some dilation δ_ϵ^r and R_i satisfies (85). Applying Proposition 2.1 with $\mu = \gamma = 1$, we obtain a homeomorphism $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $\Phi(0) = 0$, Φ is an analytic diffeomorphism from Ω into Ω, and each vector field $\tilde{\hat{X}}_i$ (for $i = 1, \cdots, m$) defined as,

$$\tilde{\hat{X}}_i(y) := D\Phi\left(\Phi^{-1}(y)\right) \cdot \hat{X}_i\left(\Phi^{-1}(y)\right), \quad \forall y \in \Omega,$$

satisfies

$$\forall y \in \Omega, \forall \epsilon > 0, \quad \tilde{\hat{X}}_i(\epsilon y) = \tilde{\hat{X}}_i(y).$$

Set for every $i = 1, \cdots, m$ and for every $y \in \Omega$, $\bar{\hat{X}}_i(y) := |y|\tilde{\hat{X}}_i(y)$ and

$$\bar{R}_i(y) := |y| D\Phi\left(\Phi^{-1}(y)\right) \cdot \left(R_i\left(\Phi^{-1}(y)\right)\right).$$

Fix $i = 1, \cdots, m$; by construction of Φ (see the proof of Proposition 2.1), we have that for every $y \in \Omega$ and every $\epsilon > 0$,

$$\begin{aligned}
\bar{R}_i(\epsilon y) &= |\epsilon y| D\Phi\left(\Phi^{-1}(\epsilon y)\right) \cdot \left(R_i\left(\Phi^{-1}(\epsilon y)\right)\right) \\
&= \epsilon |y| D\Phi\left(\delta_\epsilon^r\left(\Phi^{-1}(y)\right)\right) \cdot R_i\left(\delta_\epsilon^r\left(\Phi^{-1}(y)\right)\right) \qquad \text{(by (8))} \\
&= \epsilon^2 |y| D\Phi\left(\Phi^{-1}(y)\right) \cdot \left[\left(\delta_\epsilon^r\right)^{-1}\left(R_i\left(\delta_\epsilon^r\left(\Phi^{-1}(y)\right)\right)\right)\right] \qquad \text{(by (9))} \\
&= |y| D\Phi\left(\Phi^{-1}(y)\right) \cdot \left[\epsilon^2\left(\delta_\epsilon^r\right)^{-1}\left(R_i\left(\delta_\epsilon^r\left(\Phi^{-1}(y)\right)\right)\right)\right].
\end{aligned}$$

But we know by (85) that for every $x \in \Omega$ and for every $\epsilon > 0$,

$$\lim_{x \rightarrow 0} \epsilon \left(\delta_\epsilon^r\right)^{-1}\left(R_i\left(\delta_\epsilon^r(x)\right)\right) = 0.$$

Hence we deduce that for every $y \in \Omega$,

$$\lim_{\epsilon \rightarrow 0} \frac{\bar{R}_i(\epsilon y)}{\epsilon} = 0. \tag{87}$$

By construction, the vector fields $\bar{\hat{X}}_1, \cdots, \bar{\hat{X}}_m$ are analytic on Ω[17], homogeneous of degree zero with respect to the standard dilation and satisfy (44) for any $y \in \Omega$. Hence we can apply Theorem 6; therefore there exist a semiconcave control-Lyapunov function $\bar{V} : \mathbb{R}^3 \rightarrow \mathbb{R}$ which is homogeneous of degree 1 with respect to the standard dilation, two sets $\bar{\mathcal{S}}, \mathcal{V}_{\bar{\mathcal{S}}} \subset \mathbb{R}^3$, a

[17] As a matter of fact, the vector fields $\tilde{\hat{X}}_i$'s have polynomial coordinates and the function $y \mapsto |y|$ is analytic on Ω.

feedback $k_{\bar{\mathcal{S}}} : \mathbb{R}^3 \to \mathbb{R}^m$, and a constant $\bar{\Delta} > 0$ such that the following properties are satisfied:

(i) The set $\bar{\mathcal{S}}$ is closed, homogeneous with respect to the standard dilation and stratified with strata of dimension one and two.
(ii) The feedback $k_{\bar{\mathcal{S}}}$ is locally bounded on $\mathbb{R}^3$, smooth on $\mathbb{R}^3 \setminus \bar{\mathcal{S}}$ and for every $i = 1, \cdots, m$ the i-th component $(k_{\bar{\mathcal{S}}})_i$ of the feedback $k_{\bar{\mathcal{S}}}$ is homogeneous of degree 1 with respect to the standard dilation.
(iii) For every $y \in \bar{\mathcal{S}}$, there exists some sequence $(y_n)_n \in \mathbb{R}^3 \setminus \bar{\mathcal{S}}$ which converges to y and such that $k_{\bar{\mathcal{S}}}(y_n)$ tends to $k_{\bar{\mathcal{S}}}(y)$.
(iv) For any $y \in \Omega \setminus \bar{\mathcal{S}}$ and for any $\zeta \in \partial \bar{V}(y)$,

$$\left\langle \sum_{i=1}^{m} (k_{\bar{\mathcal{S}}}(y))_i \, \bar{\hat{X}}_i(y), \zeta \right\rangle \leq -\bar{\Delta}|y|^2.$$

(v) The set $\mathcal{V}_{\bar{\mathcal{S}}}$ is homogeneous with respect to the standard dilation and open in Ω.
(vi) For every $y \in \mathcal{V}_{\bar{\mathcal{S}}} \setminus \bar{\mathcal{S}}$ and every $\xi \in \partial d_{\bar{\mathcal{S}}}(y)$,

$$\left\langle \sum_{i=1}^{m} (k_{\bar{\mathcal{S}}}(y))_i \, \bar{\hat{X}}_i(y), \xi \right\rangle \geq \bar{\Delta}|y|^2.$$

Define the two "discontinuous" vector fields F, G on $\mathbb{R}^3$ by

$$\forall y \in \Omega \quad F(y) := \sum_{i=1}^{m} (k_{\bar{\mathcal{S}}})_i \, \bar{\hat{X}}_i(y), \quad F(0) = 0,$$

$$\forall y \in \Omega \quad G(y) := \sum_{i=1}^{m} (k_{\bar{\mathcal{S}}})_i \, \bar{R}_i(y), \quad G(0) = 0.$$

By construction, F and G are homogeneous of degree -1 with respect to the standard dilation; moreover by (iv), (vi) together with (87) , there are $\rho, \bar{\Delta}' > 0$ such that

$$\forall y \in \Omega \setminus \bar{\mathcal{S}}, |y| \leq \rho \Longrightarrow \forall \zeta \in \partial V(y), \quad \langle (F+G)(y), \zeta \rangle \leq -\bar{\Delta}'|y|^2,$$

$$\forall y \in \mathcal{V}_{\bar{\mathcal{S}}} \setminus \bar{\mathcal{S}}, |y| \leq \rho \Longrightarrow \forall \xi \in \partial d_{\bar{\mathcal{S}}}(y), \quad \langle (F+G)(y), \xi \rangle \geq \bar{\Delta}'|y|^2.$$

Using the local version of Lemma 17, we deduce that the closed-loop system

$$\dot{y} = (F+G)(y)$$

is LAS_0 (that is, locally asymptotically stable at the origin). We notice that for every $y \in \Omega$, the vector $(F+G)(y)$ writes,

$$\begin{aligned}
& (F+G)(y) \\
= \quad & \sum_{i=1}^{m} (k_{\bar{\mathcal{S}}}(y))_i \left(\bar{\hat{X}}_i(y) + \bar{R}_i(y) \right) \\
= \quad & \sum_{i=1}^{m} (k_{\bar{\mathcal{S}}}(y))_i \, |y| D\Phi \left(\Phi^{-1}(y) \right) \cdot \left(\left(\hat{X}_i + R_i \right) \left(\Phi^{-1}(y) \right) \right) \\
= \quad & D\Phi \left(\Phi^{-1}(y) \right) \cdot \left(\left(\sum_{i=1}^{m} (k)_i \left(\hat{X}_i + R_i \right) \right) \left(\Phi^{-1}(y) \right) \right) \\
= \quad & D\Phi \left(\Phi^{-1}(y) \right) \cdot \left(\left(\sum_{i=1}^{m} (k)_i X_i \right) \left(\Phi^{-1}(y) \right) \right),
\end{aligned}$$

where the function $k := (k_1, \cdots, k_m) : \mathbb{R}^3 \to \mathbb{R}^m$ is defined by,

$$(k)_i(x) := |\Phi(x)| \left(k_{\bar{\mathcal{S}}}(\Phi(x)) \right)_i, \quad \forall i = 1, \cdots, m.$$

In conclusion, if we set $k(0) := 0$, then from homogeneity of $k_{\bar{\mathcal{S}}}$ of degree 1 with respect to the standard dilation and (8), we deduce that for every $i = 1, \cdots, m$, the i-th component of k is homogeneous of degree 2 with respect to δ_ϵ^r. We conclude easily that the feedback k make the closed-loop system

$$\dot{x} = \sum_{i=1}^{m} (k(x))_i X_i(x)$$

LAS_0 with respect to the set

$$\mathcal{S} := \left\{ x \in \mathbb{R}^3 \text{ s.t. } \Phi(x) \in \bar{\mathcal{S}} \right\}.$$

We notice that if in the proof above, we define $V : \mathbb{R}^3 \to \mathbb{R}$ by $V(x) := \bar{V}\left(\Phi(x) \right)$ and if we set

$$\mathcal{V}_{\mathcal{S}} := \left\{ x \in \mathbb{R}^3 \text{ s.t. } \Phi(x) \in \mathcal{V}_{\bar{\mathcal{S}}} \right\},$$

then we obtain the following result.

Theorem 9. *Let $Y_1, \cdots, Y_m$ be smooth vector fields on $\mathbb{R}^3$ which are homogeneous of degree $k \leq 1$ with respect to some dilation δ_ϵ^r and which satisfy*

$$\mathrm{Lie}\{Y_1, \cdots, Y_m\}(0) = \mathbb{R}^3.$$

Then there exist a semiconcave control-Lyapunov function $V : \mathbb{R}^3 \to \mathbb{R}$ which is homogeneous of degree 1 with respect to δ_ϵ^r, a set $\mathcal{S} \subset \mathbb{R}^3$ and a feedback $k_{\mathcal{S}} : \mathbb{R}^3 \to \mathbb{R}^m$ such that the following properties are satisfied:

(i) *The set $\mathcal{S}$ is closed, homogeneous with respect to δ_ϵ^r and stratified with strata of dimension one and two.*

(ii) *The feedback $k_{\mathcal{S}}$ is locally bounded on $\mathbb{R}^3$, smooth on $\mathbb{R}^3 \setminus \mathcal{S}$ and for every $i = 1, \cdots, m$ the i-th component $(k_{\mathcal{S}})_i$ of the feedback $k_{\mathcal{S}}$ is homogeneous of degree 2 with respect to the standard dilation.*

(iii) *For every* $x \in \mathcal{S}$*, there exists some sequence* $(x_n)_n \in \mathbb{R}^3 \setminus \mathcal{S}$ *which converges to* x *and such that* $k_{\mathcal{S}}(x_n)$ *tends to* $k_{\mathcal{S}}(x)$*.*

(iv) *There exists a continuous, positive definite function* $W : \mathbb{R}^3 \to \mathbb{R}$ *such that,*

$$\forall x\ \mathbb{R}^3 \in \mathcal{S}, \forall \zeta \in \partial V(x), \quad \left\langle \sum_{i=1}^{m} (k_{\mathcal{S}}(x))_i\, Y_i(x), \zeta \right\rangle \leq -W(x).$$

(v) *There exists a open neighbourhood* $\mathcal{V}_{\mathcal{S}}$ *of the set* $\mathcal{S}$ *in* $\mathbb{R}^3 \setminus \{0\}$ *which is homogeneous with respect to* δ_ϵ^r *and a continuous, positive definite function* $w : \mathbb{R}^3 \to \mathbb{R}$ *such that,*

$$\forall x \in \mathcal{V}_{\mathcal{S}} \setminus \mathcal{S}, \forall \xi \in \partial d_{\mathcal{S}}(x), \quad \left\langle \sum_{i=1}^{m} (k_{\mathcal{S}}(x))_i\, Y_i(x), \xi \right\rangle \geq w(x).$$

5. Consequences for time-varying stabilizing feedbacks

In [18] (see also [19]), Coron proved that all controllable driftless control systems may be stabilized by continuous (and even smooth) time-varying feedback. In particular, his result implies that if a control system of the form (1) satisfies the Hörmander's condition (2) , then for all $T > 0$, there exists a time-varying feedback $u \in C^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ such that

$$u(t, 0) = 0, \quad \forall t \in \mathbb{R},$$

$$u(t + T, x) = u(t, x), \quad \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n,$$

and the origin is locally asymptotically stable for

$$\dot{x} = \sum_{i=1}^{m} u_i(t, x) X_i(x).$$

Later in [31], Morin, Pomet and Samson proved the homogeneous version of Coron's Theorem. In fact, given a smooth homogeneous control system verifying (2), they presented a method to construct smooth homogeneous time-varying feedback laws which achieve the stabilization of the control system to the origin. Here we base on the design method of Morin, Pomet and Samson to announce two specific results about the existence of some type of repulsive time-varying feedbacks. We notice that we just give an idea of the proof of the first result.

Theorem 10. *Let* $Y_1, \cdots, Y_m$ *be smooth vector fields on* $\mathbb{R}^3$ *which are homogeneous of degree* 1 *with respect to some dilation* δ_ϵ^r *and which satisfy*

$$\mathrm{Lie}\{Y_1, \cdots, Y_m\}(0) = \mathbb{R}^3.$$

Then there exists a closed set $\mathcal{S} \subset \mathbb{R}^3$ *which is homogeneous with respect to* δ_ϵ^r *such that for any* $T > 0$ *and for any neighbourhood* $\mathcal{V} \subset \mathbb{R}^3 \setminus \{0\}$ *of* $\mathcal{S} \setminus \{0\}$ *which is homogeneous with respect to* δ_ϵ^r*, there exists a time-varying feedback* $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^m$ *which is smooth, homogeneous with respect to* δ_ϵ^r*, and which satisfies the following properties:*

(i) *For any* $t \in \mathbb{R}, x \in \mathbb{R}^3, u(t + T, x) = u(t, x)$*.*

(ii) *For any* $t \in \mathbb{R}, u(t, 0) = 0$*.*

(iii) *For any* $x \in \mathcal{V}$*, the function* $t \in \mathbb{R} \mapsto u(t, x)$ *is constant.*

(iv) *The closed-loop system*

$$\dot{x} = \sum_{i=1}^{m} u_i(t,x) Y_i(x) \tag{88}$$

is globally asymptotically stable at the origin.

(v) *For every trajectory $x(\cdot) : [0,\infty) \rightarrow \mathbb{R}^3$ of (88) such that $x(0) \notin \mathcal{V}$ the following property is satisfied:*

$$x(t) \notin \mathcal{V}, \quad \forall t \geq 0.$$

Proof. Let us give an idea of the proof of Theorem 10. We set $\Omega := \mathbb{R}^3 \setminus \{0\}$. By Theorem 9, we know that under our assumptions, there are a semiconcave control-Lyapunov function $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ which is homogeneous of degree 1 with respect to the standard dilation δ_ϵ^r, a set $\mathcal{S} \subset \mathbb{R}^3$ and a feedback $k_{\mathcal{S}} : \mathbb{R}^3 \rightarrow \mathbb{R}^m$ such that the properties (i)-(vi) of Theorem 9 are satisfied. We set for every $x \in \mathbb{R}^3$,

$$F(x) := \sum_{i=1}^{m} \left(k_{\mathcal{S}}(x)\right)_i Y_i(x);$$

we notice that by construction, the vector field $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is homogeneous of degree $k-2$ with respect to δ_ϵ^r. Set for every $\rho > 0$,

$$\mathcal{B}_\rho := \left(\mathcal{S} + \rho \overline{B}\right) \cap N^{-1}(1),$$

$$\text{and} \quad \hat{\mathcal{B}}_\rho := \left\{ x \in \Omega \text{ s.t. } P(x) \in \mathcal{B}_\rho \right\}.$$

We notice that for any $\rho > 0$, the set $\hat{\mathcal{B}}_\rho$ is a closed neighbourhood of the set $\mathcal{S}$, besides if $\rho > 0$ is taken sufficiently small then there exists $\mu > 0$ such that,

$$\forall x \in \partial \hat{\mathcal{B}}_\rho, \forall \xi \in N_{\hat{\mathcal{B}}_\rho}^C(x), \quad \langle F(x), \xi \rangle \geq \mu N(x)^{k-2} |\xi|^{18}. \tag{89}$$

Fix $\rho > 0$ which satisfies the property above and such that

$$\mathcal{B}_\rho \subset (\mathcal{V} \cap \mathcal{V}_{\mathcal{S}}) \cap N^{-1}(1).$$

We use from now the notations for N and P that we defined in the proof of Proposition 2.1. Let $\Psi : N^{-1}(1) \rightarrow \mathbb{R}$ a nonnegative smooth function which verifies $\Psi(x) = 1$ for $x \in \mathcal{B}_{\frac{\rho}{2}}$ and $\Psi(x) = 0$ for $x \in N^{-1}(1) \setminus \mathcal{B}_\rho$; we set

$$\forall x \in \Omega, \quad a(x) := \Psi(P(x)) N(x)^{k-2}(-x) + (1 - \Psi(P(x))) F(x).$$

We leave the reader to verify that the vector field a is continuous on $\mathbb{R}^3$, smooth outside the origin and homogeneous of degree $k-2$ with respect to δ_ϵ. By homogeneity of the Y_i's and compactness of the sphere, there exist an integer $M > 0$ and M commutators $Y_{\pi_1}, \cdots, Y_{\pi_M}$ of length > 1 (we refer to Section 4.1 for the notations concerning the commutators) and there are m smooth functions $u_1, \cdots, u_m : \Omega \rightarrow \mathbb{R}$ and M smooth functions $v_1, \cdots, v_M : \Omega \rightarrow \mathbb{R}$ such that

$$a(x) = \sum_{i=1}^{m} u_i(x) Y_i(x) + \sum_{j=1}^{M} v_j(x) Y_{\pi_j}(x).$$

In addition, by construction of the function a we can assume that for every $x \in \Omega \setminus \hat{\mathcal{B}}_\rho$,

$$u_i(x) = (k_{\mathcal{S}}(x))_i, \quad \forall i = 1, \cdots, m,$$
$$\text{and} \quad v_j(x) = 0 \quad \text{for all} \quad j = 1, \cdots, M.$$

Now, by adapting the proof of Morin, Pomet and Samson to the case of a homogeneous of degree $k-2$ with respect to δ_ϵ^r, we can construct highly oscillatory functions of time $u^\epsilon : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^m$ which are homogeneous of degree 2 with respect to δ_ϵ^r, smooth outside the origin and such that for any $\tau > 0$ and any $x_0 \in \Omega$, the solutions $x^\epsilon : [0, \tau] \to \mathbb{R}^3$ of

$$\dot{x} = \sum_{i=1}^{m} u_i^\epsilon(t, x) X_i(x), \quad t \in [0, \tau], \text{ and } x(0) = x_0,$$

converge uniformly to the solution $x^\infty : [0, \tau] \to \mathbb{R}^3$ of

$$\dot{x} = a(x), \quad t \in [0, \tau], \text{ and } x(0) = x_0.$$

Moreover, we claim that we can do the construction of the $(u^\epsilon)_{\epsilon>0}$ can be made in such a way that

$$\forall \epsilon > 0, \forall t \in \mathbb{R}, \forall x \in \Omega \setminus \hat{\mathcal{B}}_\rho, \quad u^\epsilon(t, x) = k_{\mathcal{S}}(x).$$

Now, using the fact that a smooth regularization of the function V gives a smooth Lyapunov function for the system $\dot{x} = a(x)$, the homogeneity of the datas and (89), we conclude that for $\epsilon > 0$ sufficiently small the closed-loop system

$$\dot{x} = \sum_{i=1}^{m} u_i^\epsilon(t, x) X_i(x),$$

is globally asymptotically stable at the origin and that all its trajectories satisfy property (v). □

Using the classical technique of homogeneization of control systems that we recall in Section 4.1 and a Lyapunov converse theorem for homogeneous time-varying vector fields, Theorem 10 leads naturally to the following result:

Corollary 5.1. *If $n = 3$ and if the system (1) satisfies the Hörmander's condition (2), then there exist a neighbourhood of the origin $\mathcal{W}$, a dilation δ_ϵ^r and a closed set $\mathcal{S} \subset \mathbb{R}^n$ which is homogeneous with respect to δ_ϵ^r such that for any $T > 0$ and for any neighbourhood $\mathcal{V}$ of $\mathcal{S}$ which is homogeneous with respect to δ_ϵ^r, there exists a time-varying feedback $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ which is smooth, homogeneous with respect to δ_ϵ^r, and which satisfies the following properties:*

(i) *For any $t \in \mathbb{R}, x \in \mathbb{R}^n, u(t + T, x) = u(t, x)$.*
(ii) *For any $t \in \mathbb{R}, u(t, 0) = 0$.*
(iii) *For any $x \in \mathcal{V}$, the function $t \in \mathbb{R} \mapsto u(t, x)$ is constant.*
(iv) *The closed-loop system*

$$\dot{x} = \sum_{i=1}^{m} u_i(t, x) X_i(x) \tag{90}$$

is globally asymptotically stable at the origin.

(v) *For every trajectory* $x(\cdot) : [0,\infty) \to: R^3$ *of (90) such that* $x(0) \notin \mathcal{V}$ *the following property is satisfied:*

$$x(t) \notin \mathcal{V}, \quad \forall t \geq 0.$$

Appendix

Here we present the example of a control system in dimension three, which is globally asymptotically controllable at the origin and which does not admit a local smooth repulsive stabilizing feedback. The example that we present is indeed analytic and homogeneous with respect to the standard dilation of degree -1.

In the sequel, we denote by $(x_1, x_2, x_3)^*$ a vector $x \in \mathbb{R}^3$. Define two vector fields X_1, X_2 on $\mathbb{R}^3$ by

$$X_1(x) := \begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 \\ 0 \end{pmatrix} \quad \text{and} \quad X_2(x) := \begin{pmatrix} 0 \\ 0 \\ x_3^2 \end{pmatrix},$$

for any $x \in \mathbb{R}^3$. We leave the reader to verify that both these two vector fields are analytic and homogeneous with respect to the standard dilation of degree -1. Furthermore, we notice that in the control system

$$\dot{x} = u_1 X_1(x) + u_2 X_2(x), \tag{91}$$

the two controls u_1 and u_2 act independently on the coordinates (x_1, x_2) and x_3. Besides it is not difficult to see that the control system on the real line given by

$$\dot{z} = uz^2, \quad u \in \mathbb{R},$$

and that the control system in the plane defined by

$$\begin{aligned} \dot{x} &= u'(x^2 - y^2) \\ \dot{y} &= u'(2xy), \quad u' \in \mathbb{R}, \end{aligned}$$

are globally asymptotically controllable at the origin. As a matter of fact, we notice easily (as shown in Figure 8) that for every $(x, y) \neq (0, 0)$ in the plane, the set

$$\left\{ u'(x^2 - y^2, 2xy) : u' \in \mathbb{R} \right\}$$

is the tangent space to the circle passing through (x, y) and $(0, 0)$ with center on the y-axis. We conclude easily that the control system (91) is globally asymptotically controllable at the origin.

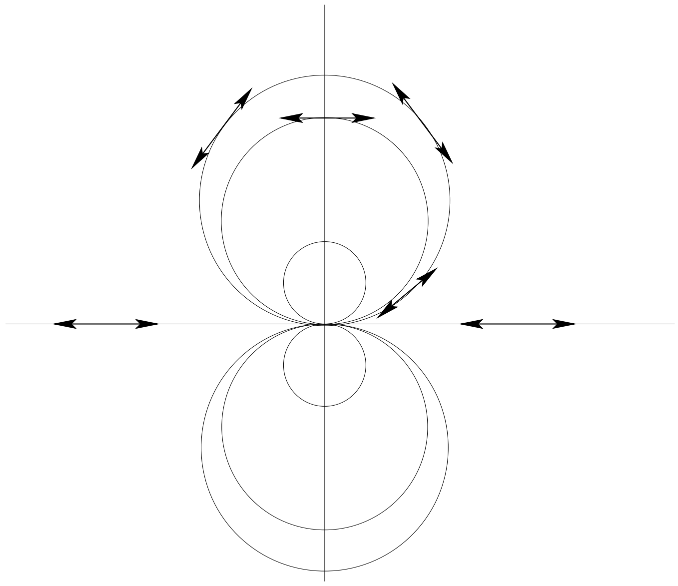

Figure 8.

Let us now prove that control system (91) does not admit a local smooth repulsive stabilizing feedback at the origin. For that we argue by contradiction.

So let us assume that such a feedback exists; this means that there is some neighbourhood of the origin $\mathcal{W}$, some set $\mathcal{S} \subset \mathcal{W}$ which contains the origin, and some feedback $k_{\mathcal{S}} : \mathcal{W} \to \mathbb{R}^2$ such that the properties (i)-(iv) of Definition 1.1 are satisfied. We first notice that the vector line span$\{(0,0,1)^*\}$ is invariant under the control system (91); this implies that this vector line cannot intersect the singular set $\mathcal{S}$. Hence if we fix some point $(0,0,\bar{x}_3)^* \in \mathcal{W}$ such that $\bar{x}_3 > 0$, since the set $\mathcal{S}$ is closed, there exists some ball $\mathcal{B}$ centered at the point $(0,0,\bar{x}_3)^*$ which is included in $\mathbb{R}^3 \setminus \mathcal{S}$. Moreover there exists necessarily some nontrivial circle $\mathcal{C}$ in the plane $\{x_3 = 0\}$ which is centered on the x_2-axis, which passes through the origin (see Figure 8) and such that the set defined by

$$\mathcal{C}_{x_3} := \left\{(x_1,x_2,x_3)^* \in \mathbb{R}^3 \text{ s.t. } (x_1,x_2) \in \mathcal{C} \text{ and } x_3 = \bar{x}_3\right\},$$

is contained in the ball $\mathcal{B}$. Let us parametrize this circle by some smooth function

$$\gamma : [0,2\pi] \longrightarrow \gamma(\theta) \in \mathcal{C}_{x_3}.$$

If we denote for every $\theta \in [0,2\pi]$ by $\gamma(\theta,\cdot)$ the unique solution of the closed-loop system

$$\dot{x} = \left(k_{\mathcal{S}}(x)\right)_1 X_1(x) + \left(k_{\mathcal{S}}(x)\right)_2 X_2(x), \tag{92}$$

such that $x(0) = \gamma(\theta)$, then we deduce by property (iv) of Definition 1.1 that for any $t \geq 0$, the point $\gamma(\theta,t)$ does not belong to $\mathcal{S}$. In addition , property (iii) implies that the set $\{\gamma(\theta,t) : \theta \in [0,2\pi]\}$ tends uniformly to the singleton $\{0\}$ as t tends to infinity. Since the circle $\mathcal{C}$ is invariant under the closed-loop system (92), this proves that the smooth mapping

$$\begin{array}{rcl} \tilde{\gamma}:[0,2\pi]\times[0,\infty) & \longrightarrow & \mathcal{C} \\ (\theta,t) & \longmapsto & (\gamma_1(\theta,t),\gamma_2(\theta,t)), \end{array}$$

satisfies

$$\tilde{\gamma}(\cdot,0)=\gamma \quad \text{and} \quad \lim_{t\to\infty}\tilde{\gamma}(\cdot,t)=0.$$

Since the circle $\mathcal{C}$ is not contractible (see for instance [7]), we obtain a contradiction.

References

[1] F. Ancona and A. Bressan. Patchy vector fields and asymptotic stabilization. *ESAIM Control Optim. Calc. Var.*, 4:445–471, 1999.

[2] F. Ancona and A. Bressan. Flow stability of patchy vector fields and robust feedback stabilization. *SIAM J. Control Optim.*, 41(5):1455-1476, 2002.

[3] J.-P. Aubin and A. Cellina. *Differential Inclusions.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 264. Springer-Verlag, Berlin, 1984.

[4] Z. Artstein. Stabilization with relaxed controls. *Nonlinear Analysis TMA*, 7:1163-1173, 1983.

[5] M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.* Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 1997.

[6] A. Bellaiche. The tangent space in sub-Riemannian geometry. in *Sub-Riemannian Geometry, Birkhäuser*, 1–78, 1996.

[7] G.E. Bredon. Topology and Geometry. Graduate Texts in Mathematics, vol. 139. Springer-Verlag, New York, 1993.

[8] A. Bressan. Local asymptotic approximation of nonlinear control systems. *Internat. J. Control*, 41(5):1331–1336, 1985.

[9] A. Bressan. Singularities of stabilizing feedbacks. *Rend. Sem. Mat. Univ. Politec. Torino*, 56(4):87–104 (2001), 1998. Control theory and its applications (Grado, 1998).

[10] R.W. Brockett. Asymptotic stability and feedback stabilization. In R.W. Brockett, R.S. Millman, and H.J. Sussmann, editors, *Differential Geometric Control Theory*, pages 181-191. Birkhäuser, Boston, 1983.

[11] P. Cannarsa and C. Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control.* Progress in Nonlinear Differential Equations and their Applications, 58. Birkhäuser Boston Inc., Boston, MA, 2004.

[12] C.-L. Chow. Über systeme von linearen partiellen differentialgleichungen ester ordnung. *Math. Ann.*, 117: 98-105, 1939.

[13] F.H. Clarke. *Optimization and Nonsmooth Analysis.* Wiley-Interscience, New York, 1983. Republished as vol.5 of *Classics in Applied Mathematics*, SIAM, 1990.

[14] F.H. Clarke, Yu.S. Ledyaev, L. Rifford, and R.J. Stern. Feedback stabilization and Lyapunov functions. *SIAM J. Control Optim.*, 39(1):25-48, 2000.

[15] F.H. Clarke, Yu.S. Ledyaev, E.D. Sontag, and A.I. Subbotin. Asymptotic controllability implies feedback stabilization. *I.E.E.E. Trans. Aut. Control*, 42:1394-1407, 1997.

[16] F.H. Clarke, Yu.S. Ledyaev, R.J. Stern, and P.R. Wolenski. *Nonsmooth Analysis and Control Theory.* Graduate Texts in Mathematics, vol. 178. Springer-Verlag, New York, 1998.

[17] J.-M. Coron. A necessary condition for feedback stabilization. *Systems and Control Lett.*, 14:227-232, 1990.

[18] J.-M. Coron. Global Asymptotic stabilization of controllable systems without drift. *Math. Cont. Sign. Sys.*, 5:295-312, 1992.

[19] J.-M. Coron. Linearized control systems and application to smooth stabilization. *SIAM J. Control Optim.*, 32(2):358-386, 1994.

[20] J.-M. Coron. On the stabilization of some nonlinear control systems: results, tools, and applications. In *Nonlinear analysis, differential equations and control (Montreal, QC, 1998)*, pages 307-367. Kluwer Acad. Publ., Dordrecht, 1999.

[21] K. Deimling. *Multivalued differential equations*. de Gruyter Series in Nonlinear Analysis and Applications, 1. Walter de Gruyter & Co., Berlin, 1992.

[22] L. Grüne. Homogeneous state feedback stabilization of homogeneous systems. *SIAM J. Control Optim.*, 38(4):1288-1308, 2000.

[23] H. Hermes. Nilpotent and high-order approximations of vector field systems. *SIAM Rev.*, 33(2):238-264, 1991.

[24] H. Hermes. Homogeneous coordinates and continuous asymptotically stabilizing feedback controls. In "Differential Equations, Stability and Control" (S. Elaydi, ed.), *Lecture Notes in Pure and Applied Math.*, 127, Marcel Dekker Inc., 249-260, 1991.

[25] H. Hermes. Asymptotically stabilizing feedback controls. *J. Differential Equations*, 92, 76-89, 1991.

[26] H. Hironaka. Introduction aux ensembles sous-analytiques. *Astérisque (Soc. Math. de France)*, 7-8, 13-20, 1973.

[27] H. Hironaka. Subanalytic sets. In *Number theory , algebraic geometry and commutative algebra, in honor of Y. Akizuki.* Kinokuniya, Tokyo, 1973.

[28] M. Hirsch and S. Smale. *Differential equations, dynamical systems, and linear algebra.* Pure and Applied Mathematics, Vol. 60, Academic Press, New York-London, 1974.

[29] S. Lojasiewicz. *Ensembles semi-analytiques.* Lecture note at IHES, Bures-sur-Yvette ; Reproduit NA66.765, École polytechnique, Paris, 1965.

[30] E. Michael. Continuous selections. I. *Ann. of Math. (2)*, 63:361-382, 1956.

[31] P. Morin, J-B. Pomet and C. Samson. Design of homogeneous time-varying stabilizing control laws for driftless controllable systems via oscillatory approximation of Lie brackets in closed loop. *SIAM J. Control Optim.*, 38(1):22-49, 1999.

[32] C. Prieur. Asymptotic controllability and robust asymptotic stabilizability. *SIAM J. Control Optim.*, 43(5):1888-1912, 2005.

[33] P.K. Rashevsky. About connecting two points of a completely nonholonomic space by admissible curve. *Uch. Zapiski Ped. Inst. Libknechta*, 2:83-94, 1938.

[34] L. Rifford. *Problèmes de stabilisation en théorie du controle.* PhD thesis, Université Claude Bernard Lyon I, 2000.

[35] L. Rifford. Stabilisation des systèmes globalement asymptotiquement commandables. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(3):211-216, 2000.

[36] L. Rifford. Existence of Lipschitz and semiconcave control-Lyapunov functions. *SIAM J. Control Optim.*, 39(4):1043-1064, 2000.

[37] L. Rifford. On the existence of nonsmooth control-Lyapunov functions in the sense of generalized gradients. *ESAIM Control Optim. Calc. Var.*, (6), 593–611, 2001.

[38] L. Rifford. Semiconcave control-Lyapunov functions and stabilizing feedbacks. *SIAM J. Control Optim.*, 41(3):659-681, 2002.

[39] L. Rifford. Singularities of viscosity solutions and the stabilization problem in the plane. *Indiana Univ. Math. J.*, 52(5):1373-1396, 2003.

[40] L. Rifford. The Stabilization Problem: AGAS and SRS Feedbacks. In *Optimal Control, Stabilization, and Nonsmooth Analysis*, Lectures Notes in Control and Information Sciences Vol. 301, Springer-Verlag, Heidelberg, pp. 173-184, 2004.

[41] L. Rifford. On the stabilization problem on surfaces. *Rend. Semin. Mat. Torino*, to appear.

[42] L. Rifford. Stratified semiconcave control-Lyapunov functions and the stabilization problem. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, to appear.

[43] L. Rifford and E. Trélat. On the stabilization problem in sub-Riemannian geometry. Work in progress.

[44] L. Rosier. Homogeneous Lyapunov function for homogeneous continuous vector field. *Systems & Control letters*, 19(6):467-473, 1992.

[45] L.P. Rothschild and E.M. Stein. Hypoelliptic differential operators and nilpotent groups. *Acta Math.*, 137:247-320, 1976.

[46] M. Shiota. *Geometry of subanalytic and semialgebraic sets.* Progress in Mathematics 150, Birkhaüser, Boston 1997.

[47] E.D. Sontag. A Lyapunov-like characterization of asymptotic controllability. *SIAM J. Control Optim.*, 21:462-471, 1983.

[48] E.D. Sontag. Stability and stabilization: discontinuities and the effect of disturbances. In *Nonlinear analysis, differential equations and control (Montreal, QC, 1998)*, pages 307–367. Kluwer Acad. Publ., Dordrecht, 1999.

[49] H. J. Sussmann. Subanalytic sets and feedback control. *J. Differential Equations*, 31(1):31-52, 1979.