REFINEMENT OF THE BENOIST THEOREM ON THE SIZE OF DINI SUBDIFFERENTIALS

Ludovic Rifford
Université de Nice-Sophia Antipolis, Laboratoire J.A. Dieudonné, Parc Valrose, 06108 Nice Cedex 02, France

(Communicated by Yacine Chitour)

Abstract. Given a lower semicontinuous function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \), we prove that the set of points of \( \mathbb{R}^n \) where the lower Dini subdifferential has convex dimension \( k \) is countably \((n-k)\)-rectifiable. In this way, we extend a theorem of Benoist (see [1, Theorem 3.3]), and as a corollary we obtain a classical result concerning the singular set of locally semiconcave functions.

1. Introduction. Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be any lower semicontinuous function, the lower Dini subdifferential of \( f \) at \( x \) in the domain of \( f \) (denoted by \( \text{dom}(f) \)) is defined by

\[
\partial^- f(x) = \left\{ \zeta \in \mathbb{R}^n \mid \liminf_{y \to x} \frac{f(y) - f(x) - \langle \zeta, y - x \rangle}{\|y - x\|} \geq 0 \right\}.
\]

As it is well-known, for every \( x \in \text{dom}(f) \), the set \( \partial^- f(x) \) is a possibly empty convex subset of \( \mathbb{R}^n \). Now let \( k \in \{1, \cdots, n\} \) be fixed; we call \( k \)-dimensional Dini singular set of \( f \), denoted by \( D^k(f) \), the set of \( x \in \text{dom}(f) \) such that \( \partial^- f(x) \) is a nonempty convex set of dimension \( k \). Moreover, we call Dini singular set of \( f \), the set defined by

\[
D(f) := \bigcup_{k \in \{1, \cdots, n\}} D^k(f).
\]

Before stating our result, we recall that, given \( r \in \{0, 1, \cdots, n\} \), the set \( C \subset \mathbb{R}^n \) is called a \( r \)-rectifiable set if there exists a Lipschitz continuous function \( \phi : \mathbb{R}^r \to \mathbb{R}^n \) such that \( C \subset \phi(\mathbb{R}^r) \). In addition, \( C \) is called countably \( r \)-rectifiable if it is the union of a countable family of \( r \)-rectifiable sets. The aim of the present short note is to extend a result by Benoist, who proved that \( D(f) \) is countably \((n-1)\)-rectifiable (see [1, Theorem 3.3]), and to obtain as a corollary a classical result on locally semiconcave functions. We prove the following result.

Theorem 1.1. Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a lower semicontinuous function. Then for every \( k \in \{1, \cdots, n\} \), the set \( D^k(f) \) is countably \((n-k)\)-rectifiable.

Let us now recall briefly the notions of semiconcave and locally semiconcave functions; we refer the reader to the book [2] for further details on semiconcavity (see also [4]). Let \( \Omega \) be an open and convex subset of \( \mathbb{R}^n \), \( u : \Omega \to \mathbb{R} \) be a continuous...
function, and $C$ be a nonnegative constant. We say that $u$ is $C$-semiconcave or semiconcave on $\Omega$ if
\[ \mu u(y) + (1 - \mu)u(x) - u(\mu x + (1 - \mu)y) \leq \frac{\mu(1 - \mu)C}{2} |x - y|^2, \]
for any $\mu \in [0, 1]$, and any $x, y \in \mathbb{R}^n$. Consider now an open subset $\Omega$ of $\mathbb{R}^n$; the function $u : \Omega \to \mathbb{R}$ is called locally semiconcave on $\Omega$, if for every $x \in \Omega$, there is an open and convex neighborhood of $x$ where $u$ is semiconcave. For every $k \in \{1, \cdots, n\}$, we call $k$-dimensional singular set of $u$, denoted by $\Sigma^k(u)$, the set of $x \in \Omega$ such that the Clarke generalized gradient of $u$ at $x$, denoted by $\partial u(x)$, is a convex set of dimension $k$ (see [2, 3]). In fact, it is easy to deduce from (1), that for any locally semiconcave function $u : \Omega \to \mathbb{R}$ on an open subset $\Omega$ of $\mathbb{R}^n$, the sets $\partial u(x)$ and $(-\partial^* u(x))$ coincide at any $x \in \Omega$ (see [2, Theorem 3.3.6 p. 59]). This implies that $\Sigma^k(u) = D^k(-u)$ for every $k \in \{1, \cdots, n\}$ and yields the following result.

**Corollary 1.** Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a locally semiconcave function. Then for every $k \in \{1, \cdots, n\}$, the set $\Sigma^k(u)$ is countably $(n - k)$-rectifiable.

Our proofs combine techniques developed by Benoist in [1] and Cannarsa, Sinestrari in [2].

Notations: Throughout this paper, we denote by $\langle \cdot, \cdot \rangle$ and $| \cdot |$, respectively, the Euclidean scalar product and norm in $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and any $r > 0$, we set $B(x, r) := \{ y \in \mathbb{R}^n \mid |y - x| < r \}$ and $\bar{B}(x, r) := \{ y \in \mathbb{R}^n \mid |y - x| \leq r \}$. Finally, we use the abbreviations $B_r := B(0, r)$, $\bar{B}_r := \bar{B}(0, r)$, $B := B_1$, and $\bar{B} := \bar{B}_1$.

2. Preliminary results. Let $k \in \{1, \cdots, n - 1\}$, we call $k$-planes the $k$-dimensional subspaces of $\mathbb{R}^n$. Given a $k$-plane $\Pi$, we denote by $\Pi^\perp$ its orthogonal complement in $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$, we denote by $p_\Pi(x)$ and $p_{\Pi^\perp}(x)$ the orthogonal projections of $x$ onto $\Pi$ and $\Pi^\perp$ respectively. If $\Pi, \Pi'$ are two given $k$-planes, we set
\[ d(\Pi, \Pi') := \| p_\Pi - p_{\Pi'} \|, \]
where $\| \cdot \|$ is the operator norm of a linear operator in $\mathbb{R}^n$. We notice that the set of $k$-planes, denoted by $\mathcal{P}^k$, equipped with the distance $d$, is a compact metric space. Hence it admits a dense countable family $\{\Pi^\perp_\nu\}_{\nu \geq 1}$. In the sequel, we denote by $B^1(\Pi, \epsilon)$ the set of $\Pi' \in \mathcal{P}^k$ such that $d(\Pi, \Pi') \leq \epsilon$.

Given a compact set $K \subset \mathbb{R}^n$, we recall that the support function $\sigma_K$ of $K$ is defined by
\[ \forall h \in \mathbb{R}^n, \quad \sigma_K(h) := \max \{ \langle w, h \rangle \mid w \in K \}. \]
We notice that if $\text{conv}(K)$ denotes the convex hull of $K$, then we have
\[ \sigma_{\text{conv}(K)} = \sigma_K. \]
Moreover if $K, K'$ are two compact sets such that $K \subset K'$, then $\sigma_K \leq \sigma_{K'}$.

Given a $k$-plane $\Pi$, we define the function $\bar{\sigma}_\Pi : \mathbb{R}^n \to \mathbb{R}$ by
\[ \forall h \in \mathbb{R}^n, \quad \bar{\sigma}_\Pi(h) := \max \{ \langle w, h \rangle \mid w \in \Pi \cap \bar{B} \}. \]
The following result is useful for the proof of our theorem.

**Lemma 2.1.** Let $\Pi, \Pi'$ be two $k$-planes and $h \in \mathbb{R}^n$, then we have
\[ |\bar{\sigma}_{\Pi}(h) - \bar{\sigma}_{\Pi'}(h)| \leq d(\Pi, \Pi')|h|. \]
Proof. There is \( w \in \Pi \cap \mathcal{B} \) such that \( \bar{\sigma}_\Pi(h) = \langle w, h \rangle \). Set
\[
d := |p_\Pi(w)|.
\]
Notice, that since \( w \in \mathcal{B} \), we have necessarily \( d \leq 1 \), which means that \( p_\Pi(w) \) belongs to \( \Pi' \cap \mathcal{B} \). Hence we have
\[
\bar{\sigma}_\Pi(h) \geq \langle p_\Pi(w), h \rangle = \langle p_\Pi(w) - p_\Pi(w), h \rangle + \langle w, h \rangle \\
\geq -\|p_\Pi(w) - p_\Pi(w)\| |h| + \bar{\sigma}_\Pi(h) \\
\geq -\|p_\Pi - p_\Pi\| |w||h| + \bar{\sigma}_\Pi(h) \\
\geq -d(\Pi, \Pi')|h| + \bar{\sigma}_\Pi(h).
\]
We deduce that \( \bar{\sigma}_\Pi(h) - \bar{\sigma}_\Pi(h) \geq -d(\Pi, \Pi')\|h\| \). By symmetry, we obtain the inequality (2). \( \square \)

3. Proof of the theorem. Let \( k \in \{1, \ldots, n\} \) be fixed. Let us choose a sequence \( (v_j)_{j \geq 1} \) which is dense in \( \mathbb{R}^k \) and let us define, for \( \omega = (r, i, j, l) \in I := (\mathbb{N}^*)^4 \), the set \( D_\omega \) constituted of elements \( x \) belonging to the closed ball \( \mathcal{B}_r \) such that \( f(x) \leq r \), and such that there exist \( \Pi \in B^k_r(\Pi, \frac{1}{2r}) \), \( \rho \geq \frac{1}{4} \) and \( \zeta \in \mathcal{B}\left( v_j, \frac{1}{2r} \right) \) satisfying:
\[
\forall y \in B\left( x, \frac{1}{7} \right), \quad f(y) \geq f(x) + \langle \zeta, y-x \rangle + \rho\bar{\sigma}_\Pi(y-x) - \frac{1}{2r}|y-x|.
\]
Lemma 3.1. We have the following inclusion
\[
\mathcal{D}^k(f) \subset \bigcup_{\omega \in I} \mathcal{D}_\omega.
\]
Proof. Denote by \( e_1^k, \ldots, e_k^k \) the standard basis in \( \mathbb{R}^k \) and choose a constant \( \nu^k > 0 \) such that
\[
\mathcal{B}^k_{\nu^k} \subset \text{conv}(\pm e_1^k, \ldots, \pm e_k^k),
\]
where \( \mathcal{B}^k_{\nu^k} \) denotes the closed ball centered at the origin with radius \( \nu^k \) in \( \mathbb{R}^k \). Let \( x \in \mathcal{D}^k(f) \); there are \( \zeta \in \mathbb{R}^n \) and \( \mu > 0 \) such that the convex set \( \partial^- f(x) \) contains the \( k \)-ball \( \mathcal{B}_r^k \) defined as,
\[
\mathcal{B}_r^k := \mathcal{B}(\zeta, \mu) \cap H,
\]
where \( H \) denotes the affine subspace of dimension \( k \) which is spanned by \( \partial^- f(x) \) in \( \mathbb{R}^n \). Choose \( r \geq 1 \) such that \( |x| \leq r, f(x) \leq r \), and \( \mu \geq \frac{1}{2r} \). By (4), there are \( k \) vectors \( e_1, \ldots, e_k \in \mathbb{R}^n \) of norm 1 such that
\[
\mathcal{B}_{\nu^k} \cap \mathcal{F} \subset \mu E \subset \mathcal{B}_r^k,
\]
where \( \Pi \) and \( E \) are defined by
\[
\Pi := \text{SPAN}\{e_1, \ldots, e_k\} \quad \text{and} \quad E := \text{conv}(\pm e_1, \ldots, \pm e_k).
\]
For every \( m \in \{1, \ldots, k\} \) and every \( \epsilon = \pm 1 \), the vector \( \zeta + \mu e_m \) belongs to \( B_r^k \), then there exists a neighborhood \( \mathcal{V}_{m, \epsilon} \) of \( x \) such that
\[
\forall y \in \mathcal{V}_{m, \epsilon}, \quad f(y) \geq f(x) + \langle \zeta + \mu e_m, y-x \rangle - \frac{1}{2r}|y-x|.
\]
Hence we deduce that for every \( y \in \bigcap_{m \in \{1, \ldots, k\}, \epsilon = \pm 1} \mathcal{V}_m \), we have
\[
f(y) \geq f(x) + \langle \zeta, y-x \rangle \\
+ \max \{\mu(e_m, y-x) \mid m = 1, \ldots, k, \epsilon = \pm 1\} - \frac{1}{2r}|y-x|.
\]
But by (5), we have for every $h \in \mathbb{R}^n$,
\[
\max \{ \mu(e_m, h) \mid m = 1, \ldots, k, \epsilon = \pm 1 \} = \sigma_{\mu E}(h) \geq \sigma_{(B_{\nu^k, \mu} \cap \Pi)}(h) = \nu^k \mu \sigma_{\Pi}(h).
\]
We conclude easily by density of the families $\{\Pi_i^k\}_{i \geq 1}$, $\{v_j\}_{j \geq 1}$.

Set for every $i \geq 1$, the cone
\[
K_i := \left\{ h \in \mathbb{R}^n \mid \tilde{\sigma}_{\Pi_i}(h) \leq \frac{1}{2} \|h\| \right\}.
\]

We have the following lemma.

**Lemma 3.2.** For every $\omega = (r, i, j, l) \in I$ and every $x \in D_\omega$, we have
\[
D_\omega \cap \bar{B} \left( x, \frac{1}{7} \right) \subset \{ x \} + K_i.
\]

**Proof.** Let $y \in D_\omega \cap \bar{B} \left( x, \frac{1}{7} \right)$ be fixed. There are $\Pi_y \in B^k_{\sigma} \left( \Pi_i, \frac{1}{7} \right)$, $\rho_y \geq \frac{2}{r}$ and $\zeta \in \bar{B} \left( v_j, \frac{1}{2r} \right)$ such that
\[
\forall z \in \bar{B} \left( y, \frac{1}{7} \right), \quad f(z) \geq f(y) + \langle \zeta_y, z - y \rangle + \rho_y \tilde{\sigma}_{\Pi_y}(z - y) - \frac{1}{2r} |z - y|.
\]
In particular, for $z = x$, this implies
\[
f(x) \geq f(y) + \langle \zeta_y, x - y \rangle + \rho_y \tilde{\sigma}_{\Pi_y}(x - y) - \frac{1}{2r} |y - x|
\]
\[
\geq f(y) + \langle \zeta_y, x - y \rangle - \frac{1}{2r} |y - x|. \quad (6)
\]
But since $x \in D_\omega$, there are $\Pi_x \in B^k_{\sigma} \left( \Pi_i, \frac{1}{7} \right)$, $\rho_x \geq \frac{2}{r}$ and $\zeta_x \in \bar{B} \left( v_j, \frac{1}{2r} \right)$ such that
\[
f(y) \geq f(x) + \langle \zeta_x, y - x \rangle + \rho_x \tilde{\sigma}_{\Pi_x}(y - x) - \frac{1}{2r} |y - x|. \quad (7)
\]
Summing the inequalities (6) and (7), we obtain
\[
0 \geq \langle \zeta_x - \zeta_y, y - x \rangle + \rho_x \tilde{\sigma}_{\Pi_x}(y - x) - \frac{1}{r} |y - x|.
\]
But $|\zeta_x - \zeta_y| \leq \frac{1}{7}$, hence
\[
\rho_x \tilde{\sigma}_{\Pi_x}(y - x) \leq \frac{2}{r} |y - x|.
\]
Which gives by (2)
\[
\tilde{\sigma}_{\Pi_x}(y - x) = (\tilde{\sigma}_{\Pi_x}(y - x) - \tilde{\sigma}_{\Pi_x}(y - x)) + \tilde{\sigma}_{\Pi_x}(y - x)
\]
\[
\leq d(\Pi_i, \Pi_x) |y - x| + \frac{2}{\rho_x^k} |y - x|
\]
\[
\leq \frac{1}{4r} |y - x| + \frac{1}{4} |y - x|
\]
\[
\leq \frac{1}{2} |y - x|.
\]

$\square$
Lemma 3.3. Let \( \omega = (r, i, j, l) \in \mathcal{I} \) and \( \bar{x} \in D_{\omega} \) be fixed; set
\[
A := p_{\Pi_i} \left( D_{\omega} \cap B \left( \bar{x}, \frac{1}{2l} \right) \right).
\]
For every \( y \in A \), there exists a unique \( z = z_y \in \Pi_i \) such that
\[
y + z \in D_{\omega} \cap B \left( \bar{x}, \frac{1}{2l} \right).
\]
Moreover, the mapping \( \psi_y : y \in A \mapsto z_y \) is Lipschitz continuous.

Proof. First of all, for every \( y \in A \), there is, by definition of \( A \), some \( x \in D_{\omega} \cap B \left( \bar{x}, \frac{1}{2l} \right) \) such that \( y = p_{\Pi_i} \left( x \right) \). Since \( x - y \in \Pi_i \), this proves the existence of \( z_y \). To prove the uniqueness, we argue by contradiction. Let \( y \in A \), assume that there are \( z \neq z' \in \Pi_i \) such that \( y + z \) and \( y + z' \) belong to \( D_{\omega} \cap B \left( \bar{x}, \frac{1}{2l} \right) \). Since \( y + z \) and \( y + z' \) are balls of radius \( \frac{1}{2l} \), we have that \( \bar{\sigma}_{\Pi_i}(z') - z \) \( \rightarrow \) \( \bar{\sigma}_{\Pi_i}(z') - z \). Which means that \( \bar{\sigma}_{\Pi_i}(z') - z \) \( \rightarrow \) \( \bar{\sigma}_{\Pi_i}(z') - z \). We find a contradiction. Let us now prove that the map \( \psi_y \) is Lipschitz continuous. Let \( y, y' \in A \) be fixed. By the proof above we know that \( \psi_y(y) = x - y \) (resp. \( \psi_y(y') = x' - y' \)) where \( x \) is such that \( y = p_{\Pi_i} \left( x \right) \) (resp. \( y = p_{\Pi_i} \left( x \right) \)). Set \( z \) := \( \psi_y(y), z' := \psi_y(y') \) and \( h := x' - x \). Since \( x = y + z \) and \( x' = y' + z' \) where \( y, y' \in \Pi_i \) and \( z, z' \in \Pi_i \), we have \( |h|^2 = |z' - z|^2 + |y' - y|^2 \). But \( \bar{\sigma}_{\Pi_i}(h) = |z' - z| \leq \frac{1}{2}|h| \). Hence we obtain that
\[
|z' - z| \leq |x' - x| = |h| \leq \frac{2}{\sqrt{3}}|y' - y|.
\]
The proof of the lemma is completed. \( \square \)

From the lemma above, for every \( \omega = (r, i, j, l) \in \mathcal{I} \) and every \( \bar{x} \in D_{\omega} \), the map \( \phi : A \rightarrow \mathbb{R}^n \) defined as,
\[
\forall y \in A, \quad \phi(y) = y + \psi_y(y),
\]
is Lipschitz continuous and satisfies
\[
D_{\omega} \cap B \left( \bar{x}, \frac{1}{2l} \right) \subset \phi(A).
\]
Since \( A \subset \Pi_i \), such a map can be extended into a Lipschitz continuous map from \( \Pi_i \) into \( \mathbb{R}^n \). Since \( \Pi_i \) has dimension \( (n - k) \), we deduce that the set \( D_{\omega} \cap B \left( \bar{x}, \frac{1}{2l} \right) \) is \( (n - k) \)-rectifiable. The fact that any set \( D_{\omega} \) can be covered by a finite number of balls of radius \( \frac{1}{2l} \) completes the proof of the theorem.

Acknowledgements. The author is very indebted to Joël Benoist for fruitful discussions.
REFERENCES


Received September 2006; revised May 2007.

E-mail address: rifford@math.unice.fr