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STABILIZATION OF THE LOGISTIC MODEL UNDER YIELD CONTROL CONSTRAINED TO INITIAL HARVEST

A. Rapaport*, L. Rifford†

INRA, Laboratoire de Biométrie, Montpellier, France.
e-mail : rapaport@montpellier.inra.fr
†Ecole Normale Supérieure de Lyon, France.
e-mail : Ludovic.Rifford@ens-lyon.fr

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Abstract

We adapt the backstepping method to provide a smooth suboptimal stabilization of an harvested stock, when the true optimal solution is bang-bang and the second variation method leads to a destabilizing controller.

1 Introduction

We consider the harvesting problem under *yield control* :

$$\dot{x} = f(x) - h\,, \quad x(0) = x_0 \tag{1}$$

The state variable $x \in [0, K]$ represents the biomass stock and the control variable $h \in [0, H]$ the harvest. In this study, the growth function f is the *logistic law* :

$$f(x) = rx\left(1 - \frac{x}{K}\right), \tag{2}$$

a model used quite often for biological modelling or population dynamics. The coefficients r and K are respectively the *intrinsic growth rate* and the *carrying capacity.*

The problem of optimal exploitation, maximizing an economical criterion :

$$J_{x_0}(h) = \int_0^{+\infty} e^{-\delta t}\left(p - \frac{c}{x(t)}\right) h(t)dt \tag{3}$$

has been intensively studied. Here δ is the *discount factor*, p the *unit price* and c the *harvesting cost.* See for instance [4] for the optimal "bang-bang" solution and [5] for a neighbouring linear-quadratic approximation. Nevertheless, the optimal harvest policy $h^*()$ always imposes a certain value $h^*(x_0)$ on the initial harvest, that may require an unrealistic jump from the current harvest. To deal with this drawback, we propose studying the optimal control of the derivative of the harvest instead of the harvest itself. This is equivalent to saying that we add an "integrator" to the control system (1) :

$$\left\{\begin{array}{rcll} \dot{x} & = & f(x) - h\,, & x(0) = x_0 \\ \dot{h} & = & u\,, & h(0) = h_0 \end{array}\right. \tag{4}$$

$u \in [-U, U]$ is now our new control.

We first show that the optimal trajectories for the modified problem $(x^*(t), h^*(t))$ has the following characteristics : over an initial period of time, a bang bang control drives $(x^*(t), h^*(t))$ to an equilibrium steady state (x_e, h_e). Thereafter $(x^*(t), h^*(t)$ is held constant at (x_e, h_e).

A bang bang solution is not desirable in practice, due to its lack of smoothness. For this reason we study suboptimal solutions based on linear quadratic approximations about the equilibrium and penalization of large deviations of u. We show that for certain values of the parameters, the linear approximation of the optimal strategy destabilizes the system.

Finally we propose a family of global stabilizing laws to be combined with the linear approximation, to provide a strategy, which is almost optimal and that steers smoothly the system towards the desired steady state.

2 The optimal solution

The original problem, in which h is treated as a control variable was solved by Clark [4] with the help of the Maximum Principle.

Let $(x^*(t), h^*(t))$ be the optimal state. Over a time interval $[t_1, t_2]$ in which $h^*(t)$ is interior to $[0, H]$, (x^*, h^*) are governed by the Euler equation :

$$\frac{d}{dt}(L_{\dot{x}}) = L_x$$

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where $L(x,\dot{x}) = e^{-\delta t}\,(p - c/x)\,(f(x) - \dot{x})$. This has only constant solution $(x^*(t), h^*(t)) = (x_e, h_e)$ where :

$$x_e = \frac{K}{4}\left\{\left(\frac{c}{pK} + 1 - \frac{\delta}{r}\right) + \sqrt{\left(\frac{c}{pK} + 1 - \frac{\delta}{r}\right)^2 + \frac{8c\delta}{pKr}}\right\}$$
$$h_e = r x_e \left(1 - \frac{x_e}{K}\right)$$

The overall strategy is to drive $(x^*(t), h^*(t))$ to (x_e, h_e) as quickly as possible, via bang bang controls and keep it there.

However, we have to be careful about the controllability of the system. If we start for instance from a very high level of harvesting, it will be impossible to drive the stock to the equilibrium value : whatever is the control, the stock will be exhausted in finite time. Let $\mathcal{T}$ be the trajectory defined by the following system :

$$\left\{\begin{array}{lcll} \dot{x} & = & -f(x) + h\,, & x(0) = 0 \\ \dot{h} & = & U\,, & h(0) = 0 \end{array}\right. \qquad (5)$$

and the region $\mathcal{R}$ by the set of points in the plane (x, h) above the projection of the graph of $\mathcal{T}$ on the plane (x, h).

We state without proof the following simple lemma :

Lemma 1 *Let $\phi, \psi : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ be two locally Lipschitz continuous functions such that :*

$$\phi(\tau, \xi) \le \psi(\tau, \xi) \quad \forall (\tau, \xi).$$

If (y, z) is solution of the system

$$\frac{dy}{d\tau} = \phi(\tau, y)\,, \quad \frac{dz}{d\tau} = \psi(\tau, z)$$

such that $y(0) = z(0)$, then :

$$y(\tau) \le z(\tau) \quad \forall \tau \ge 0.$$

If we start now with initial conditions (x_0, h_0) in the region $\mathcal{R}$ and define the control law for (4) :

$$u^+ = \left|\begin{array}{ll} -U & \text{if } h > 0 \\ 0 & \text{if } h = 0 \end{array}\right.$$

then, for any other control law $u(.)$ with $u(t) \in [-U, U]$, $\forall t \ge 0$, we have :

$$h^+(t) \le h(t), \quad \forall t \ge 0$$

where h^+ and h are the harvest variables for the system controlled respectively by u^+ and u. From lemma 1 applied to the stock variables x and x^+, we deduce that :

$$x(t) \le x^+(t), \quad \forall t \ge 0$$

It is easy to check that the trajectory (x^+, h^+) cannot leave the region $\mathcal{R}$ because of uniqueness of solutions to the Cauchy problem. Hence, there exists $t_0^+ \ge 0$ such that :

$$\begin{array}{l} x^+(t) = 0 \quad \forall t \ge t_0^+ \quad \Longrightarrow \\ \forall u(.)\ \exists t_0 \ge 0 \text{ such that } x(t) = 0, \quad \forall t \ge t_0. \end{array}$$

This proves that from any initial conditions in $\mathcal{R}$, it is impossible to attain the equilibrium if it is non-zero.

Now, we are going to show that all initial conditions outside the region $\mathcal{R}$ can be driven to the equilibrium with a "bang-bang" control. For this, we define a control law which depends on (x, h). Let $\mathcal{T}_{+U}$ (resp. $\mathcal{T}_{-U}$) be the trajectories corresponding to the dynamics :

$$\left\{\begin{array}{l} \dot{x} = -f(x) + h \\ \dot{h} = \left|\begin{array}{ll} -U\ (+U) & \text{if } h > 0\ (h < H) \\ 0 & \text{else} \end{array}\right. \\ x(0) = x_e, \quad h(0) = h_e \end{array}\right. \qquad (6)$$

These are in fact the trajectories in reversed time leaving from (x_e, h_e). So, we can now more precise our control law :

$$\bar{u} = \left|\begin{array}{ll} +U & \text{on the right side of } \mathcal{T}_{-U}, \mathcal{T}_{+U} \\ -U & \text{on the left side} \\ +U & \text{on } \mathcal{T}_{-U} \\ -U & \text{on } \mathcal{T}_{+U} \\ 0 & \text{if } h = 0 \text{ or } h = H \end{array}\right.$$

Assume that the initial condition belongs to the left side of $\mathcal{T}_{+U}$ and $\mathcal{T}_{-U}$. The control law $\bar{u}$ allows h to decrease until the trajectory reaches $\mathcal{T}_{+U}$ or the $\{h = 0\}$ axis. By uniqueness of solutions to the Cauchy problem, it cannot reaches $\mathcal{T}_{-U}$ during this first stage of the trajectory. In this latter case, the state remains on the x axis (while $\bar{u} = 0$) until it reaches $\mathcal{T}_{+U}$ (notice that under the parabola of equilibriums $\dot{x} > 0$). Then, the equilibrium is eventually attained, the state remaining on $\mathcal{T}_{+U}$ (see figure).
If the initial condition belongs to the right side, a similar behavior occurs reaching $\mathcal{T}_{-U}$ or the $\{h = H\}$ line. After a possible stay on this latter one, the equilibrium is eventually attained along $\mathcal{T}_{-U}$.

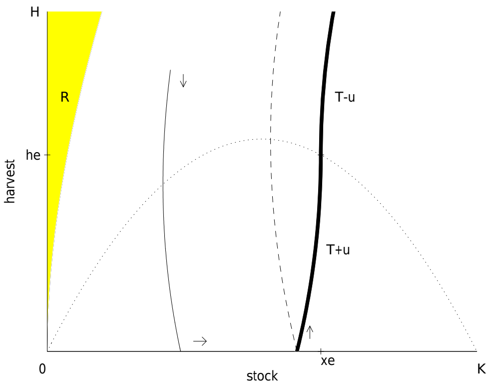
Now, we are going to show that this closed-loop control law is the fastest strategy for reaching the equilibrium from initial conditions outside the region $\mathcal{R}$.

Proposition 2 *For any initial condition (x_0, h_0) outside $\mathcal{R}$, the control law $\bar{u}$ drives the state to the equilibrium (x_e, h_e) in least time.*

Proof Let T_0 be the time when the trajectory arrives in (x_e, h_e). Suppose that there exists another control $u(.)$ such that there exists $T < T_0$, $(x(t), h(t)) = (x_e, h_e)$ $t \ge T$. Consider for instance the case where (x_0, h_0) is in the left side of $\mathcal{T}_{-\mathcal{U}}$ and $\mathcal{T}_{+\mathcal{U}}$, then :

$$\forall t \ge 0\,,\ u(t) \in [-U, +U] \Rightarrow \forall t \ge 0\,,\ h(t) \ge \bar{h}(t).$$

Hence from the lemma 1, $x(t) \le \bar{x}(t)$, $\forall t \ge 0$. And there exists $T_1 \ge T$ such that $\overline{x}(t_1) < x_e$, so $x(T_1) \le \overline{x}(t_1) <$



x_e. Contradiction. ▮

So, the proposed control law forces the state to approach the equilibrium state (x_e, h_e) as rapidly as possible. As with the classical formulation where we control h directly [4], we expect that this control is a good suboptimal strategy.

Conjecture *The most-rapid approach to the equilibrium (x_e, h_e) is also optimal with respect to the cost function J.*

Unfortunately, the "bang-bang" strategy is a discontinuous feedback which, in practice, may lead to incessant switchings about the equilibrium if the controller is not able to react promptly when the state reaches the set point. For this reason we are looking for suboptimal but smooth feedbacks.

3 The Linear-Quadratic Approximation

As bang-bang solutions are not practicable, a very well-known approximation method consist in adding to the criterion a quadratic penalty involving the control :

$$\bar{J}(u) = \int_0^{+\infty} e^{-\delta t} \left\{ \left(p - \frac{c}{x(t)}\right) h(t) - \frac{\alpha}{2} u(t)^2 \right\} dt$$

(α is a positive parameter, supposed to be small) and then study a first order approximation of the control law about the equilibrium point.

According to [3], the discounted optimal problem is equivalent to an autonomous one, with the Hamiltonian :

$$\bar{H} = \bar{\lambda}_x \left(f(x) - h\right) + \bar{\lambda}_h u + \left(p - \frac{c}{x}\right) h - \frac{\alpha}{2} u^2 - \delta \bar{V}$$

where $(\bar{\lambda}_x, \bar{\lambda}_v)$ is the adjoint vector and $\bar{V}$ the value function for the cost function $\bar{J}$. The theory of the second variation consists then of solving the linear-quadratic *auxiliary problem* :

$$\max_{\bar{u}()} \int_0^{\infty} \begin{bmatrix} \bar{x}(t) \\ \bar{h}(t) \end{bmatrix}^t \begin{bmatrix} \bar{H}_{xx} & \bar{H}_{xh} \\ \bar{H}_{xh} & \bar{H}_{hh} \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ \bar{h}(t) \end{bmatrix} - \frac{\alpha}{2} \bar{u}(t)^2 \, dt$$

$$\text{subject to : } \left\{ \begin{array}{lcl} \dot{\bar{x}} & = & f_x(x_e).\bar{x} - \bar{h} \\ \dot{\bar{h}} & = & \bar{u} \\ \bar{x}(0) & = & \bar{x}_0 = x_0 - x_e \\ \bar{h}(0) & = & \bar{h}_0 = h_0 - h_e \end{array} \right.$$

The adjoint equations of $\bar{H}$ at the equilibrium (x_e, h_e) give :

$$\left\{ \begin{array}{lcl} \bar{\lambda}_{x_e} & = & p - \dfrac{c}{x_e} \\ \bar{\lambda}_{h_e} & = & 0 \end{array} \right.$$

Writing

$$\begin{array}{rcl} A & = & \begin{bmatrix} f_x(x_e) & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r\left(1 - \frac{2x_e}{K}\right) & -1 \\ 0 & 0 \end{bmatrix} \\ B & = & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ Q & = & \begin{bmatrix} \bar{H}_{xx}(x_e, h_e) & \bar{H}_{xh}(x_e, h_e) \\ \bar{H}_{xh}(x_e, h_e) & \bar{H}_{hh}(x_e, h_e) \end{bmatrix} \\ & = & \begin{bmatrix} -2\frac{r}{K}\bar{\lambda}_{x_e} - 2c\frac{h_e}{x_e^3} & \frac{c}{x_e^2} \\ \frac{c}{x_e^2} & 0 \end{bmatrix} \\ R & = & -\alpha \end{array}$$

we seek a quadratic value function $\bar{V}(\bar{x}_0, \bar{h}_0) = (\bar{x}_0 \ \bar{h}_0) P (\bar{x}_0 \ \bar{h}_0)^t$ solution of the auxiliary problem. P should then be a symmetric matrix, solution of the Riccati algebraic equation :

$$P(A - \delta/2 I_2) + (A - \delta/2 I_2)^t P + P B R^{-1} B^t P - Q = 0 \quad (7)$$

where I_2 stands for the identity matrix in $\mathbb{R}^2$. The theory of the second variation affirms that the maximal solution of (7), if it exists, is the value function of the auxiliary problem and that the linear feedback $\bar{u}^* = K(\bar{x} \ \bar{h})^t = -R^{-1} B^t P (\bar{x} \ \bar{h})^t$ is the first order approximation of the nonlinear optimal control (see [2]).

We remark that this is not exactly the usual linear regulator problem because the matrix Q is not definite positive. Consider the Hamiltonian matrix associated with the linear quadratic approximation :

$$\mathcal{H} = \begin{bmatrix} A - \delta/2 I_2 & B R^{-1} B^t \\ -Q & -A^t + \delta/2 I_2 \end{bmatrix},$$

As $\det(\mathcal{H})$ is a fourth order polynom in δ, $\mathcal{H}$ is non singular only for some isolated values of δ and we have the following result :

Proposition 3 *If $\mathcal{H}$ is not singular and $\delta \neq 2(1 - 2x_e/K)$, the optimal solution of the auxiliary problem exists and is defined by the unique symmetric matrix P solution of the Riccati equation (7) such that $A - \delta/2 I_2 - B R^{-1} B^t P$ is Hurwitz.*

Proof The theory of the algebraic Riccati equation (see [1]) provides :

1. Characterization of the solutions by the one-to-one correspondence between the solutions P of (7) and the $\mathcal{H}$-invariant subspaces $\mathrm{Sp}\begin{bmatrix} I_2 \\ P \end{bmatrix}$ (Th. 3.1.1 p. 55).
2. Existence of at least one symmetric solution P, if the pair $[A-\delta/2I_2, B]$ is controllable, $\alpha \neq 0$ and $\mathcal{H}$ is non singular (Th. 2.32 p. 39).
3. Uniqueness of a symmetric solution P_- (resp. P_+) such that all eigenvalues of $A-\delta/2I_2-BR^{-1}B^tP_-$ (resp. $A-\delta/2I_2-BR^{-1}B^tP_+$) have non-positive (resp. non-negative) real part, if furthermore the multiplicities of the pure imaginary eigenvalues of $\mathcal{H}$ are all even (Cor. 3.2.3 and 3.2.4 p. 66).
4. Comparison of solutions : $P_- - P_+$ is definite non-negative (Th. 32.3 p. 69).

$\mathcal{C}_{[A-\delta/2I_2, B]} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ for all δ. So, the pair $[A-\delta/2I_2, B]$ is always controllable.

As $\mathcal{H}$ is similar to $-\mathcal{H}$, the spectrum of $\mathcal{H}$ is symmetric with respect to the imaginary axis. So, if $\mathcal{H}$ has any pure imaginary eigenvalue, by Cayley-Hamilton theorem, there should exists a real number μ such that $(\mathcal{H}^2+\mu^2I_4)^2=0$. But :

$$[\,1\;0\;0\;0\,]\,(\mathcal{H}^2+\mu^2I_4)^2\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=0 \quad\Rightarrow$$

$$\left\{r\left(1-\frac{2x_e}{K}-\delta/2\right)\right\}^2+\mu^2=0.$$

So, excepted for the particular value $\bar{\delta}=2(1-2x_e/K)$, (7) has exactly two solutions P_- and P_+ such that $A-\delta/2I_2-BR^{-1}B^tP_-$ (resp. $A-\delta/2I_2-BR^{-1}B^tP_+$) have non-positive (resp. non-negative) real part eigenvalues.

As the restriction of $\mathcal{H}$ to the invariant subspace $\mathrm{Sp}\begin{bmatrix} I_2 \\ P \end{bmatrix}$ is exactly $A-\delta/2I_2-BR^{-1}B^tP$ and $\mathcal{H}$ does not have any pure imaginary eigenvalues, $A-\delta/2I_2-BR^{-1}B^tP_-$ and $A-\delta/2I_2-BR^{-1}B^tP_+$ are respectively stable and unstable matrices.

Now, bearing in mind that we study a maximization problem, we must choose P_- since this is associated with higher cost. This gives a stabilizing feedback law. ∎

Unfortunately, there is no guarantee that the feedback $\bar{u}^*=-R^{-1}B^tP_-(\bar{x}\ \bar{h})^t$, which stabilizes the pair $[A-\delta/2I_2, B]$ stabilizes also $[A, B]$ and furthermore stabilizes asymptotically the system (4) for nonzero values of δ.

Proposition 4 *If $c<Kp/2$, for large enough values of δ, the linear approximation feedback $\bar{u}^*=-R^{-1}B^tP(\bar{x}\ \bar{h})^t$ destabilizes the system (4).*

Proof Let study the behavior of the system for very large values of δ :

$$\begin{cases} \lim_{\delta\to+\infty} x_e = \dfrac{c}{p} \\ \lim_{\delta\to+\infty} h_e = \dfrac{rc}{p}\left(1-\dfrac{c}{Kp}\right) \end{cases} \quad \begin{array}{c}\text{practicable if}\\ c<Kp.\end{array}$$

If $P=\dfrac{1}{\delta}\begin{bmatrix} a & b \\ b & d \end{bmatrix}$, the Riccati equation (7) is equivalent to the system of equations :

$$\begin{aligned} 2\tfrac{a}{\delta}\left(r\left(1-\tfrac{2x_e}{K}\right)-\delta/2\right)-\tfrac{b^2}{\alpha\delta^2}+2\tfrac{r}{K}(p-\tfrac{c}{x_e})+2\tfrac{ch_e}{x_e^3} &= 0\\ -\tfrac{a}{\delta}+\tfrac{b}{\delta}r\left(1-\tfrac{2x_e}{K}\right)-b-\tfrac{bd}{\alpha\delta^2}-\tfrac{c}{x_e^2} &= 0\\ -2\tfrac{b}{\delta}-d-\tfrac{d^2}{\alpha\delta} &= 0 \end{aligned}$$

So $P\sim\dfrac{1}{\delta}\begin{bmatrix} 2\frac{rp^2\left(1-\frac{c}{Kp}\right)}{c} & \frac{p}{c} \\ \frac{p}{c} & 0 \end{bmatrix}$ and $A-\delta/2I_2-BR^{-1}B^tP$ is stable as $P\to 0$ when $\delta\to+\infty$.

But $A-BR^{-1}B^tP\sim\begin{bmatrix} r\left(1-\frac{2c}{Kp}\right) & -1 \\ \frac{p}{\alpha\delta c} & 0 \end{bmatrix}$ is unstable when $c<Kp/2$.

So, by continuity of the spectrum of the eigenvalues of $A-BR^{-1}B^tP$ with respect to δ, we conclude that for δ large enough, the linear approximation feedback destabilizes the system, which is not practicable at all. ∎

4 Global stabilization

In this section, we drop, for simplicity, the notation $\bar{}$ for relative coordinates of the dynamical system (4) to the economical equilibrium (x_e, h_e).

We look now for a smooth perturbation (v_x, v_h) of the linear-quadratic law that globally asymptotically stabilizes the nonlinear system :

$$\begin{cases} \dot{x} = f(x)-h, & x(0)=x_0\\ \dot{h} = k_x x+k_h h, & h(0)=h_0\\ \dot{k_x} = v_x, & k_x(0)=G_x\\ \dot{k_h} = v_h, & k_h(0)=G_h \end{cases} \tag{8}$$

towards $(x,h)=(0,0)$, where $G=(G_x\ G_h)$ is the feedback gain of the linear quadratic approximation explicited in proposition 4.

If such a smooth stabilizing law exists, it drives asymptotically the system towards the equilibrium whatever are the initial conditions, with the same first order approximation of the control h than the linear-quadratic method

at initial time, which is the most important weight in the cost with large δ.
The determination of such a stabilizing law is inspired by the *backstepping* methods (see [6]). As we study a behavior near the equilibrium, we assume in the following that the trajectories generated by these control laws do not violate the biological constraints of the problem :

$$x \in [0, K] \quad \text{and} \quad h \in [0, H].$$

Proposition 5 *Let V be a strict controlled Lyapunov function for the system $\dot{x} = f(x) - h$ (s.t.s. V is definite positive and there exists a feedback $\xi(.)$ such that $V_x(f - \xi)$ is definite negative). If f/x, V_x/x and ξ/x are all proper smooth functions, then*

$$\begin{aligned} \hat{v}_x = \; & \xi/x - k_x(k_h - \beta_h) + (k_x - \beta_x) f/x \\ & -c_2\left(k_x + \xi_x f/x + V_x/x + c_1\xi/x\right) \\ \hat{v}_h = \; & 1 + k_x - \beta_x - k_h(k_h - \beta_h) \\ & -c_2(k_h - \xi_x f - c_1) \end{aligned}$$

with $\beta(x,h) = -c_1(h - \xi(x)) + \xi_x(x)(f(x) - h) + V_x(x)$ and c_1, c_2 positive constants, is a smooth control for the the system (8) that globally asymptotically stabilizes (x, h).

Proof Let write

$$\left\{ \begin{array}{lcl} z_1 & = & h - \xi(x) \\ z_2 & = & k_x x + k_h h - \beta(x,h) \end{array} \right.$$

and show that :

$$W(x, h, k_x, k_h) = V(x) + \frac{z_1^2}{2} + \frac{z_2^2}{2}$$

is a strict controlled Lyapunov function for the system (8).

$$\begin{aligned} \dot{z_1} &= k_x x + k_h(\xi + z_1) - \xi_x(f - \xi - z_1) \\ &= z_2 - c_1 z_1 + V_x \\ \dot{z_2} &= (k_x - \beta_x)(f - \xi - z_1) + (k_h - \beta_h)(k_x x \\ & \quad + k_h(\xi + z_1)) + v_x x + v_h(\xi + z_1) \end{aligned}$$

Then :

$$\begin{aligned} \dot{W} &= V_x(f - \xi) - V_x z_1 + z_1 \dot{z_1} + z_2 \dot{z_2} \\ &= V_x(f - \xi) - c_1 z_1^2 + z_1 z_2 + z_2 \dot{z_2} \\ &= V_x(f - \xi) - c_1 z_1^2 + z_2\{x[-\xi/x + v_x \\ & \quad + k_x(k_h - \beta_h) + f/x(k_x - \beta_x)] \\ & \quad + (\xi + z_1)[1 + v_h - (k_x - \beta_x) \\ & \quad + k_h(k_h - \beta_h)]\} \end{aligned}$$

But the function β can be factorized as follows :

$$\beta = x\left[\xi_x f/x + V_x/x + c_1\xi/x\right] - (\xi + z_1)[\xi_x f + c_1].$$

So, when $v_x = \hat{v}_x$ and $v_h = \hat{v}_h$, we have :

$$\dot{W} = V_x(f(x) - \xi(x)) - c_1(h - \xi(x))^2 - c_2 z_2^2$$

which is definite negative in (x, h). So $(x(t), h(t))$ converges towards $(0,0)$ by Lasalle theorem. ∎

5 Application

At initial time, the system is typically at an equilibrium point (x_0, h_0) of the dynamics (1), optimal for certain values of the parameters (p, c, δ) but due to some evolution of the market, the parameters might change and another set point (x_e, h_e) is desired, to be reached in a smooth manner.

For the logistic law (2), the dynamics in coordinates with origin at a given equilibrium point (x_e, h_e) is :

$$\dot{x} = rx\left(1 - \frac{x + 2x_e}{K}\right) - h.$$

Then, for any positive constant c_0, the functions :

$$V(x) = \frac{x^2}{2} \quad , \quad \xi(x) = f(x) + c_0 x,$$

fulfill the conditions of proposition 5.

We present here simulations for large values of δ, for which the method exposed in the previous section is relevant. It turns out that, due to the degree of freedom on the parameters c_0, c_1 and c_2, we can scope with the state constraints when the linear-quadratic approximated solution cannot.

Example : For $(r, K, c, p, \delta) = (1, 1, 0.4, 1, 30)$, the optimal equilibrium is : $(x_e, h_e) = (0.4079, 0.2415)$. With bounds on the harvest and its derivative $H = 0.5$, $U = 0.2$, we obtain the following results :
1. Initial condition with $x_0 = x_e + 0.2$ and $(c_0, c_1, c_2) = (50, 0.5, 0.1)$ (see figures 1 and 2).

Strategy	$J \times 10^{-2}$	$\max \mathrm{Re}(\lambda)$	*remark*
Bang-bang	33		
LQ ($\alpha = 0.001$)	30	-0.17	*const. viol.*
LQ ($\alpha = 0.055$)	28	0.042	*unstable*
Back-stepping	29		

2. Initial condition with $x_0 = x_e - 0.15$ and $(c_0, c_1, c_2) = (70, 0.25, 0.05)$ (see figures 3 and 4).

Strategy	$J \times 10^{-2}$	$\max \mathrm{Re}(\lambda)$	*remark*
Bang-bang	-28		
LQ ($\alpha = 0.001$)	-33	-0.18	*const. viol.*
LQ ($\alpha = 0.04$)	-34	0.023	*unstable*
Back-stepping	-33		

We begin with α small (figures 1 and 3) to compute the linear-quadratic approximation gain G. Unfortunately, the LQ controller violates the constraints on h. However, the "back-stepping" controller is initialized with this value

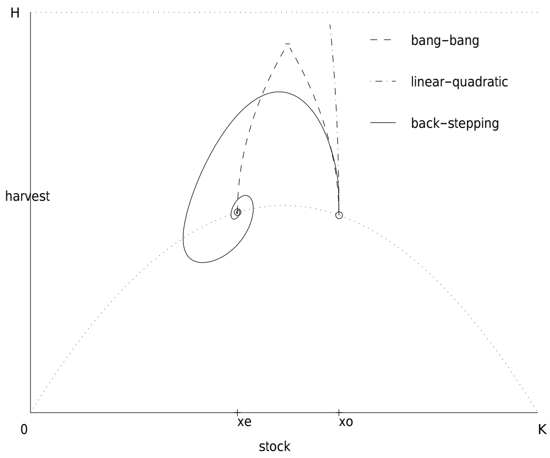


Figure 1: $x_0 - x_e = 0.2, \quad \alpha = 0.001.$

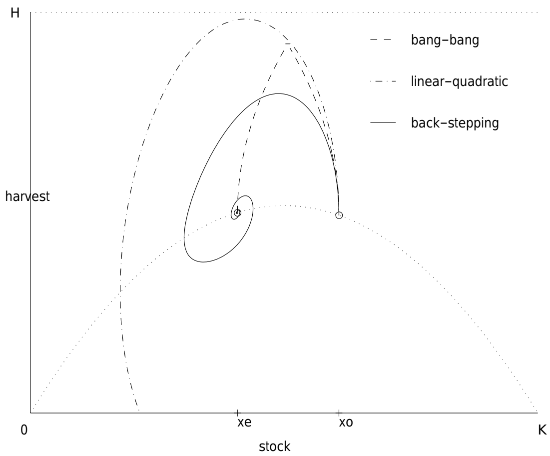


Figure 2: $x_0 - x_e = 0.2, \quad \alpha = 0.055.$

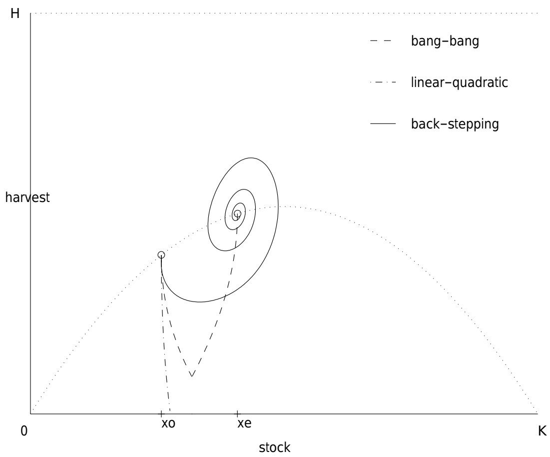


Figure 3: $x_0 - x_e = -0.15, \quad \alpha = 0.001.$

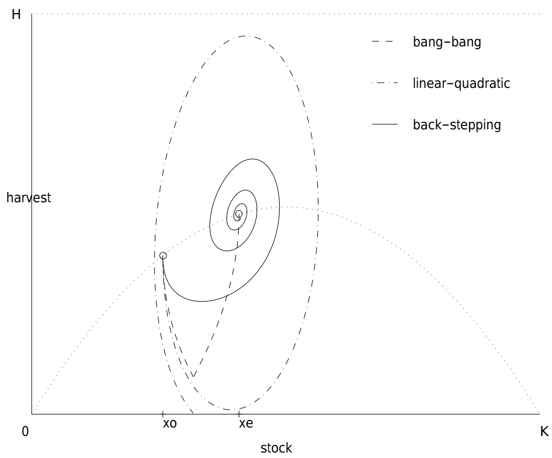


Figure 4: $x_0 - x_e = -0.15, \quad \alpha = 0.04.$

of the gain but we choose (empirically) the coefficient c_0, c_1 and c_2 such that the trajectory does not violate the constraint. If we increase the penalty α on the control u to force the LQ controller to respect the constraints (figures 2 and 4), we find a destabilizing controller with worse performance than the back-stepping one.
The choice of the parameters c_0, c_1 and c_2 and their influence on the trajectory and the cost is not yet well understood but will be the matter of a future work.

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