# Regularity of weak KAM solutions

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# Setting

Let $M$ be a smooth manifold of dimension $n \geq 2$ be fixed. Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class $C^2$ satisfying the following properties:

(H1) **Superlinear growth:**
For every $K \geq 0$, there is $C^*(K) \in \mathbb{R}$ such that

$$H(x,p) \geq K|p| + C^*(K) \qquad \forall (x,p) \in T^*M.$$

(H2) **Uniform convexity:**
For every $(x,p) \in T^*M$, $\frac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite.

(H3) **Uniform boundedness:** For every $R \geq 0$, we have

$$A^*(R) := \sup \left\{ H(x,p) \mid \|p\| \leq R \right\} < \infty.$$

Assumption (H3) holds if $M$ is compact.

# A first result of regularity

We are concerned with the regularity properties of **viscosity solutions** of the **Hamilton-Jacobi equation**

$$H(x, d_x u) = 0 \quad \text{on } M \qquad (HJ).$$

**Theorem (LR '07)**

*Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class $C^2$ satisfying (H1)-(H2) and $u : M \to \mathbb{R}$ be a viscosity solution of (HJ). Then the function $u$ is locally semiconcave on $M$. Moreover, the singular set of $u$ is nowhere dense in $M$ and $u$ is $C^{1,1}_{loc}$ on the open dense set $M \setminus \overline{\Sigma(u)}$.*

Reminder:

$$\Sigma(u) = \left\{ x \in M \,|\, u \text{ not diff. at } x \right\}$$

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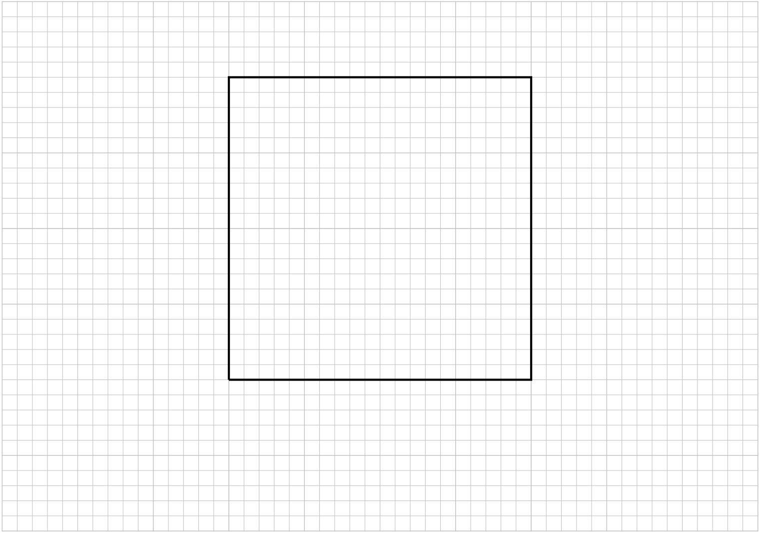
# An instructive example



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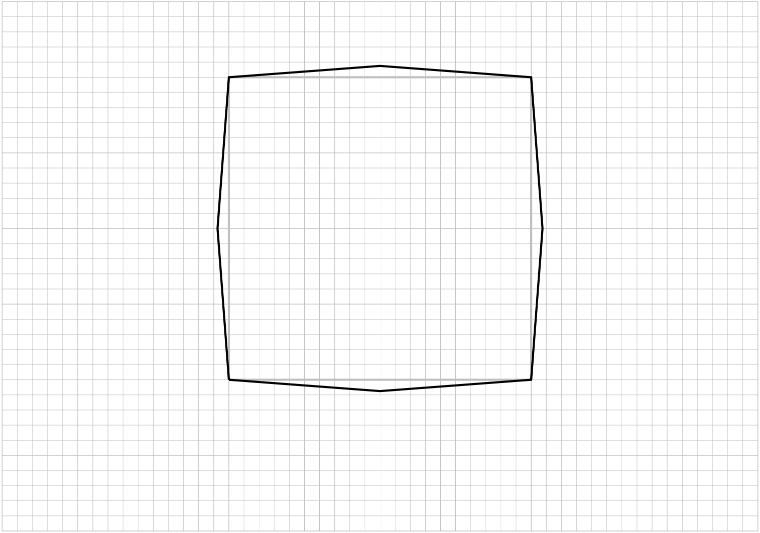

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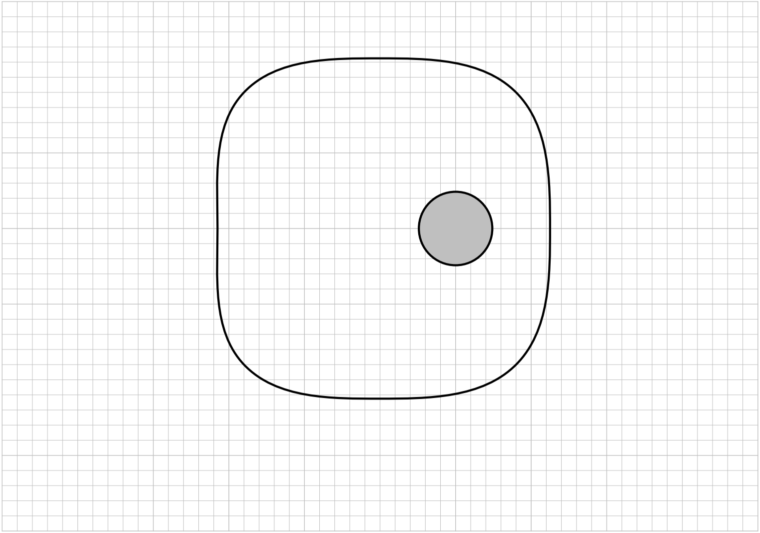


# Characterization of viscosity solutions

Let $L : TM \to \mathbb{R}$ be the Tonelli Lagrangian associated with $H$ by Legendre-Fenchel duality, that is

$$L(x,v) := \max_{p \in T_x^* M} \Big\{ p \cdot v - H(x,p) \Big\} \qquad \forall (x,v) \in TM.$$

**Proposition**

*The function $u : M \to \mathbb{R}$ is a viscosity solution of (HJ) iff:*

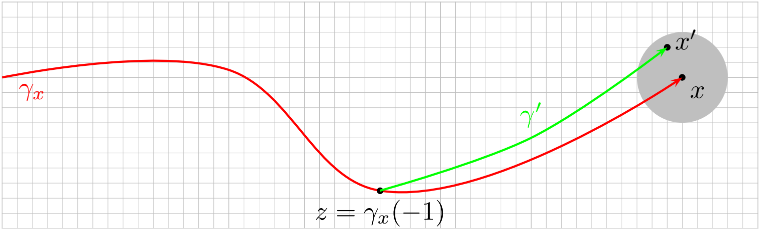
(i) *For every Lipschitz curve $\gamma : [a,b] \to M$, we have*

$$u\big(\gamma(b)\big) - u\big(\gamma(a)\big) \leq \int_a^b L\big(\gamma(t), \dot{\gamma}(t)\big)\, ds.$$

(ii) *$\forall x \in M$, there is a curve $\gamma_x : (-T, 0] \to M$ such that*

$$u\big(\gamma(b)\big) - u\big(\gamma(a)\big) = \int_a^b L\big(\gamma(t), \dot{\gamma}(t)\big)\, ds \quad \forall a < b < 0.$$

# Semiconcavity




$$u(x) = u(z) + \int_{-1}^{0} L\big(\gamma_x(t), \dot\gamma_x(t)\big)\, dt$$

$$u(x') \leq u(z) + \int_{-1}^{0} L\big(\gamma'(t), \dot\gamma'(t)\big)\, dt$$

Thus

$$u(x') \leq u(x) + \int_{-1}^{0} L\big(\gamma'(t), \dot\gamma'(t)\big) - L\big(\gamma_x(t), \dot\gamma_x(t)\big)\, dt$$

# Limiting differentials and semi-calibrated curves

We can repeat the previous argument to show that for every $x \in M$, every semi-calibrated curve $\gamma_x : (-T_x, 0] \to M$ and every $t \in (0, T)$, the graph of $u$ at $\gamma_x(-t)$ admits a support function of class $C^2$ from below.

Moreover, we can show that for every $x \in M$, there is a one-to-one correspondence between the **limiting differential** of $u$ at $x$,

$$d_x^* u := \left\{ \lim d_{x_k} u \,|\, x_k \to x, u \text{ diff at } x_k \right\},$$

and the set of semi-calibrated curves ($p = \frac{\partial L}{\partial v}(\dot{\gamma}(0))$).

# The classical Dirichlet problem

Let $M$ be an open set in $\mathbb{R}^n$ with compact boundary of class $C^{k,1}$ and $H : \mathbb{R}^n \to \mathbb{R}$ of class $C^{k,1}$ (with $k \geq 2$) satisfying (H1)-(H3) and such that $H(x,0) < 0$ for every $x \in \bar{M}$.

## Proposition

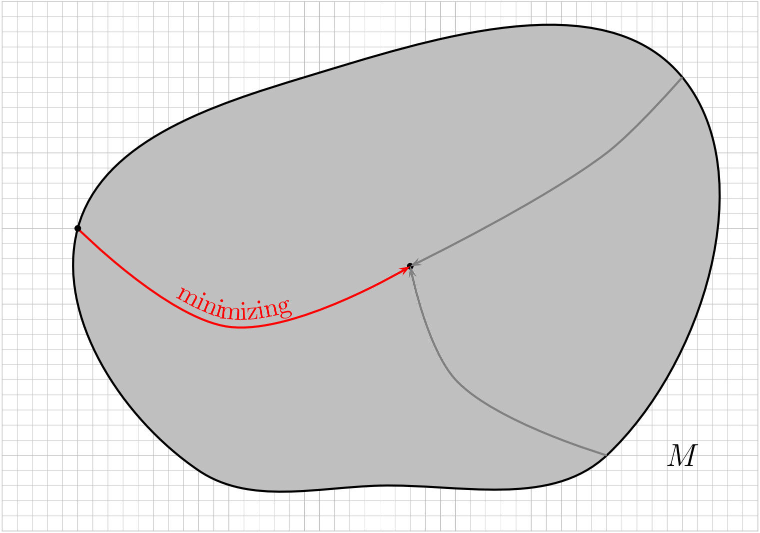
*The continuous function $u : \bar{M} \to \mathbb{R}$ given by*

$$u(x) := \inf \left\{ \int_0^t L\big(\gamma(s), \dot{\gamma}(s)\big)\, ds \right\},$$

*where the infimum is taken among Lipschitz curves $\gamma : [0,t] \to \bar{M}$ with $\gamma(0) \in \partial\Omega, \gamma(t) = x$ is the unique viscosity solution to the Dirichlet problem*

$$\begin{cases} H\big(x, du(x)\big) & = & 0 & \forall x \in M, \\ u(x) & = & 0 & \forall x \in \partial M. \end{cases}$$

# The classical Dirichlet problem (picture)



# A Sard theorem

Let $u$ be a solution to the previous Dirichlet problem. We call **critical point** of $u$, any $x \in M$ such that $0 \in \partial_x u$. Here, $\partial_x$ denotes the Clarke generalized differential of $u$ at $x$, *i.e.*

$$\partial_x u := \operatorname{conv}\left(d_x^* u\right).$$

We denote by $\mathcal{C}(u)$ the set of critical points of $u$ in $M$.

**Theorem (LR '07)**

*If $k \geq 2n^2 + 4n + 1$, then the set $u(\mathcal{C}(u))$ has Lebesgue measure zero.*

# The distance function to the cut-locus

We call **cut locus** associated with this Dirichlet problem the set

$$\mathrm{cut}(u) := \overline{\Sigma}(u).$$

The **distance functon to the cut locus** is defined as

$$\tau_{cut}(x) := \min \{ t \geq 0 \mid \exp(x,t) \in \mathrm{cut}(u) \},$$

for every $x \in \partial M$.

**Theorem (Itoh-Tanaka '01, Li-Nirenberg '05)**

*The function $t_{cut}$ is Lipschitz.*

Since $\mathrm{cut}(u) = \{ \exp(x, t_{cut}(x)) \mid x \in \partial M \}$, we get

**Corollary**

*The set $\mathrm{cut}(u)$ has a finite $(n-1)$-dimensional Hausdorff measure.*

# Weak KAM solutions

Let $M$ be a smooth compact manifold of dimension $n \geq 2$ be fixed. Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class $C^k$, with $k \geq 2$. We call **critical value** of $H$ the constant $c = c[H]$ defined as

$$c[H] := \inf_{u \in C^1(M;\mathbb{R})} \left\{ \max_{x \in M} \left\{ H\big(x, du(x)\big) \right\} \right\}.$$

**Theorem (Fathi '90s)**

*There is a viscosity solution $u : M \to \mathbb{R}$ to the critical HJ equation*

$$H(x, d_x u) = c[H] \quad \text{on } M.$$

*It is called a* ***critical*** *or a* ***weak KAM solution*** *of $H$.*

# The Aubry set

Denote by $\mathcal{S}(H)$ the set of weak KAM solutions for $H$. The **Aubry set** may be defined as

$$\tilde{\mathcal{A}}(H) = \bigcup_{u \in \mathcal{S}(H)} \mathrm{Graph}(du).$$

## Proposition

*For every $x \in M$ and every $p \in d_x^* u$ there is a semi-calibrated curve $\gamma = \gamma_{x,p} : (-\infty, 0] \to M$ such that*

$$\frac{\partial L}{\partial v}(\dot{\gamma}(0)) = (x, p).$$

*It satisfies*

$$\lim_{t \to -\infty} dist\left(\gamma(t), \mathcal{A}(H)\right) = 0.$$

# Two results of regularity

**Theorem (Bernard '07)**

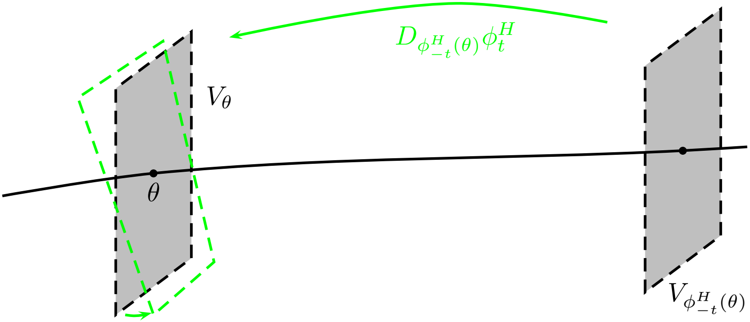
*Assume that the Aubry set is exactly one hyperbolic periodic orbit, then any critical solution is "smooth" in a neighborhood of $\mathcal{A}(H)$.*

**Theorem (Arnaud '08)**

*Let $M = \mathbb{T}^2$ and $H : T^*M \to \mathbb{R}$ be an Hamiltonian of class $C^2$ satisfying (H1)-(H2). Let $u : M \to \mathbb{R}$ be a solution of (HJ) of class $C^1$ without singularities. Then $u$ is $C^{1,1}$ and $C^2$ almost everywhere.*

We call **singularity** any equilibrium of the characteristic flow of $u$, that is any $x \in M$ such that $\frac{\partial H}{\partial p}(x, d_x u) = 0$.

# Green bundles I




For every $\theta = (x, d_x u) \in T^*M$ and every $t \in \mathbb{R}$, we define the Lagrangian subspace $G_\theta^t \subset T_\theta T^*M$ by ($V_\theta \simeq \{0\} \times \mathbb{R}^2$)

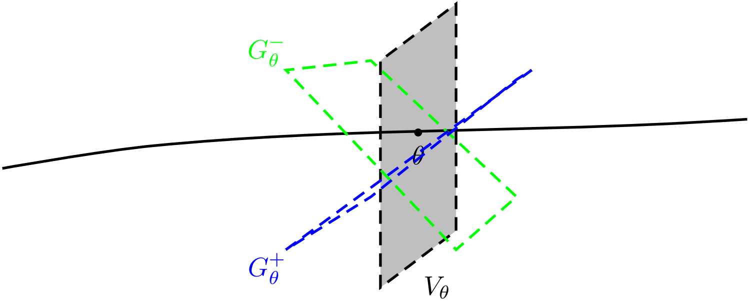
$$G_\theta^t := \left(\phi_t^H\right)_* \left(V_{\phi^H_{-t}(\theta)}\right).$$

# Green bundles II

## Definition

For every $\theta = (x, d_x u)$, we define the positive and negative Green bundles at $\theta$ as

$$G_\theta^+ := \lim_{t \to +\infty} G_\theta^t \quad \text{and} \quad G_\theta^- := \lim_{t \to -\infty} G_\theta^t$$



# Green bundles III

The following properties hold:

- For every $\theta = (x, d_x u)$, $G_\theta^- \preceq G_\theta^+$.
- The function $x \in M \mapsto G^+_{(x,d_x u)}$ is upper-semicontinuous.
- The function $x \in M \mapsto G^-_{(x,d_x u)}$ is lower-semicontinuous.
- So, if $G^+_{(x,d_x u)} = G^-_{(x,d_x u)}$ for some $x$ then both functions are continuous at $x$.
- For every $x \in M$, we have

$$G^-_{(x,d_x u)} \preceq \mathcal{H}ess_C u(x) \preceq G^+_{(x,d_x u)},$$

where $\mathcal{H}ess_C u(x)$ denotes the Clarke generalized Hessian of $u$ at $x$.

Thank you for your attention !!