Regularity of weak KAM solutions

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Analysis of Hamilton-Jacobi equation:
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Let $M$ be a smooth manifold of dimension $n \geq 2$ be fixed. Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class $C^2$ satisfying the following properties:

(H1) **Superlinear growth:**
For every $K \geq 0$, there is $C^*(K) \in \mathbb{R}$ such that

$$H(x, p) \geq K|p| + C^*(K) \quad \forall (x, p) \in T^*M.$$ 

(H2) **Uniform convexity:**
For every $(x, p) \in T^*M$, $\frac{\partial^2 H}{\partial p^2}(x, p)$ is positive definite.

(H3) **Uniform boundedness:** For every $R \geq 0$, we have

$$A^*(R) := \sup \{ H(x, p) \mid \|p\| \leq R \} < \infty.$$ 

Assumption (H3) holds if $M$ is compact.
A first result of regularity

We are concerned with the regularity properties of viscosity solutions of the Hamilton-Jacobi equation

\[ H(x, d_x u) = 0 \quad \text{on } M \quad \text{(HJ)}. \]

**Theorem (LR '07)**

Let \( H : T^* M \to \mathbb{R} \) be a Hamiltonian of class \( C^2 \) satisfying (H1)-(H2) and \( u : M \to \mathbb{R} \) be a viscosity solution of (HJ). Then the function \( u \) is locally semiconcave on \( M \). Moreover, the singular set of \( u \) is nowhere dense in \( M \) and \( u \) is \( C^{1,1}_{loc} \) on the open dense set \( M \setminus \Sigma(u) \).

Reminder:

\[ \Sigma(u) = \left\{ x \in M \mid u \text{ not diff. at } x \right\} \]
A first result of regularity

We are concerned with the regularity properties of viscosity solutions of the Hamilton-Jacobi equation

\[ H(x, d_x u) = 0 \quad \forall x \in M \] (HJ).

**Theorem (LR, 2007)**

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An instructive example
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Characterization of viscosity solutions

Let \( L : TM \to \mathbb{R} \) be the Tonelli Lagrangian associated with \( H \) by Legendre-Fenchel duality, that is
\[
L(x, v) := \max_{p \in T^*_x M} \left\{ p \cdot v - H(x, p) \right\} \quad \forall (x, v) \in TM.
\]

Proposition

The function \( u : M \to \mathbb{R} \) is a viscosity solution of (HJ) iff:

(i) For every Lipschitz curve \( \gamma : [a, b] \to M \), we have
\[
u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(t), \dot{\gamma}(t)) \, ds.
\]

(ii) \( \forall x \in M \), there is a curve \( \gamma_x : (-T, 0] \to M \) such that
\[
u(\gamma(b)) - u(\gamma(a)) = \int_a^b L(\gamma(t), \dot{\gamma}(t)) \, ds \quad \forall a < b < 0.
\]
Semiconcavity

\[ z = \gamma_x(-1) \]

\[ u(x) = u(z) + \int_{-1}^{0} L(\gamma_x(t), \dot{\gamma}_x(t)) \, dt \]

\[ u(x') \leq u(z) + \int_{-1}^{0} L(\gamma'(t), \dot{\gamma}'(t)) \, dt \]

Thus

\[ u(x') \leq u(x) + \int_{-1}^{0} L(\gamma'(t), \dot{\gamma}'(t)) - L(\gamma_x(t), \dot{\gamma}_x(t)) \, dt \]
We can repeat the previous argument to show that for every \( x \in M \), every semi-calibrated curve \( \gamma_x : (-T_x, 0] \to M \) and every \( t \in (0, T) \), the graph of \( u \) at \( \gamma_x(-t) \) admits a support function of class \( C^2 \) from below. Moreover, we can show that for every \( x \in M \), there is a one-to-one correspondence between the limiting differential of \( u \) at \( x \),

\[
d^*_x u := \left\{ \lim d_{x_k} u \mid x_k \to x, u \text{ diff at } x_k \right\},
\]

and the set of semi-calibrated curves \( (p = \frac{\partial L}{\partial \nu} (\dot{\gamma}(0))) \).
The classical Dirichlet problem

Let $M$ be an open set in $\mathbb{R}^n$ with compact boundary of class $C^{k,1}$ and $H : \mathbb{R}^n \to \mathbb{R}$ of class $C^{k,1}$ (with $k \geq 2$) satisfying (H1)-(H3) and such that $H(x,0) < 0$ for every $x \in \bar{M}$.

**Proposition**

The continuous function $u : \bar{M} \to \mathbb{R}$ given by

$$u(x) := \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s)) \, ds \right\},$$

where the infimum is taken among Lipschitz curves $\gamma : [0, t] \to \bar{M}$ with $\gamma(0) \in \partial \Omega, \gamma(t) = x$ is the unique viscosity solution to the Dirichlet problem

$$\begin{cases} H(x, du(x)) = 0 & \forall x \in M, \\ u(x) = 0 & \forall x \in \partial M. \end{cases}$$
The classical Dirichlet problem (picture)
Let $u$ be a solution to the previous Dirichlet problem. We call \textbf{critical point} of $u$, any $x \in M$ such that $0 \in \partial_x u$. Here, $\partial_x$ denotes the Clarke generalized differential of $u$ at $x$, \textit{i.e.}

$$\partial_x u := \text{conv}(d_x^* u).$$

We denote by $C(u)$ the set of critical points of $u$ in $M$.

**Theorem (LR '07)**

\textit{If $k \geq 2n^2 + 4n + 1$, then the set $u(C(u))$ has Lebesgue measure zero.}
The distance function to the cut-locus

We call **cut locus** associated with this Dirichlet problem the set

$$ \text{cut}(u) := \overline{\Sigma}(u). $$

The **distance function to the cut locus** is defined as

$$ \tau_{\text{cut}}(x) := \min \{ t \geq 0 \mid \exp(x, t) \in \text{cut}(u) \}, $$

for every $x \in \partial M$.

**Theorem (Itoh-Tanaka ’01, Li-Nirenberg ’05)**

The function $t_{\text{cut}}$ is Lipschitz.

Since $\text{cut}(u) = \{ \exp(x, t_{\text{cut}}(x)) \mid x \in \partial M \}$, we get

**Corollary**

The set $\text{cut}(u)$ has a finite $(n - 1)$-dimensional Hausdorff measure.
Let $M$ be a smooth compact manifold of dimension $n \geq 2$ be fixed. Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class $C^k$, with $k \geq 2$. We call **critical value** of $H$ the constant $c = c[H]$ defined as

$$c[H] := \inf_{u \in C^1(M;\mathbb{R})} \left\{ \max_{x \in M} \{ H(x, du(x)) \} \right\}.$$

**Theorem (Fathi ’90s)**

There is a viscosity solution $u : M \to \mathbb{R}$ to the critical HJ equation

$$H(x, d_x u) = c[H] \quad \text{on } M.$$

It is called a **critical** or a **weak KAM solution** of $H$. 
Denote by $S(H)$ the set of weak KAM solutions for $H$. The **Aubry set** may be defined as

$$
\tilde{\mathcal{A}}(H) = \bigcup_{u \in S(H)} \text{Graph}(du).
$$

**Proposition**

For every $x \in M$ and every $p \in d_x^* u$ there is a semi-calibrated curve $\gamma = \gamma_{x, p} : (-\infty, 0] \rightarrow M$ such that

$$
\frac{\partial L}{\partial v}(\dot{\gamma}(0)) = (x, p).
$$

It satisfies

$$
\lim_{t \to -\infty} \text{dist} (\gamma(t), \mathcal{A}(H)) = 0.
$$
Two results of regularity

**Theorem (Bernard ’07)**

Assume that the Aubry set is exactly one hyperbolic periodic orbit, then any critical solution is ”smooth” in a neighborhood of $A(H)$.

**Theorem (Arnaud ’08)**

Let $M = \mathbb{T}^2$ and $H : T^*M \to \mathbb{R}$ be an Hamiltonian of class $C^2$ satisfying (H1)-(H2). Let $u : M \to \mathbb{R}$ be a solution of (HJ) of class $C^1$ without singularities. Then $u$ is $C^{1,1}$ and $C^2$ almost everywhere.

We call **singularity** any equilibrium of the characteristic flow of $u$, that is any $x \in M$ such that $\frac{\partial H}{\partial p}(x, d_x u) = 0$. 
For every $\theta = (x, d_x u) \in T^* M$ and every $t \in \mathbb{R}$, we define the Lagrangian subspace $G^t_\theta \subset T_\theta T^* M$ by $(V_\theta \simeq \{0\} \times \mathbb{R}^2)$

$$G^t_\theta := \left( \phi^H_t \right)_* \left( V_{\phi^H_{-t}(\theta)} \right).$$
Definition

For every $\theta = (x, d_x u)$, we define the positive and negative Green bundles at $\theta$ as

$$G_{\theta}^+ := \lim_{t \to +\infty} G_{\theta}^t \quad \text{and} \quad G_{\theta}^- := \lim_{t \to -\infty} G_{\theta}^t$$
The following properties hold:

- For every $\theta = (x, d_x u)$, $G_{\theta}^- \preceq G_{\theta}^+$. 
- The function $x \in M \mapsto G^+_{(x, d_x u)}$ is upper-semicontinuous.
- The function $x \in M \mapsto G^-_{(x, d_x u)}$ is lower-semicontinuous.
- So, if $G^+_{(x, d_x u)} = G^-_{(x, d_x u)}$ for some $x$ then both functions are continuous at $x$.
- For every $x \in M$, we have

$$G^-_{(x, d_x u)} \preceq \mathcal{H}ess_C u(x) \preceq G^+_{(x, d_x u)};$$

where $\mathcal{H}ess_C u(x)$ denotes the Clarke generalized Hessian of $u$ at $x$. 

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Thank you for your attention!!