Franks’ Lemma for Mañé perturbations of Riemannian metrics and applications

Ludovic Rifford

Université Nice Sophia Antipolis
& Institut Universitaire de France

Workshop on Hamiltonian dynamical systems
January 4-10, 2015
Fudan University, Shanghai, China
Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 2\) be fixed.
Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 2\) be fixed.

**Definition**

We call Mañé perturbation or conformal perturbation of the metric \(g\) any perturbation of the form

\[
\tilde{g} = e^f g,
\]

where \(f : M \to \mathbb{R}\) is a smooth function.
Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 2\) be fixed.

**Definition**

We call **Mañé perturbation** or **conformal perturbation** of the metric \(g\) any perturbation of the form

\[
\tilde{g} = e^f g,
\]

where \(f : M \to \mathbb{R}\) is a smooth function.

**Remark**

*If \(f\) is close to 0 in \(C^k\) topology then the geodesic flow of \(\tilde{g} = e^f g\) is close the geodesic flow of \(g\) in \(C^{k-1}\) topology.*
Connecting geodesics

\[ \gamma \] connecting \( \gamma_1 \) and \( \gamma_2 \) within \( \text{Supp} \ (f) \).
First define the connecting trajectory by

\[ \tilde{\gamma}(t) = \alpha(t) \gamma_1(t) + (1 - \alpha(t)) \gamma_2(t). \]

and reparametrize it by arc-length w.r.t. the initial metric $g$ to get a new parametrized curve $\gamma$. 

Ludovic Rifford

Franks’ Lemma for Mañé perturbations..
First define the connecting trajectory by
\[ \tilde{\gamma}(t) = \alpha(t) \gamma_1(t) + (1 - \alpha(t)) \gamma_2(t). \]
and reparametrize it by arc-length w.r.t. the initial metric \( g \) to get a new parametrized curve \( \gamma \). Setting
\[ H(x, p) = \frac{1}{2} \| p \|_x^2 \quad \text{and} \quad \tilde{H}(x, p) = \frac{e^{-f(x)}}{2} \| p \|_x^2, \]
we would like to construct a real function \( f \) satisfying
\[
\begin{align*}
\dot{\gamma} &= \frac{\partial \tilde{H}}{\partial p} = e^{-f(\gamma)} \frac{\partial H}{\partial p}(\gamma, p) \\
\dot{p} &= -\frac{\partial \tilde{H}}{\partial x} = -e^{-f(\gamma)} \frac{\partial H}{\partial x}(\gamma, p) - \tilde{H}(\gamma, p) d_\gamma f.
\end{align*}
\]
along \( \gamma \).
A constructive method I

First define the connecting trajectory by

$$\tilde{\gamma}(t) = \alpha(t) \gamma_1(t) + (1 - \alpha(t)) \gamma_2(t).$$

and reparametrize it by arc-length w.r.t. the initial metric $g$ to get a new parametrized curve $\gamma$. Setting

$$H(x, p) = \frac{1}{2} \|p\|_x^2$$

and

$$\tilde{H}(x, p) = \frac{e^{-f(x)}}{2} \|p\|_x^2,$$

we would like to construct a real function $f$ satisfying

$$\begin{cases} 
\dot{\gamma} = \frac{\partial \tilde{H}}{\partial p} = e^{-f(\gamma)} \frac{\partial H}{\partial p}(\gamma, p) \\
\dot{p} = -\frac{\partial \tilde{H}}{\partial x} = -e^{-f(\gamma)} \frac{\partial H}{\partial x}(\gamma, p) - \tilde{H}(\gamma, p) d\gamma f.
\end{cases}$$

along $\gamma$. This can be done if we force $f = 0$ along $\gamma$. 

Ludovic Rifford
Franks’ Lemma for Mañé perturbations.
A constructive method I

As a matter of fact, if we impose $f = 0$ along $\gamma$ then we need

\[
\begin{cases}
\gamma'(t) = \frac{\partial H}{\partial p}(\gamma(t), p(t)) \\
\dot{p}(t) = -\frac{\partial H}{\partial x}(\gamma(t), p(t)) - \frac{1}{2}d_{\gamma(t)}f.
\end{cases}
\]

which can be solved.
As a matter of fact, if we impose $f = 0$ along $\gamma$ then we need

\[
\begin{align*}
\dot{\gamma}(t) &= \frac{\partial H}{\partial p}(\gamma(t), p(t)) \\
\dot{p}(t) &= -\frac{\partial H}{\partial x}(\gamma(t), p(t)) - \frac{1}{2} d_{\gamma(t)}f.
\end{align*}
\]

which can be solved. Moreover, since $H(\gamma(t), p(t)) = 1/2$ we have

\[d_{\gamma(t)}f \cdot \dot{\gamma}(t) = 0.\]
As a matter of fact, if we impose $f = 0$ along $\gamma$ then we need

\[
\begin{cases}
\dot{\gamma}(t) = \frac{\partial H}{\partial p} (\gamma(t), p(t)) \\
\dot{p}(t) = -\frac{\partial H}{\partial x} (\gamma(t), p(t)) - \frac{1}{2} d_{\gamma(t)} f.
\end{cases}
\]

which can be solved. Moreover, since $H(\gamma(t), p(t)) = 1/2$ we have

\[d_{\gamma(t)} f \cdot \dot{\gamma}(t) = 0.\]
The control approach

Define the mapping

\[ E : C^\infty([0, \tau], \mathbb{R}^n) \longrightarrow \mathbb{R}^n \times (\mathbb{R}^n)^* \]

\[ u \longmapsto (x_u(\tau), p_u(\tau)) \]

where \((x_u, p_u) : [0, \tau] \longrightarrow \mathbb{R}^n \times (\mathbb{R}^n)^*\) is the solution of

\[
\begin{align*}
\dot{x}(t) &= \frac{\partial H}{\partial p}(x(t), p(t)) \\
\dot{p}(t) &= -\frac{\partial H}{\partial x}(x(t), p(t)) - u(t),
\end{align*}
\]

starting at \((x_1(0), p_1(0))\).
The control approach

Define the mapping

\[ E : C^\infty([0, \tau], \mathbb{R}^n) \longrightarrow \mathbb{R}^n \times (\mathbb{R}^n)^* \]

\[ u \longmapsto (x_u(\tau), p_u(\tau)) \]

where \((x_u, p_u) : [0, \tau] \longrightarrow \mathbb{R}^n \times (\mathbb{R}^n)^*\) is the solution of

\[
\begin{align*}
\dot{x}(t) &= \frac{\partial H}{\partial p}(x(t), p(t)) \\
\dot{p}(t) &= -\frac{\partial H}{\partial x}(x(t), p(t)) - u(t),
\end{align*}
\]

starting at \((x_1(0), p_1(0))\).

\[ \Rightarrow \text{If } E \text{ is open at } u \equiv 0 \text{ then we can connect } \gamma_1 \text{ to the geodesics which are sufficiently close to } \gamma_1. \]
The Franks’ Lemma

Let $\varphi : M \to M$ be a $C^1$ diffeomorphism, consider a finite set of points $S = \{x_1, \ldots, x_m\}$ and set

$$\Pi = \bigoplus_{i=1}^m T_{x_i} M, \quad \Pi' = \bigoplus_{i=1}^m T_{\varphi(x_i)} M.$$
The Franks’ Lemma

Let $\varphi : M \to M$ be a $C^1$ diffeomorphism, consider a finite set of points $S = \{x_1, \ldots, x_m\}$ and set

$$\Pi = \bigoplus_{i=1}^m T_{x_i} M, \quad \Pi' = \bigoplus_{i=1}^m T_{\varphi(x_i)} M.$$ 

Lemma (Franks, 1971)

There is $\bar{\epsilon} > 0$ such that for every $\epsilon \in (0, \bar{\epsilon})$, there is $\delta = \delta(\epsilon) > 0$ such that for any isomorphism

$$L = (L_1, \ldots, L_m) : \Pi \to \Pi'$ s.t. $\|L_i - D_{x_i} \varphi\| < \delta \ \forall i,$$

there exists a $C^1$ diffeomorphism $\psi : M \to M$ satisfying

1. $\psi(x_i) = \varphi(x_i) \ \forall i,$
2. $D_{x_i} \psi = L_i \ \forall i,$
3. $\|g - f\|_{C^1} < \epsilon.$
Given $\theta_0 = (x, v) \in UM$ and $T > 0$, we consider the unit speed geodesic $\gamma_{\theta_0} : [0, T] \to M$ starting at $x$ with initial velocity $v$ and we set $\theta_1 := (\gamma_{\theta_0}(T), \dot{\gamma}_{\theta_0}(T))$. Then denoting by $N_0, N_1$ the hyperplanes in $T_{\theta_0}UM, T_{\theta_1}UM$ which are orthogonal to the flow at $\theta_0, \theta_1$, we consider the (local) **Poincaré mapping** from $\Sigma_0$ (tangent to $N_0$ at $\theta_0$) to $\Sigma_1$ (tangent to $N_1$ at $\theta_1$).
Given $\theta_0 = (x, v) \in UM$ and $T > 0$, we consider the unit speed geodesic $\gamma_{\theta_0} : [0, T] \to M$ starting at $x$ with initial velocity $v$ and we set $\theta_1 := (\gamma_{\theta_0}(T), \dot{\gamma}_{\theta_0}(T))$. Then denoting by $N_0, N_1$ the hyperplanes in $T_{\theta_0} UM, T_{\theta_1} UM$ which are orthogonal to the flow at $\theta_0, \theta_1$, we consider the (local) Poincaré mapping from $\Sigma_0$ (tangent to $N_0$ at $\theta_0$) to $\Sigma_1$ (tangent to $N_1$ at $\theta_1$).
Let \( \text{Sp}(m) \) be the symplectic group in \( M_{2m}(\mathbb{R}) (m = n - 1) \), that is the smooth submanifold of matrices \( X \in M_{2m}(\mathbb{R}) \) satisfying

\[
X^* \mathbb{J} X = \mathbb{J} \quad \text{where} \quad \mathbb{J} := \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}.
\]

Choosing a convenient set of coordinates, the differential of the Poincaré mapping \( P := P_g(\Sigma_0, \Sigma_T, \gamma_{\theta_0}) \) at \( \theta_0 \) is the symplectic matrix \( X(T) \) where \( X : [0, T] \rightarrow \text{Sp}(m) \) is solution to the Cauchy problem

\[
\begin{cases}
\dot{X}(t) = A(t)X(t) & \forall t \in [0, T], \\
X(0) = I_{2m},
\end{cases}
\]

where \( A(t) \) has the form

\[
A(t) = \begin{pmatrix} 0 & I_m \\ -K(t) & 0 \end{pmatrix} \quad \forall t \in [0, T].
\]
Problem:

Given $\epsilon > 0$, does the set of differentials of Poincaré maps $P_{\tilde{g}}(\Sigma_0, \Sigma_T, \gamma_{\theta_0})$ at $\theta_0$ associated with smooth conformal factors $f : M \to \mathbb{R}$ such that

$$\|f\|_{C^2} < \epsilon,$$

fill a ball around $d\bar{\theta}_P$ in $\text{Sp}(\mathfrak{m})$? What's the radius of that ball in terms of $\epsilon$?
Problem:

Given $\epsilon > 0$, does the set of differentials of Poincaré maps $P_{\tilde{g}}(\Sigma_0, \Sigma_T, \gamma_{\theta_0})$ at $\theta_0$ associated with smooth conformal factors $f : M \to \mathbb{R}$ such that

- $\|f\|_{C^2} < \epsilon$,
Problem:

Given $\epsilon > 0$, does the set of differentials of Poincaré maps $P_{\tilde{g}}(\Sigma_0, \Sigma_T, \gamma_{\theta_0})$ at $\theta_0$ associated with smooth conformal factors $f : M \to \mathbb{R}$ such that

- $\|f\|_{C^2} < \epsilon$,
- the curve $\gamma_{\theta_0} : [0, T] \to M$ is a unit-speed geodesic w.r.t. $\tilde{g}$,
Problem:

Given $\epsilon > 0$, does the set of differentials of Poincaré maps $P_{\tilde{g}}(\Sigma_0, \Sigma_T, \gamma_{\theta_0})$ at $\theta_0$ associated with smooth conformal factors $f: M \to \mathbb{R}$ such that

- $\|f\|_{C^2} < \epsilon$,
- the curve $\gamma_{\theta_0}: [0, T] \to M$ is a unit-speed geodesic w.r.t. $\tilde{g}$,

fill a ball around $d\tilde{\theta}P$ (in $\text{Sp}(m)$)?
Problem:

Given $\epsilon > 0$, does the set of differentials of Poincaré maps $P_{\tilde{g}}(\Sigma_0, \Sigma_T, \gamma_{\theta_0})$ at $\theta_0$ associated with smooth conformal factors $f : M \to \mathbb{R}$ such that

\begin{itemize}
  \item $\|f\|_{C^2} < \epsilon$,
  \item the curve $\gamma_{\theta_0} : [0, T] \to M$ is a unit-speed geodesic w.r.t. $\tilde{g}$,
\end{itemize}

fill a ball around $d_{\theta}P$ (in $\text{Sp}(m)$)?

What’s the radius of that ball in term of $\epsilon$?
Set $\gamma := \gamma_{\theta_0}$. We are looking for a smooth function

$$f : M \to \mathbb{R}$$

satisfying the following properties

$$f(\gamma(t)) = 0 \quad \text{and} \quad d_{\gamma(t)}f = 0 \quad \forall t \in [0, T],$$

with

$$d^2V(\gamma(t)) \quad \text{free.}$$
Set $\gamma := \gamma_{\theta_0}$. We are looking for a smooth function $f : M \to \mathbb{R}$ satisfying the following properties

$$f(\gamma(t)) = 0 \quad \text{and} \quad d_{\gamma(t)} f = 0 \quad \forall t \in [0, T],$$

with

$$d^2 V(\gamma(t)) \quad \text{free}.$$ 

$$\implies \quad d^2 V(\gamma(t)) \quad \text{is the control.}$$
A controllability problem on $\text{Sp}(m)$

The differential of the Poincaré map $P_{\tilde{g}}(\Sigma_0, \Sigma_T, \gamma_{\theta_0})$ at $\theta_0$ associated with the metric $\tilde{g} = e^{f}g$ is given by $X_u(T)$ where $X_u : [0, T] \to \text{Sp}(m)$ is solution to the control problem

\[
\begin{cases}
\dot{X}_u(t) = A(t)X_u(t) + \sum_{i \leq j = 1}^{m} u_{ij}(t)E(ij)X_u(t), & \forall t \in [0, T], \\
X(0) = I_{2m},
\end{cases}
\]

where the $2m \times 2m$ matrices $E(ij)$ are defined by

\[
E(ij) := \begin{pmatrix} 0 & 0 \\ E(ij) & 0 \end{pmatrix},
\]

with

\[
\begin{cases}
(E(ij))_{k,l} := \delta_{ik}\delta_{il} \forall i = 1, \ldots, m, \\
(E(ij))_{k,l} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \forall i < j = 1, \ldots, m.
\end{cases}
\]
Local controllability on $\text{Sp}(m)$

We are considering a bilinear control system on $M_{2m}(\mathbb{R})$ of the form

$$\dot{X}(t) = A(t)X(t) + \sum_{i=1}^{k} u_i(t)B_iX(t) \quad \forall t \in [0, T].$$
Local controllability on $\text{Sp}(m)$

We are considering a bilinear control system on $M_{2m}(\mathbb{R})$ of the form

$$\dot{X}(t) = A(t)X(t) + \sum_{i=1}^{k} u_i(t)B_iX(t) \quad \forall t \in [0, T].$$

Moreover, if we assume that $A(t), B_1, \ldots, B_k$ satisfy

$$\mathbb{J}A(t), \mathbb{J}B_1, \ldots, \mathbb{J}B_k \in S(2m) \quad \forall t \in [0, T],$$

then any solution $X : [0, T] \rightarrow M_{2m}(\mathbb{R})$ starting at $\bar{X} \in \text{Sp}(m)$ satisfies

$$X(t) \in \text{Sp}(m) \quad \forall t \in [0, T].$$
Local controllability on $\text{Sp}(m)$

**Proposition**

Define the $k$ sequences of smooth mappings

$$\{B_1^j\}, \ldots, \{B_k^j\} : [0, T] \to T_{l_2m}\text{Sp}(m)$$

by

$$\begin{align*}
B_0^i(t) &:= B_i, \\
B_i^j(t) &:= B_i^{j-1}(t) + B_i^{j-1}(t)A(t) - A(t)B_i^{j-1}(t),
\end{align*}$$

for every $t \in [0, T]$ and every $i \in \{1, \ldots, k\}$. Assume that there exists some $\bar{t} \in [0, T]$ such that

$$\text{Span}\left\{B_i^j(\bar{t}) \mid i \in \{1, \ldots, k\}, j \in \mathbb{N}\right\} = T_{l_2m}\text{Sp}(m).$$

Then for every $\bar{X} \in \text{Sp}(m)$, the control system is controllable at first order around $\bar{u} \equiv 0$. 

Ludovic Rifford  
Franks’ Lemma for Mañé perturbations..
Sketch of proof.

Let $\bar{X} \in \text{Sp}(m)$ be fixed, we define the mapping $E : L^2([0, T], \mathbb{R}^k) \to M_{2m}(\mathbb{R})$ by

$$E(u) := X_u(T) \quad \forall u \in L^2([0, T], \mathbb{R}^k),$$

where $X_u$ is the solution to the control system starting at $\bar{X}$.  

If $E$ is not a submersion at $\bar{u} \equiv 0$, then there is a nonzero matrix $Y$ such that

$$X_0(T)JY \in S(2m)$$

and

$$\text{Tr} (Y^* D_0 E(v)) = 0 \quad \forall v \in L^2([0, T], \mathbb{R}^k).$$

The latter can be written as

$$(\sum_{i=1}^k \int_T^0 v_i(t) \text{Tr}(Y^* S(T)S(t) - 1^i B_i X_0(t)) \, dt = 0 \quad \forall v.$$
Sketch of proof.

Let \( \bar{X} \in \text{Sp}(m) \) be fixed, we define the mapping

\[
E : L^2([0, T], \mathbb{R}^k) \to M_{2m}(\mathbb{R})
\]

by

\[
E(u) := X_u(T) \quad \forall u \in L^2([0, T], \mathbb{R}^k),
\]

where \( X_u \) is the solution to the control system starting at \( \bar{X} \). If \( E \) is not a submersion at \( \bar{u} \equiv 0 \), then there is a nonzero matrix \( Y \) such that \( X_0(T)JY \in S(2m) \) and

\[
\text{Tr} \left( Y^* D_0 E(v) \right) = 0 \quad \forall v \in L^2([0, T], \mathbb{R}^k).
\]
Local controllability on $Sp(m)$

Sketch of proof.

Let $\bar{X} \in Sp(m)$ be fixed, we define the mapping $E : L^2([0, T], \mathbb{R}^k) \to M_{2m}(\mathbb{R})$ by

$$E(u) := X_u(T) \quad \forall u \in L^2([0, T], \mathbb{R}^k),$$

where $X_u$ is the solution to the control system starting at $\bar{X}$. If $E$ is not a submersion at $\bar{u} \equiv 0$, then there is a nonzero matrix $Y$ such that $X_0(T)\mathbb{J}Y \in S(2m)$ and

$$\text{Tr} (Y^*D_0E(v)) = 0 \quad \forall v \in L^2([0, T], \mathbb{R}^k).$$

The latter can be written as (with $\dot{S} = AS$, $S(0) = I_{2m}$)

$$\sum_{i=1}^k \int_0^T v_i(t) \text{Tr} (Y^*S(T)S(t)^{-1}B_iX_0(t)) \, dt = 0 \quad \forall v.$$
In our case, we have

\[
\begin{align*}
\dot{X}_u(t) &= A(t)X_u(t) + \sum_{i \leq j = 1}^m u_{ij}(t)E(ij)X_u(t), \quad \forall t \in [0, T], \\
X(0) &= I_{2m},
\end{align*}
\]

with

\[
A(t) = \begin{pmatrix} 0 & I_m \\ -K(t) & 0 \end{pmatrix} \quad \text{and} \quad E(ij) := \begin{pmatrix} 0 & 0 \\ E(ij) & 0 \end{pmatrix}.
\]
In our case, we have

\[
\begin{align*}
\dot{X}_u(t) &= A(t)X_u(t) + \sum_{i \leq j = 1}^m u_{ij}(t)\mathcal{E}(ij)X_u(t), \quad \forall t \in [0, T], \\
X(0) &= I_{2m},
\end{align*}
\]

with

\[
A(t) = \begin{pmatrix}
0 & I_m \\
-K(t) & 0
\end{pmatrix}
\quad \text{and} \quad
\mathcal{E}(ij) := \begin{pmatrix}
0 & 0 \\
E(ij) & 0
\end{pmatrix}.
\]

**Corollary (Contreras-Paternain, Contreras, Visscher, Lazrag)**

Assume that there is \( \bar{t} \in [0, T] \) such that the \( m \times m \) symmetric matrix \( K \) has simple eigenvalues, then the Franks' Lemma for Mané perturbations holds at first order.
In our case, we have

\[
\begin{align*}
\dot{X}_u(t) &= A(t)X_u(t) + \sum_{i\leq j=1}^m u_{ij}(t)E(ij)X_u(t), \quad \forall t \in [0, T], \\
X(0) &= I_{2m},
\end{align*}
\]

with

\[
A(t) = \begin{pmatrix} 0 & I_m \\ -K(t) & 0 \end{pmatrix} \quad \text{and} \quad E(ij) := \begin{pmatrix} 0 & 0 \\ E(ij) & 0 \end{pmatrix}.
\]

**Corollary (Contreras-Paternain, Contreras, Visscher, Lazrag)**

Assume that there is \( \bar{t} \in [0, T] \) such that the \( m \times m \) symmetric matrix \( K \) has simple eigenvalues, then the Franks’ Lemma for Mané perturbations holds at first order.

What happens if the algebraic condition on \( K \) is not satisfied?
Local controllability on $\text{Sp}(m)$

**Proposition**

Assume that $B_i B_j = 0$ for all $i, j$ and define the $k$ sequences of smooth mappings $\{B^1_j\}, \ldots, \{B^k_j\} : [0, T] \rightarrow T_{I_2m} \text{Sp}(m)$ as before. If the following properties are satisfied with $\bar{t} = 0$:

$$\left[ B^i_j(\bar{t}), B_i \right] \in \text{Span}\left\{ B^s_r(\bar{t}) \mid r = 1, \ldots, k, s \geq 0 \right\} \quad \forall i, \forall j = 1, 2,$$

and

$$\text{Span}\left\{ B^i_j(\bar{t}), [B^1_i(\bar{t}), B^1_i(\bar{t})] \mid i, l = 1, \ldots, k \text{ and } j = 0, 1, 2 \right\} = T_{I_2m} \text{Sp}(m).$$

Then, for every $\bar{X} \in \text{Sp}(m)$, the control system is controllable at second order around $\bar{u} \equiv 0$. 

---

Ludovic Rifford  
Franks’ Lemma for Mañé perturbations..
If $Q : \mathcal{U} \to \mathbb{R}$ is a quadratic form, its negative index is defined by

$$\text{ind}_-(Q) := \max\left\{\dim(L) \mid Q|_{L\setminus\{0\}} < 0\right\}.$$ 

**Theorem**

Let $F : \mathcal{U} \to \mathbb{R}^N$ be a mapping of class $C^2$ on an open set $\mathcal{U} \subset X$ and $\bar{u} \in \mathcal{U}$ be a critical point of $F$ of corank $r$. If

$$\text{ind}_- \left( \lambda^* \left( D_{\bar{u}}^2 F \right) | \ker(D_{\bar{u}}F) \right) \geq r \quad \forall \lambda \in \left( \text{Im}(D_{\bar{u}}F) \right)^\perp \setminus \{0\},$$

then the mapping $F$ is locally open at second order at $\bar{u}$. 

Ludovic Rifford

Franks’ Lemma for Mañé perturbations.
Applications of Franks’ Lemma

Theorem

Let $(M, g)$ be a smooth compact Riemannian manifold of dimension $\geq 2$ such that the periodic orbits of the geodesic flow are $C^2$-persistently hyperbolic from Mañé’s viewpoint. Then the closure of the set of periodic orbits of the geodesic flow is a hyperbolic set.
Applications of Franks’ Lemma

Theorem

Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(\geq 2\) such that the periodic orbits of the geodesic flow are \(C^2\)-persistently hyperbolic from Mañé’s viewpoint. Then the closure of the set of periodic orbits of the geodesic flow is a hyperbolic set.

Corollary

Let \((M, g)\) be a smooth compact Riemannian manifold, suppose that either \(M\) is a surface or \(\dim M \geq 3\) and \((M, g)\) has no conjugate points. Assume that the geodesic flow is \(C^2\) persistently expansively from Mañé’s viewpoint, then the geodesic flow is Anosov.
Thank you for your attention !!