Mañé's Conjecture from the control viewpoint

Ludovic Rifford

Université de Nice - Sophia Antipolis

Setting

Let M be a smooth compact manifold of dimension $n \geq 2$ be fixed. Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class C^k, with $k \geq 2$, satisfying the following properties:

(H1) **Superlinear growth:**
For every $K \geq 0$, there is $C^*(K) \in \mathbb{R}$ such that

$$H(x,p) \geq K|p| + C^*(K) \qquad \forall (x,p) \in T^*M.$$

(H2) **Uniform convexity:**
For every $(x,p) \in T^*M$, $\frac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite.

Critical value of H

Definition

We call **critical value** of H the constant $c = c[H]$ defined as

$$c[H] := \inf_{u \in C^1(M;\mathbb{R})} \left\{ \max_{x \in M} \left\{ H\big(x, du(x)\big) \right\} \right\}.$$

In other terms, $c[H]$ is the infimum of numbers $c \in \mathbb{R}$ such that there is a C^1 function $u : M \to \mathbb{R}$ satisfying

$$H\big(x, du(x)\big) \leq c \qquad \forall x \in M.$$

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Note that

$$\min_{x \in M} \left\{ H(x, 0) \right\} \leq c[H] \leq \max_{x \in M} \left\{ H(x, 0) \right\}.$$

Critical subsolutions of H

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Let $L : TM \to \mathbb{R}$ be the Tonelli Lagrangian associated with H by Legendre-Fenchel duality, that is

$$L(x, v) := \max_{p \in T_x^* M} \Big\{ p \cdot v - H(x, p) \Big\} \qquad \forall (x, v) \in TM.$$

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A Lipschitz function $u : M \to \mathbb{R}$ is a critical subsolution if and only if

$$u\big(\gamma(b)\big) - u\big(\gamma(a)\big) \leq \int_a^b L\big(\gamma(t), \dot{\gamma}(t)\big)\, ds + c\,(b - a),$$

for every Lipschitz curve $\gamma : [a, b] \to M$.

The Fathi-Siconolfi-Bernard Theorem

Theorem (Fathi-Siconolfi, 2004; Bernard, 2007)

The set $\mathcal{SS}^1(H)$ (resp. $\mathcal{SS}^{1,1}(H)$) of critical subsolutions of class C^1 (resp. $C^{1,1}$) is nonempty.

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As a consequence, the set of $x \in M$ such that

$$u \text{ critical subsolution of class } C^1 \quad \Longrightarrow \quad H\big(x, du(x)\big) = c[H]$$

is **nonempty**.

Projected Aubry set and Aubry set

Definition and Proposition

- The **projected Aubry set** of H defined as

$$\mathcal{A}(H) = \left\{ x \in M \,|\, H\big(x, du(x)\big) = c[H], \, \forall u \in \mathcal{SS}^1(H) \right\},$$

is compact and nonempty.

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- Any critical subsolution u is C^1 at any point of $\mathcal{A}(H)$ and satisfies $H\big(x, du(x)\big) = c[H], \forall x \in \mathcal{A}(H)$.
- For every $x \in \mathcal{A}(H)$, the differential of a critical subsolution at x does not depend on u.
- The **Aubry set** of H defined by

$$\tilde{\mathcal{A}(H)} := \left\{ \big(x, du(x)\big) \,|\, x \in \mathcal{A}(H), u \text{ crit. subsol.} \right\} \subset T^*M$$

  is compact, invariant by ϕ_t^H, and is a Lipschitz graph over $\mathcal{A}(H)$.

Back to the critical value

Proposition

The critical value of H satisfies

$$c[H] = \min_{u \in C^1(M;\mathbb{R})} \left\{ \max_{x \in M} \left\{ H\big(x, du(x)\big) \right\} \right\}.$$

Proposition

The critical value of H satisfies

$$c[H] = -\inf \left\{ \frac{1}{T} \int_0^T L\big(\gamma(t), \dot{\gamma}(t)\big)\, dt \right\},$$

where the infimum is taken over the Lipschitz curves $\gamma : [0, T] \to M$ such that $\gamma(0) = \gamma(T)$.

Examples

- Let $V : M \to \mathbb{R}$ be a potential of class C^2 and $H : T^*M \to \mathbb{R}$ be the Hamiltonian defined by

$$H(x,p) = \frac{1}{2}|p|^2 + V(x) \qquad \forall (x,p) \in T^*M.$$

Then $c[H] = \max_M V$ and

$$\tilde{\mathcal{A}}(H) = \left\{ (x,0) \,|\, V(x) = \max_M V \right\}.$$

- Let X be a smooth vector field on M and $L : TM \to \mathbb{R}$ defined by

$$L_X(x,v) = \frac{1}{2}|v - X(x)|^2 \qquad \forall (x,v) \in TM.$$

Then $c[H] = 0$ and the projected Aubry set always contains the set of recurrent points of the flow of X.

The Mañé Conjecture

Conjecture (Mañé, 96)

For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^k (with $k \geq 2$), there is a residual subset (i.e., a countable intersection of open and dense subsets) $\mathcal{G}$ of $C^k(M)$ such that, for every $V \in \mathcal{G}$, the Aubry set of the Hamiltonian $H_V := H + V$ is either an equilibrium point or a periodic orbit.

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Strategy of proof:

- Density result.
- Stability result.

Mañé's density Conjecture

Proposition (Contreras-Iturriaga, 1999)

Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class C^k (with $k \geq 3$) whose Aubry set is an equilibrium point (resp. a periodic orbit). Then, there is a smooth potential $V : M \to \mathbb{R}$, with $\|V\|_{C^k}$ as small as desired, such that the Aubry set of H_V is a hyperbolic equilibrium (resp. a hyperbolic periodic orbit).

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Conjecture (Mañé's density conjecture)

For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^k (with $k \geq 2$) there exists a dense set $\mathcal{D}$ in $C^k(M)$ such that, for every $V \in \mathcal{D}$, the Aubry set of the Hamiltonian H_V is either an equilibrium point or a periodic orbit.

The C^1 closing Lemma

Theorem (Pugh, 1967)

Let M be a smooth compact manifold. Suppose that some vector field X has a nontrivial recurrent trajectory through $x \in M$ and suppose that $\mathcal{U}$ is a neighborhood of X in the C^1 topology. Then there exists $Y \in \mathcal{U}$ such that Y has a closed orbit through x.

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Theorem (Pugh-Robinson, 1983)

Let (N, ω) be a symplectic manifold of dimension $2n \geq 2$ and $H : N \to \mathbb{R}$ be a given Hamiltonian of class C^2. Let X be the Hamiltonian vector field associated with H and ϕ^H the Hamiltonian flow. Suppose that X has a nontrivial recurrent trajectory through $x \in N$ and that $\mathcal{U}$ is a neighborhood of X in the C^1 topology. Then there exists $Y \in \mathcal{U}$ such that Y is a Hamiltonian vector field and Y has a closed orbit through x.

Strategy of proof

Given a "Tonelli" Hamiltonian, we need to find:

- a potential $V : M \to \mathbb{R}$ **small**,
- a periodic orbit $\gamma : [0, T] \to M$ $(\gamma(0) = \gamma(T))$,
- a Lipschitz function $v : M \to \mathbb{R}$,

in such a way that the following properties are satisfied:

- $H_V\big(x, dv(x)\big) \leq 0$ for a.e. $x \in M$,
- $\int_0^T L_V\big(\gamma(t), \dot{\gamma}(t)\big)\, dt = 0.$

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- $\int_0^T L_V\big(\gamma(t), \dot{\gamma}(t)\big)\, dt = 0.$ $(\Rightarrow c[H_V] \geq 0)$

A connecting problem

Let be given two solutions

$$\big(x_i, p_i\big) \ : \ [0,\tau] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \qquad i = 1, 2,$$

of the Hamiltonian system

$$\left\{ \begin{array}{lcl} \dot{x}(t) & = & \nabla_p H\big(x(t), p(t)\big) \\ \dot{p}(t) & = & -\nabla_x H\big(x(t), p(t)\big). \end{array} \right.$$

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Question

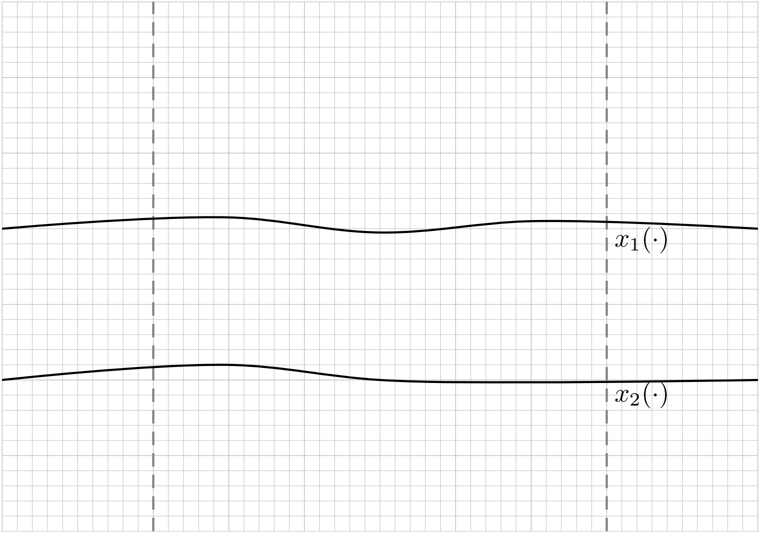
Can I add a potential V to the Hamiltonian H in such a way that the solution of the new Hamiltonian system

$$\begin{cases} \dot{x}(t) = \nabla_p H\big(x(t), p(t)\big) \\ \dot{p}(t) = -\nabla_x H\big(x(t), p(t)\big) - \nabla V\big(x(t)\big), \end{cases}$$

starting at $\big(x_1(0), p_1(0)\big)$ satisfies

$$\big(x(\tau), p(\tau)\big) = \big(x_2(\tau), p_2(\tau)\big)?$$

Picture



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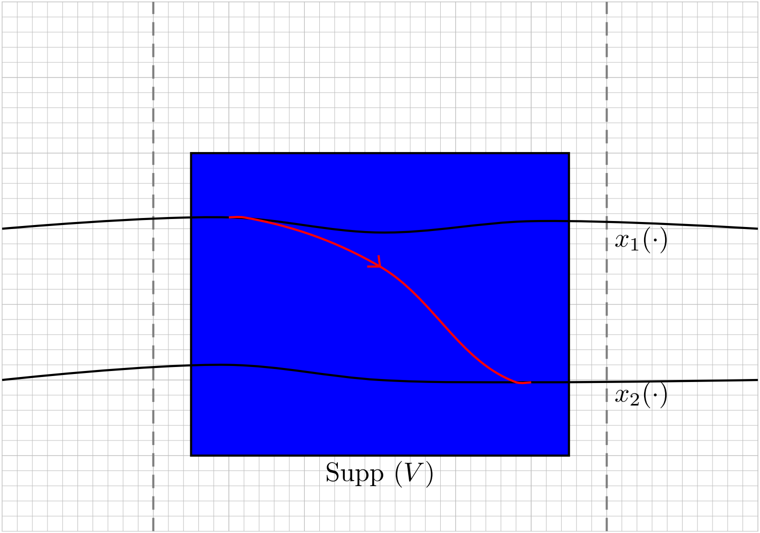


Control approach

Study the mapping

$$\begin{array}{rcl} E \ : \ L^1([0,\tau];\mathbb{R}^n) & \longrightarrow & \mathbb{R}^n \times \mathbb{R}^n \\ u & \longmapsto & \big(x_u(\tau), p_u(\tau)\big) \end{array}$$

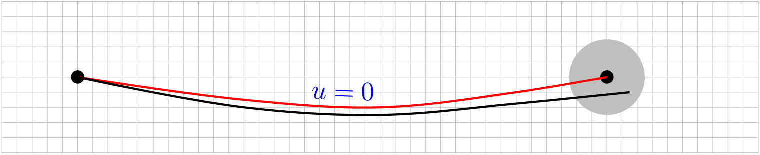
where

$$\big(x_u, p_u\big) : [0,\tau] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$$

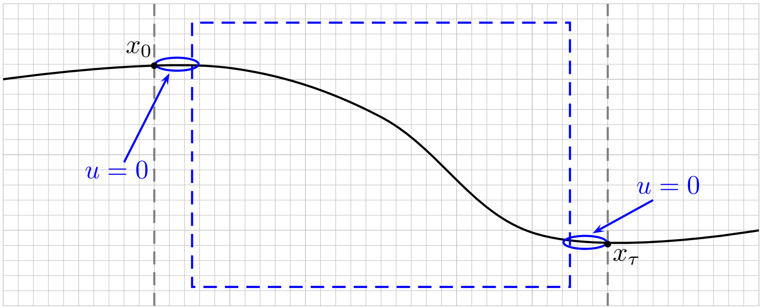
is the solution of

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \nabla_p H\big(x(t), p(t)\big) \\ \dot{p}(t) & = & -\nabla_x H\big(x(t), p(t)\big) - u(t), \end{array} \right.$$

starting at $\big(x_1(0), p_1(0)\big)$.



Exercise



Exercise

Given $x, u : [0, \tau] \to \mathbb{R}^n$ *as above, does there exists a function* $V : \mathbb{R}^n \to \mathbb{R}$ *whose the support is included in the dashed blue square above and such that*

$$\nabla V(x(t)) = u(t) \qquad \forall t \in [0, \tau]\,?$$

Exercise (solution)

There is a necessary condition

$$\int_0^\tau \langle \dot{x}(t), u(t) \rangle dt = 0.$$

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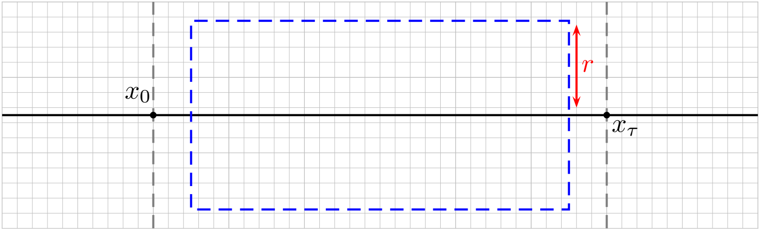
Proposition

If the above necessary condition is satisfied, then there is $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the desired properties such that

$$\|V\|_{C^1} \leq \frac{K}{r} \|u\|_{\infty}.$$

Exercise (solution)

If $x(t) = (t, 0)$, that is



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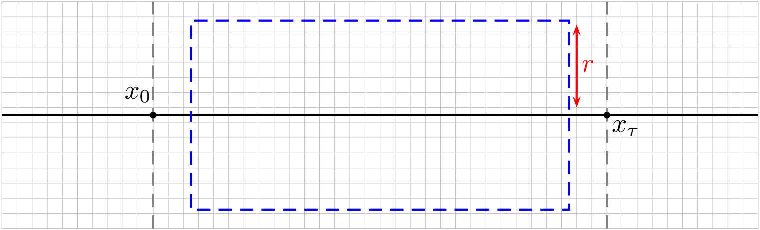

then we set

$$V(t,y) := \phi(|y|/r) \left[\int_0^t u_1(s)\,ds + \sum_{i=1}^{n-1} \int_0^{y_i} u_{i+1}(t+s)\,ds \right],$$

for every (t, y), with $\phi : [0, \infty) \to [0, 1]$ satisfying

$$\phi(s) = 1 \quad \forall s \in [0, 1/3] \quad \text{and} \quad \phi(s) = 0 \quad \forall s \geq 2/3.$$

Closing Aubry sets in C^1 topology

Theorem (Figalli-R, 2010)

Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^k with $k \geq 4$, and fix $\epsilon > 0$. Then there exists a potential $V : M \to \mathbb{R}$ of class C^{k-2}, with $\|V\|_{C^1} < \epsilon$, such that $c[H_V] = c[H]$ and the Aubry set of H_V is either an (hyperbolic) equilibrium point or a (hyperbolic) periodic orbit.

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The above result is not satisfactory. The property "having an Aubry set which is an hyperbolic closed orbit" is not stable under C^1 perturbations.

Toward a proof of Mañé's Conjecture in C^2 topology

Theorem (Figalli-R, 2010)

Assume that $\dim M \geq 3$. *Let* $H : T^*M \to \mathbb{R}$ *be a Tonelli Hamiltonian of class* C^k *with* $k \geq 4$, *and fix* $\epsilon > 0$. *Assume that there are a recurrent point* $\bar{x} \in \mathcal{A}(H)$, *a critical viscosity subsolution* $u : M \to \mathbb{R}$, *and an open neighborhood* $\mathcal{V}$ *of* $\mathcal{O}^+(\bar{x})$ *such that*

$$u \text{ is at least } C^{k+1} \text{ on } \mathcal{V}.$$

Then there exists a potential $V : M \to \mathbb{R}$ *of class* C^{k-1}, *with* $\|V\|_{C^2} < \epsilon$, *such that* $c[H_V] = c[H]$ *and the Aubry set of* H_V *is either an (hyperbolic) equilibrium point or a (hyperbolic) periodic orbit.*

Application to Mañé's Lagrangians

Recall that given X a C^k-vector field on M with $k \geq 2$, the Mañé Lagrangian $L_X : TM \to \mathbb{R}$ associated to X is defined by

$$L_X(x,v) := \frac{1}{2}\left\|v - X(x)\right\|_x^2 \qquad \forall\,(x,v) \in TM,$$

while the Mañé Hamiltonian $H_X : TM \to \mathbb{R}$ is given by

$$H_X(x,p) = \frac{1}{2}\left\|p\right\|_x^2 + \langle p, X(x)\rangle \qquad \forall\,(x,p) \in T^*M.$$

Corollary (Figalli-R, 2010)

Let X be a vector field on M of class C^k with $k \geq 2$. Then for every $\epsilon > 0$ there is a potential $V : M \to \mathbb{R}$ of class C^k, with $\|V\|_{C^2} < \epsilon$, such that the Aubry set of $H_X + V$ is either an equilibrium point or a periodic orbit.

Discussion

Theorem (Bernard, 2007)

Assume that the Aubry set is exactly one hyperbolic periodic orbit, then any critical solution is "smooth" in a neighborhood of $\mathcal{A}(H)$. As a consequence, there is a "smooth" critical subsolution.

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Conjecture (Regularity Conjecture for critical subsolutions)

For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^∞ there is a set $\mathcal{D} \subset C^\infty(M)$ which is dense in $C^2(M)$ (with respect to the C^2 topology) such that the following holds: For every $V \in \mathcal{D}$, there is a smooth critical subsolution.

Thank you for your attention !!