Mañé’s Conjecture from the control viewpoint

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Let $M$ be a smooth compact manifold of dimension $n \geq 2$ be fixed. Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class $C^k$, with $k \geq 2$, satisfying the following properties:

**H1)** Superlinear growth:
For every $K \geq 0$, there is $C^*(K) \in \mathbb{R}$ such that

$$H(x, p) \geq K|p| + C^*(K) \quad \forall (x, p) \in T^*M.$$

**H2)** Uniform convexity:
For every $(x, p) \in T^*M$, $\frac{\partial^2 H}{\partial p^2}(x, p)$ is positive definite.
Critical value of $H$

**Definition**

We call **critical value** of $H$ the constant $c = c[H]$ defined as

$$c[H] := \inf_{u \in C^1(M; \mathbb{R})} \left\{ \max_{x \in M} \left\{ H(x, du(x)) \right\} \right\}.$$  

In other terms, $c[H]$ is the infimum of numbers $c \in \mathbb{R}$ such that there is a $C^1$ function $u : M \to \mathbb{R}$ satisfying

$$H(x, du(x)) \leq c \quad \forall x \in M.$$
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$$H(x, du(x)) \leq c \quad \forall x \in M.$$ 

Note that

$$\min_{x \in M} \{ H(x, 0) \} \leq c[H] \leq \max_{x \in M} \{ H(x, 0) \}.$$
Critical subsolutions of $H$

**Definition**

We call **critical subsolution** any Lipschitz function $u : M \rightarrow \mathbb{R}$ such that

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Let $L : TM \rightarrow \mathbb{R}$ be the Tonelli Lagrangian associated with $H$ by Legendre-Fenchel duality, that is

$$L(x, v) := \max_{p \in T^*_x M} \left\{ p \cdot v - H(x, p) \right\} \quad \forall (x, v) \in TM.$$
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A Lipschitz function $u : M \rightarrow \mathbb{R}$ is a critical subsolution if and only if

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(t), \dot{\gamma}(t)) \, ds + c(b - a),$$

for every Lipschitz curve $\gamma : [a, b] \rightarrow M$. 

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The Fathi-Siconolfi-Bernard Theorem

**Theorem (Fathi-Siconolfi, 2004; Bernard, 2007)**

The set $SS^1(H)$ (resp. $SS^{1,1}(H)$) of critical subsolutions of class $C^1$ (resp. $C^{1,1}$) is nonempty.
The Fathi-Siconolfi-Bernard Theorem

Theorem (Fathi-Siconolfi, 2004; Bernard, 2007)

The set $SS^1(H)$ (resp. $SS^{1,1}(H)$) of critical subsolutions of class $C^1$ (resp. $C^{1,1}$) is nonempty.

As a consequence, the set of $x \in M$ such that

$$u \text{ critical subsolution of class } C^1 \implies H(x, du(x)) = c[H]$$

is nonempty.
The **projected Aubry set** of $H$ defined as

$$\mathcal{A}(H) = \{ x \in M \mid H(x, du(x)) = c[H], \forall u \in SS^1(H) \},$$

is compact and nonempty.
Projected Aubry set and Aubry set

Definition and Proposition

- The **projected Aubry set** of $H$ defined as

$$\mathcal{A}(H) = \{ x \in M \mid H(x, du(x)) = c[H], \forall u \in SS^1(H) \} ,$$

is compact and nonempty.

- Any critical subsolution $u$ is $C^1$ at any point of $\mathcal{A}(H)$ and satisfies $H(x, du(x)) = c[H], \forall x \in \mathcal{A}(H)$. 
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- For every $x \in \mathcal{A}(H)$, the differential of a critical subsolution at $x$ does not depend on $u$. 

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The **projected Aubry set** of $H$ defined as

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\mathcal{A}(H) = \left\{ x \in M \mid H(x, du(x)) = c[H], \forall u \in SS^1(H) \right\},
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is compact and nonempty.

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For every $x \in \mathcal{A}(H)$, the differential of a critical subsolution at $x$ does not depend on $u$.

The **Aubry set** of $H$ defined by

$$
\mathcal{A}(H) := \left\{ \left( x, du(x) \right) \mid x \in \mathcal{A}(H), u \text{ crit. subsol.} \right\} \subset T^*M
$$

is compact, invariant by $\phi^H_t$, and is a Lipschitz graph over $\mathcal{A}(H)$.
Proposition

The critical value of $H$ satisfies

$$c[H] = \min_{u \in C^1(M; \mathbb{R})} \left\{ \max_{x \in M} \left\{ H(x, du(x)) \right\} \right\}.$$  

Proposition

The critical value of $H$ satisfies

$$c[H] = -\inf \left\{ \frac{1}{T} \int_0^T L(\gamma(t), \gamma'(t)) \, dt \right\},$$

where the infimum is taken over the Lipschitz curves $\gamma : [0, T] \to M$ such that $\gamma(0) = \gamma(T)$. 

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Examples

Let $V : M \to \mathbb{R}$ be a potential of class $C^2$ and $H : T^*M \to \mathbb{R}$ be the Hamiltonian defined by

$$H(x, p) = \frac{1}{2} |p|^2 + V(x) \quad \forall (x, p) \in T^*M.$$ 

Then $c[H] = \max_M V$ and

$$\tilde{A}(H) = \left\{ (x, 0) \mid V(x) = \max_M V \right\}.$$ 

Let $X$ be a smooth vector field on $M$ and $L : TM \to \mathbb{R}$ defined by

$$L_X(x, v) = \frac{1}{2} |v - X(x)|^2 \quad \forall (x, v) \in TM.$$ 

Then $c[H] = 0$ and the projected Aubry set always contains the set of recurrent points of the flow of $X$. 
Conjecture (Mañé, 96)

For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class $C^k$ (with $k \geq 2$), there is a residual subset (i.e., a countable intersection of open and dense subsets) $\mathcal{G}$ of $C^k(M)$ such that, for every $V \in \mathcal{G}$, the Aubry set of the Hamiltonian $H_V := H + V$ is either an equilibrium point or a periodic orbit.
The Mañé Conjecture

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Strategy of proof:
- Density result.
- Stability result.
Mañé’s density Conjecture

Proposition (Contreras-Iturriaga, 1999)

Let $H : T^* M \to \mathbb{R}$ be a Hamiltonian of class $C^k$ (with $k \geq 3$) whose Aubry set is an equilibrium point (resp. a periodic orbit). Then, there is a smooth potential $V : M \to \mathbb{R}$, with $\|V\|_{C^k}$ as small as desired, such that the Aubry set of $H_V$ is a hyperbolic equilibrium (resp. a hyperbolic periodic orbit).
Mañé’s density Conjecture

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Conjecture (Mañé’s density conjecture)

For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class $C^k$ (with $k \geq 2$) there exists a dense set $\mathcal{D}$ in $C^k(M)$ such that, for every $V \in \mathcal{D}$, the Aubry set of the Hamiltonian $H_V$ is either an equilibrium point or a periodic orbit.
The $C^1$ closing Lemma

**Theorem (Pugh, 1967)**

Let $M$ be a smooth compact manifold. Suppose that some vector field $X$ has a nontrivial recurrent trajectory through $x \in M$ and suppose that $\mathcal{U}$ is a neighborhood of $X$ in the $C^1$ topology. Then there exists $Y \in \mathcal{U}$ such that $Y$ has a closed orbit through $x$. 

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The $C^1$ closing Lemma

Theorem (Pugh, 1967)

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Theorem (Pugh-Robinson, 1983)

Let $(N, \omega)$ be a symplectic manifold of dimension $2n \geq 2$ and $H : N \to \mathbb{R}$ be a given Hamiltonian of class $C^2$. Let $X$ be the Hamiltonian vector field associated with $H$ and $\phi^H$ the Hamiltonian flow. Suppose that $X$ has a nontrivial recurrent trajectory through $x \in N$ and that $\mathcal{U}$ is a neighborhood of $X$ in the $C^1$ topology. Then there exists $Y \in \mathcal{U}$ such that $Y$ is a Hamiltonian vector field and $Y$ has a closed orbit through $x$. 

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Mañé's Conjecture from the control viewpoint
Given a "Tonelli" Hamiltonian, we need to find:

- a potential $V : M \rightarrow \mathbb{R}$ small,
- a periodic orbit $\gamma : [0, T] \rightarrow M \ (\gamma(0) = \gamma(T))$,
- a Lipschitz function $v : M \rightarrow \mathbb{R}$,

in such a way that the following properties are satisfied:

- $H_V(x, dv(x)) \leq 0$ for a.e. $x \in M$,
- $\int_0^T L_V(\gamma(t), \dot{\gamma}(t)) \, dt = 0$. 
Strategy of proof

Given a "Tonelli" Hamiltonian, we need to find:

- a potential $V : M \to \mathbb{R}$ small,
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in such a way that the following properties are satisfied:

- $H_V(x, dv(x)) \leq 0$ for a.e. $x \in M$, ($\Rightarrow c[H_V] \leq 0$)
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Strategy of proof

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- a potential $V : M \to \mathbb{R}$ small,
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in such a way that the following properties are satisfied:

- $H_V(x, dv(x)) \leq 0$ for a.e. $x \in M$, ($\Rightarrow c[H_V] \leq 0$)
- $\int_0^T L_V(\gamma(t), \dot{\gamma}(t)) \, dt = 0$. ($\Rightarrow c[H_V] \geq 0$)
A connecting problem

Let be given two solutions

\[(x_i, p_i) : [0, \tau] \rightarrow \mathbb{R}^n \times \mathbb{R}^n \quad i = 1, 2,\]

of the Hamiltonian system

\[
\begin{align*}
\dot{x}(t) &= \nabla_p H(x(t), p(t)) \\
\dot{p}(t) &= -\nabla_x H(x(t), p(t)).
\end{align*}
\]
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\end{align*}
\]

Question

Can I add a potential \(V\) to the Hamiltonian \(H\) in such a way that the solution of the new Hamiltonian system

\[
\begin{align*}
\dot{x}(t) &= \nabla_p H(x(t), p(t)) \\
\dot{p}(t) &= -\nabla_x H(x(t), p(t)) - \nabla V(x(t)),
\end{align*}
\]

starting at \((x_1(0), p_1(0))\) satisfies

\[(x(\tau), p(\tau)) = (x_2(\tau), p_2(\tau))?\]
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Control approach

Study the mapping

\[ E : L^1([0, \tau]; \mathbb{R}^n) \rightarrow \mathbb{R}^n \times \mathbb{R}^n \]

\[ u \mapsto (x_u(\tau), p_u(\tau)) \]

where

\[ (x_u, p_u) : [0, \tau] \rightarrow \mathbb{R}^n \times \mathbb{R}^n \]

is the solution of

\[
\begin{cases}
\dot{x}(t) = \nabla_p H(x(t), p(t)) \\
\dot{p}(t) = -\nabla_x H(x(t), p(t)) - u(t),
\end{cases}
\]

starting at \((x_1(0), p_1(0))\).
Exercise

Given $x, u : [0, \tau] \rightarrow \mathbb{R}^n$ as above, does there exists a function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ whose the support is included in the dashed blue square above and such that

$$\nabla V(x(t)) = u(t) \quad \forall t \in [0, \tau]?$$
Exercise (solution)

There is a necessary condition

\[ \int_0^T \langle \dot{x}(t), u(t) \rangle dt = 0. \]
Exercise (solution)

There is a necessary condition

$$\int_{0}^{T} \langle \dot{x}(t), u(t) \rangle dt = 0.$$ 

As a matter of fact,

$$\int_{0}^{T} \langle \dot{x}(t), u(t) \rangle dt = \int_{0}^{T} \langle \dot{x}(t), \nabla V(x(t)) \rangle dt = V(x_{\tau}) - V(x_{0}) = 0.$$
Exercise (solution)

There is a necessary condition

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As a matter of fact,

$$\int_0^{\tau} \langle \dot{x}(t), u(t) \rangle dt = \int_0^{\tau} \langle \dot{x}(t), \nabla V(x(t)) \rangle dt = V(x_{\tau}) - V(x_0) = 0.$$

Proposition

*If the above necessary condition is satisfied, then there is $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the desired properties such that*

$$\| V \|_{C^1} \leq \frac{K}{r} \| u \|_{\infty}.$$
Exercise (solution)

If \( x(t) = (t, 0) \), that is

\[
V(t, y) := \phi \left( \frac{|y|}{r} \right) \left[ \int_{t_0}^{t} u_1(s) \, ds + (n-1) \sum_{i=1}^{\infty} \int_{y_i(t_0)}^{y_{i+1}(t+s)} u_i(t+s) \, ds \right ],
\]

for every \((t, y)\), with \( \phi : [0, \infty) \to [0, 1] \) satisfying \( \phi(s) = 1 \quad \forall \quad s \in [0, 1/3] \) and \( \phi(s) = 0 \quad \forall \quad s \geq 2/3 \).
Exercise (solution)

If \( x(t) = (t, 0) \), that is

then we set

\[
V(t, y) := \phi\left(\frac{|y|}{r}\right) \left[ \int_0^t u_1(s) \, ds + \sum_{i=1}^{n-1} \int_0^{y_i} u_{i+1}(t + s) \, ds \right],
\]

for every \((t, y)\), with \( \phi : [0, \infty) \to [0, 1] \) satisfying

\[
\phi(s) = 1 \quad \forall s \in [0, 1/3] \quad \text{and} \quad \phi(s) = 0 \quad \forall s \geq 2/3.
\]
Theorem (Figalli-R, 2010)

Let $H : T^* M \to \mathbb{R}$ be a Tonelli Hamiltonian of class $C^k$ with $k \geq 4$, and fix $\epsilon > 0$. Then there exists a potential $V : M \to \mathbb{R}$ of class $C^{k-2}$, with $\|V\|_{C^1} < \epsilon$, such that $c[H_V] = c[H]$ and the Aubry set of $H_V$ is either an (hyperbolic) equilibrium point or a (hyperbolic) periodic orbit.
Theorem (Figalli-R, 2010)

Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class $C^k$ with $k \geq 4$, and fix $\epsilon > 0$. Then there exists a potential $V : M \to \mathbb{R}$ of class $C^{k-2}$, with $\|V\|_{C^1} < \epsilon$, such that $c[H_V] = c[H]$ and the Aubry set of $H_V$ is either an (hyperbolic) equilibrium point or a (hyperbolic) periodic orbit.

The above result is not satisfactory. The property "having an Aubry set which is an hyperbolic closed orbit" is not stable under $C^1$ perturbations.
Toward a proof of Mañé’s Conjecture in $C^2$ topology

**Theorem (Figalli-R, 2010)**

Assume that $\dim M \geq 3$. Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class $C^k$ with $k \geq 4$, and fix $\epsilon > 0$. Assume that there are a recurrent point $\bar{x} \in A(H)$, a critical viscosity subsolution $u : M \to \mathbb{R}$, and an open neighborhood $\mathcal{V}$ of $O^+(\bar{x})$ such that

$$u \text{ is at least } C^{k+1} \text{ on } \mathcal{V}.$$ 

Then there exists a potential $V : M \to \mathbb{R}$ of class $C^{k-1}$, with $\|V\|_{C^2} < \epsilon$, such that $c[H_V] = c[H]$ and the Aubry set of $H_V$ is either an (hyperbolic) equilibrium point or a (hyperbolic) periodic orbit.
Recall that given $X$ a $C^k$-vector field on $M$ with $k \geq 2$, the Mañé Lagrangian $L_X : TM \to \mathbb{R}$ associated to $X$ is defined by

$$L_X(x, v) := \frac{1}{2} \| v - X(x) \|^2_x \quad \forall (x, v) \in TM,$$

while the Mañé Hamiltonian $H_X : TM \to \mathbb{R}$ is given by

$$H_X(x, p) = \frac{1}{2} \| p \|^2_x + \langle p, X(x) \rangle \quad \forall (x, p) \in T^* M.$$

**Corollary (Figalli-R, 2010)**

Let $X$ be a vector field on $M$ of class $C^k$ with $k \geq 2$. Then for every $\epsilon > 0$ there is a potential $V : M \to \mathbb{R}$ of class $C^k$, with $\| V \|_{C^2} < \epsilon$, such that the Aubry set of $H_X + V$ is either an equilibrium point or a periodic orbit.
**Theorem (Bernard, 2007)**

Assume that the Aubry set is exactly one hyperbolic periodic orbit, then any critical solution is "smooth" in a neighborhood of $A(H)$. As a consequence, there is a "smooth" critical subsolution.
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Assume that the Aubry set is exactly one hyperbolic periodic orbit, then any critical solution is ”smooth” in a neighborhood of $A(H)$. As a consequence, there is a ”smooth” critical subsolution.

Conjecture (Regularity Conjecture for critical subsolutions)

For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class $C^\infty$ there is a set $\mathcal{D} \subset C^\infty(M)$ which is dense in $C^2(M)$ (with respect to the $C^2$ topology) such that the following holds: For every $V \in \mathcal{D}$, there is a smooth critical subsolution.
Thank you for your attention!!