A Kupka-Smale Theorem for Hamiltonian systems from Mañé’s viewpoint, a control approach

Ludovic Rifford

Université de Nice - Sophia Antipolis
&
Institut Universitaire de France

PICOF ’12
Theorem (Kupka ’63, Smale ’63)

Let $M$ be a smooth compact manifold. For $C^k$ ($k \geq 1$) vector fields on $M$, the following properties are generic:

1. All closed orbits are hyperbolic.

2. Heteroclinic orbits are transversal, i.e. the intersections of stable and unstable manifolds of closed hyperbolic orbits are transversal.
Let $M$ be a smooth compact manifold and let $T^*M$ be its cotangent bundle equipped with the canonical symplectic form.

Let $H : T^*M \to \mathbb{R}$ be an Hamiltonian of class at least $C^2$ and $X_H$ be the associated Hamiltonian vector field which reads (in local coordinates)

$$X_H(x, p) = \left( \frac{\partial H}{\partial p}(x, p), -\frac{\partial H}{\partial x}(x, p) \right).$$
Let $M$ be a smooth compact manifold and let $T^*M$ be its cotangent bundle equipped with the canonical symplectic form.

Let $H : T^*M \rightarrow \mathbb{R}$ be an Hamiltonian of class at least $C^2$ and $X_H$ be the associated Hamiltonian vector field which reads (in local coordinates)

$$X_H(x, p) = \left( \frac{\partial H}{\partial p}(x, p), -\frac{\partial H}{\partial x}(x, p) \right).$$

**Definition**

Given an Hamiltonian $H : T^*M \rightarrow \mathbb{R}$, a property is called $C^k$ Mañé generic if there is a residual set $\mathcal{G}$ in $C^k(M; \mathbb{R})$ such that the property holds for any $H + V$ with $V \in \mathcal{G}$. 
Statement of the result

Theorem (Rifford-Ruggiero ’10)

Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class $C^k$ with $k \geq 2$. The following properties are $C^k$ Mañé generic:

1. Each closed orbit is either hyperbolic or no eigenvalue of the Poincaré transform of any closed orbit is a root of unity.

2. Heteroclinic orbits are transversal, i.e. the intersections of stable and unstable manifolds of closed hyperbolic orbits are transversal.

Recall that $H$ is Tonelli if it is superlinear and uniformly convex in the fibers.
Let $\bar{\theta} = (\bar{x}, \bar{p})$ be a periodic point for the Hamiltonian flow of positive period $T > 0$. Fix a local section transversal to the flow at $\bar{\theta}$ and contained in the energy level of $\bar{\theta}$.
Let \( \tilde{\theta} = (\tilde{x}, \tilde{p}) \) be a periodic point for the Hamiltonian flow of positive period \( T > 0 \). Fix a local section transversal to the flow at \( \tilde{\theta} \) and contained in the energy level of \( \tilde{\theta} \).

Then consider the **Poincaré first return map**

\[
P : \Sigma \quad \mapsto \quad \Sigma
\]

\[
\theta \quad \mapsto \quad \phi_{\tau(\theta)}^H(\theta),
\]

which is a local diffeomorphism and for which \( \tilde{\theta} \) is a fixed point.
Let $\bar{\theta} = (\bar{x}, \bar{p})$ be a periodic point for the Hamiltonian flow of positive period $T > 0$. Fix a local section transversal to the flow at $\bar{\theta}$ and contained in the energy level of $\bar{\theta}$.

Then consider the **Poincaré first return map**

$$P : \Sigma \to \Sigma$$

$$\theta \mapsto \phi^{H}_{\tau(\theta)}(\theta),$$

which is a local diffeomorphism and for which $\bar{\theta}$ is a fixed point.

The Poincaré map is symplectic, i.e. it preserves the restriction of the symplectic form to $T_{\theta}\Sigma$. 

Ludovic Rifford
The symplectic group

Let $\text{Sp}(m)$ be the symplectic group in $M_{2m}(\mathbb{R})$ ($m = n - 1$), that is the smooth submanifold of matrices $X \in M_{2m}(\mathbb{R})$ satisfying

$$X^*JX = J \quad \text{where } J := \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}.$$ 

Choosing a convenient set of coordinates, the differential of the Poincaré map is the symplectic matrix $X(T)$ where $X : [0, T] \rightarrow \text{Sp}(m)$ is solution to the Cauchy problem

$$\begin{cases} \dot{X}(t) = A(t)X(t) & \forall t \in [0, T], \\ X(0) = I_{2m}, \end{cases}$$

where $A(t)$ has the form

$$A(t) = \begin{pmatrix} 0 & I_m \\ -K(t) & 0 \end{pmatrix} \quad \forall t \in [0, T].$$
Let $\gamma$ be the projection of the periodic orbit passing through $\bar{\theta}$, we are looking for a potential $V: M \rightarrow \mathbb{R}$ satisfying the following properties

$$V(\gamma(t)) = 0, \quad dV(\gamma(t)) = 0,$$

with

$$d^2V(\gamma(t)) \text{ free.}$$
Let $\gamma$ be the projection of the periodic orbit passing through $\bar{\theta}$, we are looking for a potential $V : M \rightarrow \mathbb{R}$ satisfying the following properties

$$V(\gamma(t)) = 0, \quad dV(\gamma(t)) = 0,$$

with

$$d^2V(\gamma(t)) \text{ free.}$$

$$\Rightarrow d^2V(\gamma(t)) \text{ is the control.}$$
A controllability problem on $\text{Sp}(m)$

The Poincaré map at time $T$ associated with the new Hamiltonian

$$H + V$$

is given by $X(T)$ where $X : [0, T] \rightarrow \text{Sp}(m)$ is solution to the control problem

$$\begin{cases}
\dot{X}(t) = A(t)X(t) + \sum_{i \leq j = 1}^{m} u_{ij}(t)E(ij)X(t), & \forall t \in [0, T], \\
X(0) = I_{2m},
\end{cases}$$

where the $2m \times 2m$ matrices $E(ij)$ are defined by

$$E(ij) := \begin{pmatrix}
0 & 0 \\
E(ij) & 0
\end{pmatrix},$$

with

$$\begin{cases}
(E(ii))_{k,l} := \delta_{ik}\delta_{il} & \forall i = 1, \ldots, m, \\
(E(jj))_{k,l} := \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} & \forall i < j = 1, \ldots, m.
\end{cases}$$
Assume that there is $\bar{t} \in [0, T]$ such that

$$\dim \left( \text{Span} \left\{ [E(ij), K(\bar{t})] \mid i, j \in \{1, \ldots, m\}, i < j \right\} \right) = \frac{m(m-1)}{2}.$$ 

Then we can reach a neighborhood of $X_0(T)$. 

\[ \text{Lemma} \]
First-order controllability

**Lemma**

Assume that there is $\bar{t} \in [0, T]$ such that

$$\dim \left( \text{Span}\left\{ [E(ij), K(\bar{t})] \mid i, j \in \{1, \ldots, m\}, i < j \right\} \right) = \frac{m(m-1)}{2}.$$ 

Then we can reach a neighborhood of $X_0(T)$.

**Lemma**

The set of matrices $K \in S(m)$ such that

$$\dim \left( \text{Span}\left\{ [E(ij), K] \mid i, j \in \{1, \ldots, m\}, i < j \right\} \right) = \frac{m(m-1)}{2}$$

is open and dense in $S(m)$. 

Ludovic Rifford

A Kupka-Smale Theorem for Hamiltonian systems
Thank you for your attention !!