Optimal Transportation on Sub-Riemannian Manifolds

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(Joint work with A. Figalli)
Let $M$ be a separable metric space equipped with its Borel $\sigma$-algebra, $c : M \times M \to \mathbb{R}$ be a cost function and $\mu, \nu$ be two compactly supported probability measures in $M$. Find a measurable map $T : M \to M$ satisfying

$$T_\# \mu = \nu,$$

and in such a way that $T$ minimizes the transportation cost given by

$$\int_M c(x, T(x)) d\mu(x).$$

When the transport condition $T_\# \mu = \nu$ is satisfied, we say that $T$ is a transport map.
Two questions

- Existence of an optimal transport map?
- Uniqueness?
Example 1: The Euclidean case

Assume that $M = \mathbb{R}^n$ and that the cost $c$ is given by

$$c(x, y) = |x - y|^2.$$  

**Theorem (Brenier’s Theorem, 1991)**

If $\mu$ is absolutely continuous with respect to the Lebesgue measure, there is a unique optimal transport map $T$. It is characterized by the existence of a convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that

$$T(x) = \nabla \psi(x) \quad \text{for } \mu \text{ a.e. } x \in \mathbb{R}^n.$$
Example 2: The Riemannian case

Assume that $(M, g)$ is a smooth complete Riemannian manifold and denote by $d_g(\cdot, \cdot)$ the Riemannian distance on $M \times M$. Assume that the cost $c$ is given by

$$c(x, y) = d_g(x, y)^2.$$ 

**Theorem (McCann’s Theorem, 2001)**

If $\mu$ is absolutely continuous with respect to the Lebesgue measure on $M$, there is a unique optimal transport map $T$. It is characterized by the existence of a semiconvex function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that

$$T(x) = \exp_x (\nabla \psi(x)) \quad \text{for $\mu$ a.e. } x \in \mathbb{R}^n.$$
References

Papers:


Books:

Let $M$ be a separable metric space equipped with its Borel $\sigma$-algebra, $c : M \times M \to \mathbb{R}$ be a cost function and $\mu, \nu$ be two compactly supported probability measures in $M$. Find a probability measure $\gamma$ on $M \times M$ having marginals $\mu$ and $\nu$, i.e.

$$(\pi_1)_\# \gamma = \mu \quad \text{and} \quad (\pi_2)_\# \gamma = \nu,$$

(where $\pi_1 : M \times M \to M$ and $\pi_2 : M \times M \to M$ are the canonical projections), which minimizes the transportation cost given by

$$\int_{M \times M} c(x, y) d\gamma(x, y).$$

When the transport condition $(\pi_1)_\# \gamma = \mu, (\pi_2)_\# \gamma = \nu$ is satisfied, we say that $\gamma$ is a transport plan, and if $\gamma$ minimizes also the cost we call it an optimal transport plan.
Kantorovitch’s Duality

Theorem

There are two continuous functions \( \phi_1, \phi_2 : M \rightarrow \mathbb{R} \) satisfying

\[
\phi_1(x) = \inf_{y \in M} \{ c(x, y) - \phi_2(y) \} \quad \forall x \in M,
\]

\[
\phi_2(y) = \inf_{x \in M} \{ c(x, y) - \phi_1(x) \} \quad \forall y \in M.
\]

such that the following holds: a transport plan \( \gamma \) is optimal if and only if one has

\[
\phi_1(x) - \phi_2(y) = c(x, y) \quad \text{for } \gamma \text{ a.e. } (x, y) \in M \times M.
\]

As a consequence, to obtain that an optimal transport plan corresponds to a Monge’s optimal transport map, we have to show that \( \gamma \) is concentrated on a graph.
The function $x \mapsto d_g(x, y)^2$ is locally Lipschitz on $M$. 

The function $\phi_1$ is locally Lipschitz on $M$. As a consequence, by Rademacher’s Theorem, it is differentiable $\mu$-a.e.

Let $\bar{x} \in \text{supp}(\mu)$ be such that $\phi_1$ is differentiable at $\bar{x}$. Let $\bar{y}$ be such that

$$\phi_1(\bar{x}) = d_g(\bar{x}, \bar{y})^2 - \phi_2(\bar{y}).$$

Then we have,

$$d_g(x, \bar{y})^2 \geq \phi_1(x) + \phi_2(\bar{y}) \quad \forall x \in M.$$ 

Which implies that $\bar{y} = \exp_{\bar{x}} (-\frac{1}{2} \nabla \phi_1(\bar{x}))$. We set

$$\psi := -\frac{1}{2} \phi_1.$$
TWO ISSUES

- Show that $\phi_1$ is differentiable $\mu$-a.e. (for instance, by showing that $\phi_1$ is locally Lipschitz on $M$).
- Deduce that, if $\phi_1$ is differentiable at $\bar{x} \in \text{supp}(\mu)$, then there is a unique $\bar{y} \in M$ such that

$$\phi_1(\bar{x}) = c(\bar{x}, \bar{y}) - \phi_2(\bar{y}).$$
Let \((M, \Delta, g)\) be a complete sub-Riemannian structure of dimension \(n\) and rank \(m < n\). Let \(d_{SR}(\cdot, \cdot)\) be the sub-Riemannian distance on \(M \times M\). Let \(\mu, \nu\) be two compactly supported probability measures on \(M\). Find a measurable map \(T : M \to M\) satisfying

\[
T \# \mu = \nu,
\]

and in such a way that \(T\) minimizes the transportation cost given by

\[
\int_M d_{SR}(x, T(x))^2 d\mu(x).
\]
Let us denote by $D$ the diagonal in $M \times M$.

**Theorem (A. Figalli, L. Rifford, 2008)**

Assume that there exists an open set $\Omega \subset M \times M$ such that $\text{supp}(\mu \times \nu) \subset \Omega$, and $d_{SR}^2$ is locally Lipschitz on $\Omega \setminus D$. Let $\phi$ be the function provided by Kantorovitch's duality. Then, there is an open set $M^{\phi}$ such that and $\phi$ is locally Lipschitz in a neighborhood of $M^{\phi} \cap \text{supp}(\mu)$. There exists a unique optimal transport map which is defined $\mu$-a.e. by

$$T(x) := \begin{cases} 
\exp_x(-\frac{1}{2} d_{\phi}(x)) & \text{if } x \in M^{\phi} \cap \text{supp}(\mu), \\
x & \text{if } x \in (M \setminus M^{\phi}) \cap \text{supp}(\mu).
\end{cases}$$
Examples

- **Example 1**: Two generating distributions

**Proposition (A. Agrachev, P. Lee, 2008)**

If $\Delta$ is two-generating on $M$, then the squared sub-Riemannian distance function is locally Lipschitz on $M \times M$.

- **Example 2**: Generic sub-Riemannian structures

**Proposition (Y. Chitour, F. Jean, E. Trélat, 2006)**

Let $(M, g)$ be a complete Riemannian manifold of dim $\geq 4$. Then, for any generic distribution of rank $\geq 3$, the squared sub-Riemannian distance function is locally semiconcave (hence locally Lipschitz) on $M \times M \setminus D$. 
**Example 3: Medium-fat distributions**
The distribution $\Delta$ is called *medium-fat* if, for every $x \in M$ and every vector field $X$ on $M$ such that $X(x) \in \Delta(x) \setminus \{0\}$, there holds

$$T_x M = \Delta(x) + [\Delta, \Delta](x) + [X, [\Delta, \Delta]](x).$$

**Proposition**

Assume that $\Delta$ is medium-fat. Then the squared sub-Riemannian distance function is locally Lipschitz on $M \times M \setminus D$. 
Thank you for your attention!