Optimal Transportation on Sub-Riemannian Manifolds

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I. Statement of our optimal transportation problem

Sub-Riemannian Manifolds

A (complete) sub-Riemannian manifold of dimension $n \geq 3$ and rank $m < n$ is given by a triple (M, Δ, g) where:

- M is a smooth connected Riemannian manifold of dimension n;
- g is a complete smooth Riemannian metric on M;
- Δ is a *nonholonomic* distribution of rank m on M, that is, for every $x \in M$, there is a local parametrization of Δ by m linearly independant smooth vector fields $f_1^x, \cdots, f_m^x$ defined on an open neighborhood $\mathcal{V}_x$ such that

$$\text{Lie}\left\{f_1^x, \cdots, f_m^z\right\}(z) = T_z M \qquad \forall z \in \mathcal{V}_x.$$

The Chow-Rashevsky Theorem

A curve $\gamma : [0,1] \longrightarrow M$ is called *horizontal* if it belongs to $W^{1,2}([0,1], M)$ and satisfies

$$\dot{\gamma}(t) \in \Delta(\gamma(t)) \qquad \forall t \in [0,1].$$

Theorem (Chow 1939, Rashevsky 1938)

Let (M, g, Δ) be a sub-Riemannian manifold. Then, for any $x, y \in M$, there is an horizontal path $\gamma : [0,1] \longrightarrow M$ such that

$$\gamma(0) = x \quad \text{and} \quad \gamma(1) = y.$$

Two Examples in $\mathbb{R}^3$

- **The Heisenberg distribution:** Let f_1, f_2 be the vector fields in $\mathbb{R}^3$ defined by

$$f_1 = \frac{\partial}{\partial x_1}, \quad f_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}.$$

  The distribution Δ spanned by f_1, f_2 is nonholonomic.

- **The Martinet distribution:** Let f_1, f_2 be the vector fields in $\mathbb{R}^3$ defined by

$$f_1 = \frac{\partial}{\partial x_1}, \quad f_2 = \frac{\partial}{\partial x_2} + x_1^2 \frac{\partial}{\partial x_3}.$$

  The distribution Δ spanned by f_1, f_2 is nonholonomic.

The sub-Riemannian distance

The *length* of an horizontal path $\gamma : [0,1] \longrightarrow M$ is defined by

$$\text{length}_g(\gamma) := \int_0^1 \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma(t)})}dt.$$

The *sub-Riemannian distance* $d_{SR}(x,y)$ between two points x, y of M is defined as

$$d_{SR}(x,y) := \inf \left\{ \text{length}_g(\gamma) \right\}$$

where the infimum is taken over the horizontal paths $\gamma : [0,1] \longrightarrow M$ joining x to y.

Proposition

The function d_{SR} is continuous on $M \times M$.

Monge's Optimal Transportation Problem

Let (M, Δ, g) be a complete sub-Riemannian structure of dimension n and rank $m < n$. Let $d_{SR}(\cdot, \cdot)$ be the sub-Riemannian distance on $M \times M$. Let μ, ν be two compactly supported probability measures on M. Find a measurable map $T : M \to M$ satisfying

$$T_\sharp \mu = \nu,$$

and in such a way that T minimizes the transportation cost given by

$$\int_M d_{SR}(x, T(x))^2 d\mu(x).$$

References

- "Optimal mass transportation in the Heisenberg group" by L. Ambrosio and S. Rigot. *J. Funct. Anal.*, 2004.
- "Optimal transportation under nonholonomic constraints" by A. Agrachev and P. Lee. *Trans. Amer. Math. Soc*, to appear. Available on *http://people.sissa.it/~agrachev/*.
- "Optimal Transportation on Sub-Riemannian Manifolds" by A. Figalli and L. Rifford. Preprint, 2008. Available on *http://math.unice.fr/~rifford/*.

II. Sketch of proof of the McCann Theorem

Statement of the McCann Theorem

Let (M, g) be a smooth complete Riemannian manifold and $d_g(\cdot,\cdot)$ denote the Riemannian distance on $M \times M$. Let μ, ν be two compactly supported probability measures in M, find a measurable map $T : M \to M$ satisfying $T_\sharp \mu = \nu$, and in such a way that T minimizes the transportation cost given by

$$\int_M d_g(x, T(x))^2 d\mu(x).$$

Theorem (McCann's Theorem, 2001)

If μ is absolutely continuous with respect to the Lebesgue measure on M, then there is a unique optimal transport map T. It is characterized by the existence of a locally semiconvex function $\psi : M \to \mathbb{R}$ such that

$$T(x) = \exp_x(\nabla\psi(x)) \qquad \text{for } \mu \text{ a.e. } x \in \mathbb{R}^n.$$

Kantorovitch's Duality

Proposition

There are two continuous function $\phi_1, \phi_2 : M \to \mathbb{R}$ *satisfying*

$$\phi_1(x) = \inf_{y \in M} \left\{ d_g(x,y)^2 - \phi_2(y) \right\} \qquad \forall x \in M,$$

$$\phi_2(y) = \inf_{x \in M} \left\{ d_g(x,y)^2 - \phi_1(x) \right\} \qquad \forall y \in M.$$

such that the following holds: a transport plan γ *is optimal if and only if one has*

$$\phi_1(x) + \phi_2(y) = d_g(x,y)^2 \qquad \text{for } \gamma \text{ a.e. } (x,y) \in M \times M.$$

As a consequence, to obtain that an optimal transport plan corresponds to a Monge's optimal transport map, we have to show that γ is concentrated on a graph.

Proof of McCann's Theorem

- The function $x \mapsto d_g(x,y)^2$ is locally Lipschitz on M.
- The function ϕ_1 is locally Lipschitz on M. As a consequence, by Rademacher's Theorem, it is differentiable μ-a.e.
- Let $\bar{x} \in \mathrm{supp}(\mu)$ be such that ϕ_1 is differentiable at $\bar{x}$. Let $\bar{y}$ be such that

$$\phi_1(\bar{x}) + \phi_2(\bar{y}) = d_g(\bar{x},\bar{y})^2.$$

Then we have,

$$d_g(x,\bar{y})^2 \geq \phi_1(x) + \phi_2(\bar{y}) \qquad \forall x \in M.$$

Which implies that $\bar{y} = \exp_{\bar{x}}\left(-\frac{1}{2}\nabla\phi_1(\bar{x})\right)$. We set

$$\psi := -\frac{1}{2}\phi_1.$$

Strategy of proof

TWO ISSUES

- **Issue 1:** Show that ϕ_1 is differentiable μ-a.e. (for instance, by showing that ϕ_1 is locally Lipschitz on M).
- **Issue 2:** Deduce that, if ϕ_1 is differentiable at $\bar{x} \in \mathrm{supp}(\mu)$, then there is a unique $\bar{y} \in M$ such that

$$\phi_1(\bar{x}) = c(\bar{x}, \bar{y}) - \phi_2(\bar{y}).$$

III. The sub-Riemannian world

Sub-Riemannian minimizing geodesics I

Let $x \neq y \in M$ be fixed. By completeness, there is an horizontal path $\bar{\gamma} : [0,1] \to M$ joining x to y such that

$$d_{SR}(x,y) = \text{length}_g(\bar{\gamma}).$$

If $\bar{\gamma}$ is parametrized by arc-length, then it minimizes the quantity

$$\int_0^1 g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))dt = \text{energy}_g(\gamma).$$

over all horizontal paths γ joining x to y. One has

$$e_{SR}(x,y) = d_{SR}(x,y)^2 \qquad \forall x, y \in M.$$

Sub-Riemannian minimizing geodesics II

Denote by $\Omega_\Delta(x)$ the set of horizontal paths starting from x. The End-Point Mapping from x is defined by

$$\begin{array}{rcl} E_x : \Omega_\Delta(x) & \longrightarrow & M \\ \gamma & \longmapsto & \gamma(1). \end{array}$$

The Cost Function is given by

$$\begin{array}{rcl} C : \Omega_\Delta(x) & \longrightarrow & \mathbb{R} \\ \gamma & \longmapsto & \text{energy}_g(\gamma). \end{array}$$

By the Lagrange Multiplier Theorem, there is $\lambda \in T_x^*M$ and $\lambda_0 \in \{0,1\}$ such that

$$\lambda \cdot dE_x(\bar{\gamma}) = \lambda_0 dC(\bar{\gamma}).$$

Sub-Riemannian minimizing geodesics III

Two Cases may appear:

- $\lambda_0 = 1.$
- $\lambda_0 = 0$

Sub-Riemannian minimizing geodesics IV

- **First case: $\lambda_0 = 1$.**

The *Sub-Riemannian Hamiltonian* $H : T^*M \to \mathbb{R}$ associated with $M, \Delta, g)$ is defined by

$$H(x,p) := \max\left\{ p(v)^2 - \frac{1}{2} g_x(v,v) \mid v \in \Delta(x) \right\}.$$

Proposition

There is an smooth extremal $\psi = (\bar{\gamma}, p) : [0,1] \to T^*M$ *with* $p(1) = \lambda/2$ *such that*

$$\dot{\bar{\gamma}}(t) = \frac{\partial H}{\partial p}(\bar{\gamma}(t), p(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial x}(\bar{\gamma}(t), p(t)),$$

In particular, the path $\bar{\gamma}$ *is smooth on* $[0,1]$.

Sub-Riemannian minimizing geodesics V

- **Second case: $\lambda_0 = 0$.**

The path $\bar{\gamma}$ must be a critical point of the mapping E_x. Such an horizontal path is called *singular*.

Examples:

- **The Heisenberg distribution:** No nontrivial horizontal path is singular. Any minimizing horizontal path is a projection of a smooth extremal.
- **The Martinet distribution:** All the horizontal paths which are contained in the plane $\{x_1 = 0\}$ are singular. For any smooth Riemannian metric g, such horizontal paths are locally minimizing !

The exponential mapping

The *sub-Riemannian exponential mapping* from x is defined by

$$\begin{array}{rccc} \exp_x : & T_x^*M & \longrightarrow & M \\ & p & \longmapsto & \pi(\psi(1)) \end{array}$$

where ψ is the extremal such that $\psi(0) = (x, p)$ in local coordinates.

Proposition (Agrachev, Rifford, Trélat, 2008)

For every $x \in M$, the set $\exp_x(T_x^*M)$ contains an open dense subset of M.

Sketch of proof of our result I

- **Issue 2:** Let $\bar{x}$ be such that ϕ_1 is differentiable at $\bar{x}$. Let $\bar{y}$ be such that

$$\phi_1(\bar{x}) + \phi_2(\bar{y}) = d_{SR}(\bar{x}, \bar{y})^2.$$

Let $\bar{\gamma} : [0, 1 \to M$ be a minimizing horiazontal path between x and y. Then we have,

$$d_{SR}(x, \bar{y})^2 - \phi_1(x) \geq \phi_2(\bar{y}) \qquad \forall x \in M.$$

Which yields that for every $\gamma \in \Omega_\Delta(y)$,

$$\begin{aligned} \text{energy}_g(\gamma) - \phi_1(\gamma(1)) &\geq d_{SR}(\gamma(1), y)^2 - \phi_1(\gamma(1)) \\ &\geq \phi_2(\bar{y}) = \text{energy}_g(\bar{\gamma}) - \phi_1(\bar{\gamma}(1)). \end{aligned}$$

Therefore, $\bar{y} = \exp_{\bar{x}}\left(-\frac{1}{2} d_{\bar{x}}\phi_1\right)$.

Sketch of proof of our result II

- **Issue 1:** Outside the diagonal in $M \times M$ and in absence of singular minimizing horizontal paths, the sub-Riemannian distance d_{SR} shares the same regularity properties as the Riemannian distance.

  Let us denote by D the diagonal in $M \times M$.

Proposition

Let (M, Δ, g) be a (complete) sub-Riemannian manifold admitting no nontrivial singular minimizing horizontal paths. Then, the sub-Riemannian distance is locally semiconcave on $M \times M \setminus D$.

IV. Statements of our results

A Theorem of Existence and Uniqueness

Theorem (A. Figalli, L. Rifford, 2008)

Assume that there exists an open set $\Omega \subset M \times M$ such that $supp(\mu \times \nu) \subset \Omega$, and d_{SR}^2 is locally Lipschitz on $\Omega \setminus D$. Let ϕ be the function provided by Kantorovitch's duality. Then, there is an open set $\mathcal{M}^\phi$ such that and ϕ is locally Lipschitz in a neighborhood of $\mathcal{M}^\phi \cap supp(\mu)$. There exists a unique optimal transport map which is defined μ-a.e. by

$$T(x) := \begin{cases} \exp_x(-\frac{1}{2}\, d\phi(x)) & \text{if } x \in \mathcal{M}^\phi \cap supp(\mu), \\ x & \text{if } x \in \left(M \setminus \mathcal{M}^\phi\right) \cap supp(\mu). \end{cases}$$

Examples

- Example 1: Two generating distributions

Proposition (A. Agrachev, P. Lee, 2008)

If Δ is two-generating on M, then the squared sub-Riemannian distance function is locally Lipschitz on $M \times M$.

- Example 2: Generic sub-Riemannian structures

Proposition (Y. Chitour, F. Jean, E. Trélat, 2006)

Let (M, g) be a complete Riemannian manifold of dim ≥ 4. Then, for any generic distribution of rank ≥ 3, the squared sub-Riemannian distance function is locally semiconcave (hence locally Lipschitz) on $M \times M \setminus D$.

Examples again

- Example 3: Medium-fat distributions
  The distribution Δ is called *medium-fat* if, for every $x \in M$ and every vector field X on M such that $X(x) \in \Delta(x) \setminus \{0\}$, there holds

$$T_xM = \Delta(x) + [\Delta, \Delta](x) + [X, [\Delta, \Delta]](x).$$

Proposition

Assume that Δ is medium-fat. Then the squared sub-Riemannian distance function is locally Lipschitz on $M \times M \setminus D$.

Thank you for your attention !