# Morse-Sard type results in sub-Riemannian geometry

L. Rifford*, E. Trélat†

**Abstract**

Let $(M, \Delta, g)$ be a sub-Riemannian manifold and $x_0 \in M$. Assuming that Chow's condition holds and that $M$ endowed with the sub-Riemannian distance is complete, we prove that there exists a dense subset $N_1$ of $M$ such that for every point $x$ of $N_1$, there is a unique minimizing path steering $x_0$ to $x$, this trajectory admitting a normal extremal lift. If the distribution $\Delta$ is everywhere of corank one, we prove the existence of a subset $N_2$ of $M$ of full Lebesgue measure such that for every point $x$ of $N_2$, there exists a minimizing path steering $x_0$ to $x$ which admits a normal extremal lift, is nonsingular, and the point $x$ is not conjugate to $x_0$. In particular, the image of the sub-Riemannian exponential mapping is dense in $M$, and in the case of corank one is of full Lebesgue measure in $M$.

## 1 Introduction and main results

The following general definition of a sub-Riemannian distance is due to [3]. Let $M$ be a connected smooth $n$-dimensional manifold, $m$ an integer such that $1 \leqslant m \leqslant n$, and $f_1, \ldots, f_m$ be smooth vector fields on the manifold $M$. For all $x \in M$ and $v \in T_xM$, set

$$g(x, v) := \inf \left\{ \sum_{i=1}^{m} u_i^2 \mid u_1, \ldots, u_m \in \mathbb{R}, \sum_{i=1}^{m} u_i f_i(x) = v \right\}.$$

Then $g(x, \cdot)$ is a positive definite quadratic form on the subspace of $T_xM$ spanned by $f_1(x), \ldots, f_m(x)$. Outside this subspace we set $g(x, v) = +\infty$. The form $g$ is called *sub-Riemannian metric* associated to the $m$-tuple of vector fields $(f_1, \ldots, f_m)$. Let $\mathcal{AC}([0,1], M)$ denote the set of absolutely continuous paths in $M$ defined on $[0,1]$, we define the *length* of $\gamma \in \mathcal{AC}([0,1], M)$ as

$$l(\gamma) := \int_0^1 \sqrt{g(\gamma(t), \dot{\gamma}(t))}\, dt.$$

*Institut Girard Desargues, Université Lyon I, 43 Bd du 11 Novembre 1918, 69622 Villeurbanne cedex, France. E-mail: rifford@igd.univ-lyon1.fr.

†Univ. Paris-Sud, Lab. AN-EDP, Math., UMR 8628, Bat. 425, 91405 Orsay Cedex, France. E-mail: emmanuel.trelat@math.u-psud.fr.

We say that *Chow's condition* holds if the Lie algebra spanned by the vector fields $f_1, \dots, f_m$, is equal to the tangent space $T_xM$ at every point $x$ of $M$. It is well-known that under this condition any two points of $M$ can be joined by an absolutely continuous path with finite length.
The *sub-Riemannian distance* associated to the $m$-tuple of vector fields $(f_1, \dots, f_m)$, between two points $x_0, x_1$ in $M$, is defined as

$$d_{SR}(x_0, x_1) := \inf \{ l(\gamma) \mid \gamma \in \mathcal{AC}([0,1], M),\ \gamma(0) = x_0, \gamma(1) = x_1 \}.$$

The *sub-Riemannian sphere* $S_{SR}(x_0, r)$ (resp. the *sub-Riemannian ball* $B_{SR}(x_0, r)$) centered at $x_0$ with radius $r$ as the set of points $x \in M$ such that $d_{SR}(x_0, x) = r$ (resp. $d_{SR}(x_0, x) < r$). A path $\gamma \in \mathcal{AC}([0,1], M)$ is said to be *minimizing* if it realizes the sub-Riemannian distance between its extremities.

*Remark* 1.1. If Chow's condition holds, then:

- the topology defined by the sub-Riemannian distance $d_{SR}$ coincides with the original topology of $M$,
- sufficiently near points can be joined by a minimizing path,
- if the manifold $M$ is moreover a complete metric space for the sub-Riemannian distance $d_{SR}$, then any two points can be joined by a minimizing path.

Consider on the other part the differential system on the tangent bundle $TM$ of $M$

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) f_i(x(t)) \quad \text{a.e. on } [0,1], \tag{1}$$

where the function $u(\cdot) = (u_1(\cdot), \dots, u_m(\cdot))$, called *control function*, belongs to $L^2([0,1], \mathbb{R}^m)$. Let $x_0 \in \mathbb{R}^n$, and let $\mathcal{U}$ denote the (open) subset of $L^2([0,1], \mathbb{R}^m)$ such that the solution of (1) starting at $x_0$ and associated to a control $u(\cdot) \in \mathcal{U}$ is well-defined on $[0,1]$. The mapping

$$\begin{array}{rccc} E_{x_0} : & \mathcal{U} & \longrightarrow & \mathbb{R}^n \\ & u(\cdot) & \longmapsto & x(1), \end{array}$$

which to a control $u(\cdot)$ associates the extremity $x(1)$ of the corresponding solution $x(\cdot)$ of (1) starting at $x_0$, is called *end-point mapping* at the point $x_0$; it is a smooth mapping. The trajectory $x(\cdot)$ is said to be *singular* if the associated control $u(\cdot)$ is a singular point of the end-point mapping (*i.e.* if the Fréchet derivative of $E_{x_0}$ at $u(\cdot)$ is not onto); it is *minimizing* if it realizes the sub-Riemannian distance between its extremities.

*Remark* 1.2. A sub-Riemannian manifold is often defined as a triple $(M, \Delta, g)$, where $M$ is a $n$-dimensional manifold, $\Delta$ is a distribution of rank $m \leqslant n$, and $g$ is a Riemannian metric on $\Delta$. If the vector fields $(f_1, \dots, f_m)$ are everywhere linearly independent, then controlled paths solutions of (1) coincide with absolutely continuous paths tangent to the distribution $\Delta$, where

$$\Delta(x) = \text{Span } \{ f_1(x), \dots, f_m(x) \},$$

for all $x \in M$. These paths are said $\Delta$-*horizontal*.

On the other part, for $x_0 \in M$, let $\Omega(x_0, \Delta)$ be the set of $\Delta$-horizontal paths starting from $x_0$ whose derivative is square integrable for the metric $g$ (and hence for any Riemannian metric on $\Delta$). Endowed with the $H^1$-topology, $\Omega(x_0, \Delta)$ inherits of a Hilbert manifold structure, see [4]. For $(x_0, x_1) \in M \times M$, let $\Omega(x_0, x_1, \Delta)$ be the subset of paths $x(\cdot) \in \Omega(x_0, \Delta)$ such that $x(1) = x_1$. The set $\Omega(x_0, x_1, \Delta)$ is a submanifold of $\Omega(x_0, \Delta)$ in a neighborhood of any nonsingular path, but might fail to be a (global) manifold due to the possible existence of singular paths.

Let $x_0$ and $x_1$ in $M$. The sub-Riemannian problem of determining a minimizing trajectory steering $x_0$ to $x_1$ can be easily seen (up to reparametrization, and using the Cauchy-Schwarz inequality) to be equivalent to the *optimal control problem* of finding a control $u(\cdot) \in \mathcal{U}$ such that the solution of the control system (1) steers $x_0$ to $x_1$ in time 1, and minimizes the *cost function*

$$C(u(\cdot)) := \int_0^1 \sum_{i=1}^m u_i(t)^2 \, dt. \tag{2}$$

If a control $u(\cdot)$ associated to a trajectory $x(\cdot)$ such that $x(0) = x_0$ is optimal, then there exists a nontrivial *Lagrange multiplier* $(\psi, \psi^0) \in T^*_{x(1)}M \times \mathbb{R}$ such that

$$\psi . dE_{x_0}(u(\cdot)) = -\psi^0 dC(u(\cdot)), \tag{3}$$

where $dE_{x_0}(u(\cdot))$ (resp. $dC(u(\cdot))$) denotes the Fréchet derivative of $E_{x_0}$ (resp. $C$) at the point $u(\cdot)$. The well-known Pontryagin maximum principle (see [8]) parametrizes this condition and asserts that the trajectory $x(\cdot)$ is the projection of an *extremal*, that is a quadruple $(x(\cdot), p(\cdot), p^0, u(\cdot))$, solution of the constrained Hamiltonian system

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), p^0, u(t)), \;\; \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), p^0, u(t)),$$

$$\frac{\partial H}{\partial u}(x(t), p(t), p^0, u(t)) = 0,$$

almost everywhere on $[0, 1]$, where

$$H(x, p, p^0, u) := \langle p, \sum_{i=1}^m u_i f_i(x) \rangle + p^0 \sum_{i=1}^m u_i^2$$

is the *Hamiltonian* of the optimal control problem, $p(\cdot)$ (called *adjoint vector*) is an absolutely continuous mapping on $[0, 1]$ such that $p(t) \in T^*_{x(t)}M$, and $p^0$ is a real nonpositive constant. Moreover there holds

$$(p(1), p^0) = (\psi, \psi^0), \tag{4}$$

up to a multiplying scalar. If $p^0 < 0$ then the extremal is said to be *normal*, and in this case it is normalized to $p^0 = -1/2$. If $p^0 = 0$ then the extremal is said to be *abnormal*.

*Remark* 1.3. Any singular trajectory is the projection of an abnormal extremal, and conversely.

Furthermore, a singular trajectory is said to be *strict* (or *strictly singular*) if it does not admit a normal extremal lift; equivalently in that case we say that its abnormal extremal lift is *strictly abnormal.*

The *sub-Riemannian wave-front* $W_{SR}(x_0, r)$ centered at $x_0$ and with radius $r$ is defined as the set of end-points $x(1)$, where $(x(\cdot), p(\cdot), p^0, u(\cdot))$ is an extremal such that $x(0) = x_0$ and $C(u(\cdot)) = r^2$. Under Chow's condition, it is clear from Remark 1.1 that $S_{SR}(x_0, r)$ is a subset of $W_{SR}(x_0, r)$.

Using the previous normalization, controls associated to normal extremals can be computed as

$$u_i(t) = \langle p(t), f_i(x(t)) \rangle, \; i = 1, \dots, m.$$

Hence normal extremals are solutions of the Hamiltonian system

$$\dot{x}(t) = \frac{\partial H_1}{\partial p}(x(t), p(t)), \; \dot{p}(t) = -\frac{\partial H_1}{\partial x}(x(t), p(t)), \tag{5}$$

where

$$H_1(x, p) = \frac{1}{2} \sum_{i=1}^{m} \langle p, f_i(x) \rangle^2.$$

Notice that $H_1(x(t), p(t))$ is constant along each normal extremal and that the length of the path $x(\cdot)$ equals $(2\, H_1(x(0), p(0)))^{1/2}$. Actually, given a point $x_0$ of $M$, the differential system (5) has a well-defined smooth solution on $[0, 1]$ such that $x(0) = x_0$ and $p(0) = p_0$, for $p_0 \in U$, where $U$ is a connected open subset of $T^*_{x_0} M$. In what follows, the point $x_0$ is fixed.

**Definition 1.1.** The smooth mapping

$$\begin{array}{rccc} \exp_{x_0} : & U & \longrightarrow & M \\ & p_0 & \longmapsto & x(1) \end{array}$$

where $(x(\cdot), p(\cdot))$ is the solution of the system (5) such that $x(0) = x_0$ and $p(0) = p_0$, is called *exponential mapping* at the point $x_0$.

The exponential mapping parametrizes normal extremals. Notice that every minimizing trajectory steering $x_0$ to a point of $M \setminus \exp_{x_0}(U)$ is necessarily strictly singular.

*Remark* 1.4. Using notations of Definition 1.1, it is easy to see by reparametrization that $x(t) = \exp_{x_0}(tp_0)$, for all $t \in [0, 1]$.

*Remark* 1.5. For all $p_0 \in U$ such that $H_1(x_0, p_0) = \frac{r^2}{2}$, one has $\exp_{x_0}(p_0) \in W_{SR}(x_0, r)$. The space of normal extremals with length $r$ is parametrized by the manifold $U_r = U \cap H_1^{-1}(\frac{r^2}{2})$, which is diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}^{n-m}$ if the distribution $\Delta$ has rank $m$ at $x_0$.

A point $x \in \exp_{x_0}(U)$ is said *conjugate* to $x_0$ if it is a critical value of the mapping $\exp_{x_0}$, *i.e.* if there exists $p_0 \in U$ such that $x = \exp_{x_0}(p_0)$ and the differential $d\exp_{x_0}(p_0)$ is not onto. The *conjugate locus*, denoted by $\mathcal{C}(x_0)$, is defined as the set of all points conjugate to $x_0$.

*Remark* 1.6. By Sard Theorem applied to the mapping $\exp_{x_0}$, it is clear that the conjugate locus $\mathcal{C}(x_0)$ has Lebesgue measure zero in $M$.

*Remark* 1.7. Let $x \in \exp_{x_0}(U)$, and $p_0 \in U$ such that $x = \exp_{x_0}(p_0)$. We denote by $(x(\cdot,p_0), p(\cdot,p_0), -\frac{1}{2}, u(\cdot,p_0))$ the associated normal extremal. Then we have

$$\exp_{x_0}(p_0) = E_{x_0}(u(\cdot,p_0)).$$

Therefore if $x$ is not conjugate to $x_0$ then the control $u(\cdot,p_0)$ is nonsingular. In particular, the set of endpoints of nonstrictly singular trajectories starting from $x_0$ has Lebesgue measure zero in $M$.

*Remark* 1.8. With notations of the previous remark, if $x$ is not conjugate to $x_0$ then the path $x(\cdot) := x(\cdot,p_0)$ associated to the control $u(\cdot) := u(\cdot,p_0)$ admits a unique normal extremal lift. Indeed if it had two distinct normal extremals lifts $(x(\cdot), p_1(\cdot), -\frac{1}{2}, u(\cdot))$ and $(x(\cdot), p_2(\cdot), -\frac{1}{2}, u(\cdot))$, then the extremal $(x(\cdot), p_1(\cdot) - p_2(\cdot), 0, u(\cdot))$ would be an abnormal extremal lift of the path $x(\cdot)$, which is a contradiction since $u(\cdot)$ is nonsingular.

In the present paper we prove the two following theorems.

**Theorem 1.1.** *Suppose Chow's condition holds, and that the manifold $M$ is complete for the sub-Riemannian distance $d_{SR}$. There exists a dense subset $N_1$ of $M$ such that, for every point $x \in N_1$, there is a unique minimizing path joining $x_0$ to $x$; moreover this trajectory admits a normal extremal lift. In particular the image $exp_{x_0}(U)$ of the exponential mapping is dense in $M$.*

For all $x \in M$, let $\Delta(x) := \text{Span }\{f_1(x), \cdots, f_m(x)\}$, and let $\mu$ denote the Lebesgue measure on $M$. Regarding the previous result, one can wonder whether almost every point of $M$ belongs to $\exp_{x_0}(U)$. The following result gives a positive answer in the case of a corank-one distribution.

**Theorem 1.2.** *Suppose Chow's condition holds, and that the manifold $M$ is complete for the sub-Riemannian distance $d_{SR}$. If the distribution $\Delta$ is everywhere of corank one, then there exists a subset $N_2$ of $M$ of full Lebesgue measure such that, for every point $x \in N_2$, there exists a minimizing path joining $x_0$ to $x$ and having a normal extremal lift. Moreover this trajectory is nonsingular, and $x$ is not conjugate to $x_0$. In particular, the set $exp_{x_0}(U)$ is of full measure in $M$, i.e. $\mu(M \setminus exp_{x_0}(U)) = 0$.*

The next two sections are devoted to the proof of the latter results. In a last section we discuss some consequences and open problems.

# 2 Proof of Theorem 1.1

## 2.1 The proximal sub-differential

Let $M$ be a smooth manifold of dimension $n$ and $\Omega$ be an open subset of $M$. Let $f : \Omega \to \mathbb{R}$ be a continuous function on $\Omega$; we call *proximal sub-differential* of the function $f$ at the point $x \in \Omega$ the subset of $T_x^*M$ defined by

$$\partial_P f(x) := \{d\phi(x) \mid \phi \in C^\infty(M) \text{ and } f - \phi \text{ attains a local minimum at } x\}.$$

Note that since every local $C^\infty$ function can be extended to a $C^\infty$ function on $M$, the proximal sub-differential of $f$ at $x$ depends only on the local behavior of the function $f$ near $x$. In addition, remark that $\partial_P f(x)$ is a convex subset of $T_x^*M$ which may be empty; for instance the proximal sub-differential of the real function $t \mapsto -|t|$ at $t = 0$ is empty.

*Remark* 2.1. Notice that when $M = \mathbb{R}^n$, a vector $\zeta$ belongs to the proximal sub-differential of $f$ at a point $x$ if and only if there exists $\sigma$ and $\delta > 0$ such that

$$f(y) - f(x) + \sigma\|y - x\|^2 \geqslant \langle \zeta, y - x \rangle, \quad \forall y \in x + \delta B.$$

This is the usual definition of proximal sub-differentials in Hilbert spaces; we refer the reader to [6] for further details on that subject.

In fact, an immediate application of the smooth variational principle of Borwein-Preiss (see [5]) implies the following result.

**Theorem 2.1.** *The proximal sub-differential of a continuous function $f : \Omega \to \mathbb{R}$ is nonempty on a dense subset of $\Omega$.*

The proximal sub-differential of $f$ defines a multivalued mapping from $\Omega$ into the cotangent bundle $T^*M$. It is said to be locally bounded on $\Omega$ if for each $x \in \Omega$ there exists a neighborhood $\mathcal{V}$ of $x$ such that $\partial_P f(\mathcal{V})$ is relatively compact in $T^*M$. The following result is standard.

**Proposition 2.2.** *The function $f$ is Lipschitz continuous on $\Omega$ if and only if the proximal sub-differentials of $f$ are locally bounded on $\Omega$.*

*Remark* 2.2. Notice that the Fréchet (or viscosity) sub-differential of $f$ at $x$, defined by

$$D^- f(x) := \left\{d\phi(x) \mid \phi \in C^1(M) \text{ and } f - \phi \text{ attains a local minimum at } x\right\},$$

is larger than the proximal sub-differential, but in fact both notions coincide locally; we refer the reader to [6, Prop. 4.5 p. 138, Prop. 4.12 p. 142] for a precise statement.

To conclude this preliminary section, we remark that there exists a complete calculus of proximal sub-differentials, one that extends all the theorems of the usual smooth calculus, see [6].

## 2.2 Application to the proof of Theorem 1.1

In what follows we denote $e(\cdot) := d_{SR}(x_0, \cdot)^2$.

**Proposition 2.3.** *Let $x \in M$ such that $\partial_P e(x) \neq \emptyset$. Then there exists a unique minimizing path $x(\cdot)$ joining $x_0$ to $x$. Moreover for every $\zeta \in \partial_P e(x)$, the path $x(\cdot)$ admits a normal extremal lift $(x(\cdot), p(\cdot), -\frac{1}{2}, u(\cdot))$ such that $p(1) = \frac{1}{2}\zeta$.*

*Proof.* We adopt the following notation: for every control $u(\cdot) \in \mathcal{U}$, we denote by $x_u(\cdot)$ the trajectory solution of (1) associated to the control $u(\cdot)$ and such that $x_u(0) = x_0$. Let $x \in M$ and $\zeta \in \partial_P e(x)$. We first prove that every minimizing path steering $x_0$ to $x$ admits a normal extremal lift such that $p(1) = \frac{1}{2}\zeta$. Let $u(\cdot) \in \mathcal{U}$ be an optimal control such that the associated trajectory $x_u(\cdot)$ joins $x_0$ to $x$; there holds

$$e(x) = \int_0^1 \sum_{i=1}^m u_i(t)^2 dt.$$

On the other hand, since $\zeta \in \partial_P e(x)$, there exists a function $\phi$ of class $C^\infty$ with $d\phi(x) = \zeta$ and such that $e - \phi$ attains a local minimum at $x$. Thus there exists a neighborhood $\mathcal{V}$ of $u(\cdot)$, contained in $\mathcal{U}$, such that

$$e(x) \leqslant e(x_v(1)) - \phi(x_v(1)) + \phi(x),$$

for every control $v(\cdot) \in \mathcal{V}$. Moreover it can be easily seen by definition of the distance function, that

$$e(x_v(1)) \leqslant \int_0^1 \sum_{i=1}^m v_i(t)^2 dt.$$

Therefore we obtain

$$e(x) \leqslant \int_0^1 \sum_{i=1}^m v_i(t)^2 dt - \phi(x_v(1)) + \phi(x),$$

for every control $v(\cdot) \in \mathcal{V}$. In particular, this means that $u(\cdot)$ is a solution of the minimization problem

$$\min_{v \in \mathcal{V}} \left( \int_0^1 \sum_{i=1}^m v_i(t)^2 dt - \phi(x_v(1)) + \phi(x) \right).$$

Hence $u(\cdot)$ is a critical point of the function

$$v(\cdot) \in \mathcal{V} \mapsto C(v(\cdot)) - \phi(E_{x_0}(v(\cdot))) + \phi(x),$$

and thus

$$dC(u(\cdot)) - \zeta . dE_{x_0}(u(\cdot)) = 0.$$

This leads to the existence of a normal extremal lift $(x_u(\cdot), p_u(\cdot), -\frac{1}{2}, u(\cdot))$ such that $(x_u(1), p_u(1)) = (x, \frac{1}{2}\zeta)$. In particular, uniqueness of a minimizing path joining $x_0$ to $x$ follows. □

Th. 1.1 is a straightforward consequence of Prop. 2.3 together with Th. 2.1.

# 3 Proof of Theorem 1.2

## 3.1 The limiting sub-differential

Let $M$ be a smooth manifold of dimension $n$ and $\Omega$ be an open subset of $M$. Let $f : \Omega \to \mathbb{R}$ be a continuous function on $\Omega$; we call *limiting sub-differential* of the function $f$ at the point $x \in \Omega$ the subset of $T_x^*M$ defined by

$$\partial_L f(x) := \{\lim \zeta_n \mid \zeta_n \in \partial_P f(x_n), x_n \to x\}.$$

As the proximal sub-differential, the limiting sub-differential of $f$ at $x$ depends only on the local behavior of $f$ near $x$. Moreover by construction, $\partial_L f(x)$ is a closed subset of $T_x^*M$ which contains $\partial_P f(x)$, which is not necessarily convex and which may be empty. In some situations, the limiting sub-differential of $f$ at $x$ can be proven to be nonempty; the result is as follows.

**Proposition 3.1.** *Let $x \in \Omega$. If there exists a Lipschitz continuous $\phi$ defined in a neighborhood of $x$ such that $f - \phi$ attains a local minimum at $x$, then $\partial_L f(x)$ is nonempty.*

*Proof.* Without loss of generality, we can assume to be in $\mathbb{R}^n$. By assumption, the function $f - \phi$ attains a local minimum at $x$; this implies that $0 \in \partial_L(f - \phi)(x)$. By the sum rule on limiting sub-differentials (see [6, Proposition 10.1 p. 62]), the function $-\phi$ being Lipschitz continuous, there holds

$$\partial_L(f - \phi)(x) \subset \partial_L f(x) + \partial_L(-\phi)(x),$$

and hence $\partial_L f(x)$ is necessarily nonempty. □

This proposition will be the key result to prove Th. 1.2. Notice that there exist some continuous functions $f : \mathbb{R}^n \to \mathbb{R}$, $n \geqslant 2$, such that their limiting sub-differential is empty on a subset of positive Lebesgue measure. However if $n = 1$, it can be proven that the limiting sub-differential of any continuous function $f : \mathbb{R} \to \mathbb{R}$ is nonempty almost everywhere. Our proof of Th. 1.2 for corank-one distributions is in some way related to this latter result, but is not a consequence of it.

## 3.2 Application to the proof of Theorem 1.2

In what follows, we denote $e(\cdot) := d_{SR}(x_0, \cdot)^2$.

**Proposition 3.2.** *Let $x \in M$ such that $\partial_L e(x) \neq \emptyset$ and let $\zeta \in \partial_L e(x)$. Then there exists a minimizing trajectory joining $x_0$ to $x$ which admits a normal extremal lift $(x(\cdot), p(\cdot), -\frac{1}{2}, u(\cdot))$ such that $p(1) = \frac{1}{2}\zeta$.*

*Proof.* By definition of the limiting sub-differential, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points in $M$ converging to $x$ and a sequence $(\zeta_n)_{n\in\mathbb{N}} \in \partial_P e(x_n)$ such that $\lim \zeta_n = \zeta$. For each integer $n$, we denote by $u_n(\cdot)$ a minimizing control joining $x_0$ to $x_n$, and by $x_{u_n}(\cdot)$ its associated trajectory. From Prop.

2.3, for each integer $n$, we know that $x_{u_n}(\cdot)$ admits a normal extremal lift $(x_{u_n}(\cdot), p_{u_n}(\cdot), -\frac{1}{2}, u_n(\cdot))$ such that $p_{u_n}(1) = \frac{1}{2}\zeta_n$. Since the sub-Riemannian distance is continuous, the sequence of controls $(u_n(\cdot))_{n\in\mathbb{N}}$ is clearly bounded in $L^2([0,1],\mathbb{R}^m)$, and then up to a subsequence, it converges towards an element $u(\cdot)$ for the weak $L^2$-topology. As a consequence, since the end-point mapping $E_{x_0}$ is continuous for the weak $L^2$-topology (see [9] for a proof), we deduce, passing to the limit, that $E_{x_0}(u(\cdot)) = x$. Furthermore, up to a subsequence the sequence $(x_{u_n}(\cdot))_{n\in\mathbb{N}}$ converges uniformly towards a minimizimg path $x_u(.)$. This implies that the sequence $(p_{u_n}(\cdot))_{n\in\mathbb{N}}$ converges uniformly towards some $p_u(\cdot)$, where $p_u(\cdot)$ is an adjoint vector associated to the trajectory $x_u(\cdot)$, and $p_u(1) = \frac{1}{2}\lim_{n\to\infty}\zeta_n$. Finally the quadruple $(x_u(\cdot), p_u(\cdot), -\frac{1}{2}, u(\cdot))$ is a normal extremal lift of $x_u(\cdot)$. □

Analogously to Th. 2.1, we have the following result.

**Proposition 3.3.** *If the distribution is everywhere of corank one, then $\partial_L e(x) \neq \emptyset$ for almost every $x \in M$.*

*Proof.* In what follows, our point of view being local, we can assume to work in $\mathbb{R}^n$. Denote by $P$ the set of points $x$ of $M$ such that

$$\liminf_{y\to x} \frac{e(y)-e(x)}{\|y-x\|} = -\infty.$$

We have $M = P \cup P^c$, where $P^c$ denotes the complement of the set $P$ in $M$. Note that if $x \in P^c$ then there exists $\alpha \in \mathbb{R}$ such that $\liminf_{y\to x} \frac{e(y)-e(x)}{\|y-x\|} = \alpha$, which means that there exists a neighborhood $\mathcal{V}$ of $x$ such that

$$e(y) \geqslant e(x) + (\alpha - 1)\|y-x\|, \quad \forall y \in \mathcal{V}.$$

We infer that the function $e$ has a Lipschitz continuous support function at $x$ and hence from Prop. 3.1 that $\partial_L e(x)$ is nonempty. The rest of the proof is devoted to show that the set $\partial_L e(x)$ is nonempty for almost every point $x \in P$. We argue by contradiction: denote by $A$ the subset of $P$ where the limiting sub-differential of $f$ is empty, and suppose that $\mu(A) > 0$.

For all $x \in M$, let $\nu(x)$ denote a vector of $T_xM$ transverse to the distribution $\Delta(x)$. We may assume the vector field $\nu(\cdot)$ to be smooth on $M$. Let us consider integral curves of the differential system

$$\dot{y}(t) = \nu(y(t)). \tag{6}$$

¿From Fubini's theorem, there exists an interval $I \subset \mathbb{R}$ and an integral curve $(y(t))_{t\in I}$ of (6) such that the set

$$T := \{t \in I \mid y(t) \in A\},$$

satisfies $\lambda(T) > 0$, where $\lambda$ denotes the Lebesgue measure on $\mathbb{R}$. We are going to prove that some $\bar{t} \in I$ is the limit of local minima of the function $e(\cdot)$ restricted to the curve $y(t)$. To this aim we need different lemmas.

**Lemma 3.4.** *For all $x \in M$, there exist a neighborhood $\mathcal{V}_x$ of $x$ in $M$, a neighborhood $U_x$ of $0$ in $T_x^*M$, and a submanifold $D_x$ of codimension $1$ in $M$, such that*

$$\mathcal{V}_x \cap D_x \subset exp_x(U_x).$$

*Proof.* Clearly the mapping $\exp_x$ is smooth on its domain of definition, and its differential at 0, denoted $d\exp_x(0)$, can be computed as

$$d\exp_x(0)\,.\,\delta p_0 = \delta x(1),$$

where $(\delta x(\cdot), \delta p(\cdot))$ is the solution of the linearized system of system (5) at the equilibrium point $(x, 0)$, such that $\delta x(0) = 0$ and $\delta p(0) = \delta p_0$. This linearized system writes

$$\delta\dot{x}(t) = \sum_{i=1}^{n-1} \langle \delta p(t), f_i(x)\rangle f_i(x), \quad \delta\dot{p}(t) = 0,$$

and thus $\delta p(t)$ is constant, equal to $\delta p_0$, whence

$$\delta x(1) = \sum_{i=1}^{n-1} \langle \delta p_0, f_i(x)\rangle f_i(x). \tag{7}$$

Therefore the mapping $\exp_x$ has rank $n-1$ at the point 0, and the conclusion follows. □

For each $x \in M$, let $(p_i^*(x))_{i=1,\dots,n}$ denote the dual basis in $T_x^*M$ of the basis $(f_1(x), \dots, f_{n-1}(x), \nu(x))$ in $T_xM$. We define the mapping $\Phi : I \times O \to M$, where $O$ is a neighborhood of 0 in $\mathbb{R}^{n-1}$, by the formula

$$\Phi(t, \alpha_1, \dots, \alpha_{n-1}) := \exp_{y(t)}\left(\sum_{i=1}^{n-1} \alpha_i p_i^*(y(t))\right).$$

Using (7), it is quite easy to see that, for all $t_0 \in I$, the mapping $\Phi$ is a local diffeomorphism at $(t_0, 0)$. Thus the following lemma is straightforward.

**Lemma 3.5.** *Let $t_0 \in T$. There exist a neighborhood $\mathcal{V}$ of $y(t_0)$ in $M$ and a smooth function $\rho : \mathcal{V} \longrightarrow I$ such that for every $z \in \mathcal{V}$, one has $z \in D_{y(\rho(z))}$, and such that for every $t \in T$ with $y(t) \in \mathcal{V}$, there holds $\rho(y(t)) = t$. Moreover, there exists a real number $\delta > 0$ such that*

$$|e(z) - e(y(\rho(z)))| \leqslant \delta \|z - y(\rho(z))\|. \tag{8}$$

*for all $z \in \mathcal{V}$.*

Define the continuous function $g : I \to \mathbb{R}$ by $g(t) := e(y(t))$.

**Lemma 3.6.** *There exists $\bar{t} \in T$ and a sequence $(t_n)_{n\in\mathbb{N}}$ of $I$ converging towards $\bar{t}$, such that the function $g$ attains a local minimum at $t_n$, for every integer $n$.*

*Proof.* We argue by contradiction. If the conclusion of the lemma does not hold, this means that for every $t \in T$, there exists a neighborhood $\mathcal{V}_t$ of $t$ in $I$ on which $g$ is monotonous. In particular $g$ has bounded variations on $\mathcal{V}_t$, and hence $g$ is differentiable almost everywhere in $\mathcal{V}_t$. We get a contradiction whenever $\lambda(\mathcal{V}_t \cap T) > 0$; but since $\lambda(T) > 0$ there exists $t \in T$ such that $\lambda(\mathcal{V} \cap T) > 0$ for any neighborhood $\mathcal{V}$ of $t$ in $I$. □

**Lemma 3.7.** *There exists some constant $K > 0$ such that for every integer $n$, the limiting sub-differential $\partial_L e(y(t_n))$ contains an element with norm less than $K$.*

*Proof.* By construction of the sequence $(t_n)_{n\in\mathbb{N}}$, for every integer $n$ the function $g$ attains a minimum at $t_n$. This means that there exists an interval $(a_n, b_n)$ containing $t_n$ such that

$$\forall t \in (a_n, b_n) \quad g(t) \geqslant g(t_n).$$

On the other hand, by Lemma 3.5, there exists a neighborhood $\mathcal{V}$ of $y(\bar{t})$ such that for $n$ large enough, any $x$ close enough to $y(t_n)$ belongs to $D_{y(\rho(x))}$ where $\rho(x) \in (a_n, b_n)$. By (8), we deduce that for $x$ close enough to $y(t_n)$, there holds

$$\begin{aligned} e(x) &\geqslant e(y(\rho(x))) - \delta\|x - y(\rho(x))\| \\ &\geqslant e(y(t_n)) - \delta\|x - y(\rho(x))\|. \end{aligned}$$

Therefore if we define locally $\phi(x) := -\delta\|x - y(\rho(x))\|$, the function $e - \phi$ attains a local minimum at $y(t_n)$. Since $\phi$ is Lipschitz continuous, the sum rule on limiting sub-differentials (see [6, Prop. 10.1 p. 62]) implies that

$$0 \in \partial_L(e - \phi)(y(t_n)) \subset \partial_L e(y(t_n)) + \partial_L(-\phi)(y(t_n)).$$

Hence there exists $\zeta \in \partial_L e(y(t_n))$ and $\zeta' \in \partial_L(-\phi)(y(t_n))$ such that $0 = \zeta + \zeta'$. Finally $\|\zeta\| = \|\zeta'\|$ where $\|\zeta'\|$ is less than the Lipschitz constant of the function $\phi$. This concludes the proof of the lemma. □

Returning to the proof of Prop. 3.3, we infer easily that $\partial_L e(y(\bar{t}))$ is nonempty. This yields a contradiction with the fact that $y(\bar{t}) \in A$, and ends the proof of the proposition. □

Propositions 3.2 and 3.3 imply the existence of a subset $N$ of full Lebesgue measure in $M$ such that, for every $x \in N$, there exists a minimizing trajectory steering $x_0$ to $x$ and having a normal extremal lift. Let $N_2 := N \setminus \mathcal{C}(x_0)$. It is the set of points $x \in M$ which are not conjugate to $x_0$, and such that there exists a minimizing path $x(\cdot)$ joining $x_0$ to $x$ and having a normal extremal lift. Remark 1.7 implies that the trajectory $x(\cdot)$ is moreover nonsingular. From Remark 1.6 it is clear that $N_2$ is of full Lebesgue measure in $M$. This ends the proof of Th. 1.2.

# 4 Consequences and open questions

In what follows, we assume that Chow's condition holds, and that the manifold $M$ is complete for the sub-Riemannian distance. Let $x_0 \in M$ be fixed.

## 4.1 A formula for the sub-Riemannian distance

¿From Th. 1.1, there exists a dense subset $N_1$ of $M$ such that every point of $N_1$ can be joined from $x_0$ by a unique minimizing trajectory, which moreover admits a normal extremal lift. This yields the following result.

**Corollary 4.1.** *For all point $x \in N_1$ one has*

$$d_{SR}(x_0, x) = \inf \left\{ (2\, H_1(x_0, p))^{1/2} \ \mid p \in U \ s.t. \ exp_{x_0}(p) = x \right\}.$$

*Remark* 4.1. Actually Th. 1.1 implies that for every $x \in N_1$ there exists a unique $p \in U$ such that the above infimum is attained.

As a consequence, we deduce that the function $g : M \to \mathbb{R} \cup \{\infty\}$ defined by

$$g(x) := \inf \left\{ (2\, H_1(x_0, p))^{1/2} \ \mid p \in U \text{ s.t. } \exp_{x_0}(p) = x \right\},$$

for all $x \in M$, coincides with the mapping $d_{SR}(x_0, \cdot)$ on a dense subset of the manifold $M$. In particular, since $g$ is continuous on $M$, there holds

$$d_{SR}(x_0, x) = \inf \{ \lim g(x_n) \mid x_n \to x \}$$

for all $x \in M$.

*Remark* 4.2. If the sub-Riemannian distance to $x_0$ is Lipschitz continuous outside $x_0$, then from Prop 2.2 the limiting sub-differentials of $d_{SR}(x_0, \cdot)$ are always nonempty; hence the set of points $x$ of $M$ such that every minimizing trajectory joining $x_0$ to $x$ is strictly singular, is empty. The converse is false; a counterexample is given by the so-called Martinet flat case, see [2]. To get a converse statement, the assumption has to be strengthened as follows: if there does not exist any nontrivial singular minimizing trajectory, then $d_{SR}(x_0, \cdot)$ is Lipschitz continuous outside $x_0$, see [1].

## 4.2 On the sub-Riemannian wave-front and sphere

The following result is a direct consequence of Th. 1.1.

**Corollary 4.2.** *The sub-Riemannian wave-front $W_{SR}(x_0, r)$ is connected, for all $r > 0$.*

*Proof.* Using notations of Remark 1.5, and from Th. 1.1, we have the inclusions

$$\exp_{x_0}(U_r) \subset W_{SR}(x_0, r) \subset \overline{\exp_{x_0}(U_r)},$$

where $U_r$ is diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}^{n-m}$, and thus is connected. The conclusion follows readily. □

**Proposition 4.3.** *If the distribution $\Delta$ is everywhere of corank one, then the sub-Riemannian wave-front $W_{SR}(x_0, r)$, and thus the sub-Riemannian sphere $S_{SR}(x_0, r)$, has Lebesgue measure zero, for all $r > 0$.*

*Proof.* It suffices to notice that the image by a locally lipschitzian mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ of a set of zero measure has zero measure, and to apply Th. 1.2. □

## 4.3 Further comments

Let $N_3$ be the set of points $x \in M$ such that there exists a unique minimizing path $x(\cdot)$ joining $x_0$ to $x$, which moreover admits a normal extremal lift, and such that $x$ is not conjugate to $x_0$. Remark 1.7 implies that $x(\cdot)$ is nonsingular. Notice that, with notations of Th. 1.1, one has $N_3 = N_1 \setminus \mathcal{C}(x_0)$.

**Proposition 4.4.** *The subset $N_3$ is a nonempty open subset of $M$.*

*Remark* 4.3. From Cor. 4.1, the mapping $d_{SR}(x_0, \cdot)$ is smooth on $N_3$.

*Proof.* Let $x \in N_3$ and $p_0 \in U$ such that $x = \exp_{x_0}(p_0)$. The proof follows from the three following lemmas.

**Lemma 4.5.** *There exists a neighborhood $V_1$ of $x$ such that every minimizing control steering $x_0$ to a point of $V_1$ is nonsingular.*

*Proof of Lemma 4.5.* By contradiction, let us assume that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $M$ converging towards $x$, such that for each integer $n$ there is a singular minimizing control $v_n(\cdot)$ steering $x_0$ to $x_n$. In particular, for each $n$ there exists a Lagrange multiplier $\psi_n \in T^*_{x_n}M$ such that

$$x_n = E_{x_0}(v_n(\cdot)) \text{ and } \psi_n.dE_{x_0}(v_n(\cdot)) = 0,$$

where the sequence $(\psi_n)_{n\in\mathbb{N}}$ may be assumed to be bounded; thus up to a subsequence it converges to a covector $\psi$. On the other part, by continuity of the sub-Riemannian distance, the sequence $(v_n(\cdot))_{n\in\mathbb{N}}$ is clearly bounded in $L^2([0,1], \mathbb{R}^m)$, and hence up to a subsequence it converges towards an element $v(\cdot)$ for the weak $L^2$-topology. Since the mappings $E_{x_0}$ and $dE_{x_0}$ are continuous for the weak $L^2$-topology (see [9]), passing to the limit yields

$$x = E_{x_0}(v(\cdot)) \text{ and } \psi.dE_{x_0}(v(\cdot)) = 0.$$

Moreover the control $v(\cdot)$ is minimizing, and therefore the point $x$ is joined from $x_0$ by a minimizing singular control. This is a contradiction with the definition of $N_3$. □

In particular this lemma implies that $V_1 \subset \exp_{x_0}(U)$. Set $U_1 := \exp_{x_0}^{-1}(V_1)$.

**Lemma 4.6.** *The mapping $exp_{x_0}$ is proper from $U_1$ into $V_1$.*

*Proof of Lemma 4.6.* We argue by contradiction, and suppose that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $M$ converging towards $x$, such that for each integer $n$ there exists $p_n \in T^*_{x_0}M$, with $x = \exp_{x_0}(p_n)$, such that $(p_n)_{n\in\mathbb{N}}$ is not bounded. For each $n$, denote by $(x_{u_n}(\cdot), p_{u_n}(\cdot), -\frac{1}{2}, u_n(\cdot))$ the associated normal extremal, and set $\psi_n = p_{u_n}(1)$. It is easy to see that the sequence $(\psi_n)_{n\in\mathbb{N}}$ is not bounded (see [9] for further details). Moreover one has, according to the Lagrange multipliers rule

$$x_n = E_{x_0}(u_n(\cdot)) \quad \text{and} \quad \psi_n.dE_{x_0}(u_n(\cdot)) = \frac{1}{2}dC(u_n(\cdot)).$$

Up to a subsequence we may assume that $\psi_n/\|\psi_n\|$ tends to $\psi$, and passing to the limit as in the proof of the previous lemma one infers the existence of a control $u(\cdot)$ such that

$$x = E_{x_0}(u(\cdot)) \quad \text{and} \quad \psi.dE_{x_0}(u(\cdot)) = 0.$$

In particular $x$ is joined from $x_0$ by a singular control, and thus is conjugate to $x_0$; this is a contradiction with the definition of $N_3$. □

Lemma 4.6 implies that the set of $\{p \mid \exp_{x_0}(p) = x\}$ is compact in $U$. Moreover since $x$ is not conjugate (and from Remark 1.8), this set has no cluster point, hence it is finite. As a consequence, up to reducing $V_1$ we can assume that $V_1$ is a connected open subset of $\exp_{x_0}(U)$, and that $U_1$ is a finite union of disjoint connected open sets, all of which being diffeomorphic to $V_1$ by the mapping $\exp_{x_0}$. We infer that every point $y \in V_1$ is not conjugate to $x_0$ and that it is joined from $x_0$ by a finite number of normal extremals. From Lemma 4.5, every minimizing trajectory joining $x_0$ to $y$ admits a normal extremal lift. On the other hand, when $y$ tends to $x$, every minimizing trajectory joining $x_0$ to $y$ tends to a minimizing trajectory joining $x_0$ to $x$. Therefore, if $y$ is close enough to $x$, there exists a unique minimizing trajectory steering $x_0$ to $y$, which moreover admits a normal extremal lift. Therefore $V_1 \subset N_3$, and thus $N_3$ is open in $M$. The fact that $N_3$ is nonempty is a consequence of the Morse theory on optimality, according to which every minimizing path starting from $x_0$, with small enough length, and having a normal extremal lift, is the unique minimizing trajectory between its endpoints $x_0$ and $x$, and moreover the point $x$ is not conjugate to $x_0$. □

Let us now analyze in more details the set $N_2 \setminus N_1$.

**Lemma 4.7.** *Let $x \in N_2 \setminus N_1$. Then either $x$ belongs to the subset $R_2$, where $R_2$ denotes the set of points $x$ such that $x$ is not conjugate to $x_0$, and there exist (at least) two minimizing trajectories joining $x_0$ to $x$ and having a normal extremal lift; or $x$ belongs to $R_1$, where $R_1$ denotes the set of points $x$ such that $x$ is not conjugate to $x_0$, and on the one part there exists a unique minimizing trajectory $x_1(\cdot)$ joining $x_0$ to $x$ and having a normal extremal lift, and on the other part there exists a strictly singular minimizing path $x_2(\cdot)$ steering $x_0$ to $x$.*

We have the following result on $R_2$.

**Lemma 4.8.** *The subset $R_2$ has Lebesgue measure zero in $M$.*

*Proof.* Let $x_1 \in R_2$, and let $p_1, p_2 \in U$ so that $x_1 = \exp_{x_0}(p_1) = \exp_{x_0}(p_2)$. Since $x_1$ is not conjugate to $x_0$, the mapping $\exp_{x_0}$ is a diffeomorphism from a neighborhood $U_1$ of $p_1$ (resp. a neighborhood $U_2$ of $p_2$) into a neighborhood $V$ of $x_1$, denoted $\varphi_1$ (resp. $\varphi_2$). For all $x \in V$, we set $h_i(x) = 2H_1(x_0, \varphi_i^{-1}(x))$, $i = 1, 2$. Using notations of Introduction and of Remark 1.7, $h_i(x)$ is equal to the length of the path $x(\cdot, \varphi_i^{-1}(x))$, and thus $h_i(E_{x_0}(u(\cdot))) = C(u(\cdot))$, for all control $u(\cdot)$ in a neighborhood of $u(\cdot, p_i)$. By differentiating, one gets

$$\nabla h_i(x_1).dE_{x_0}(u(\cdot, p_i)) = dC(u(\cdot, p_i)),$$

and hence from (3) and (4) we can assume that

$$p(1, p_i) = \frac{1}{2}\nabla h_i(x_1).$$

On the other part, one has

$$R_2 \cap V \subset \{x \in V \mid h_1(x) = h_2(x)\}.$$

From Remark 1.4 it is clear that $p_1$ and $p_2$ are independent, and hence so are $\nabla h_1(x_1)$ and $\nabla h_2(x_1)$. The conclusion follows easily. □

## 4.4 Sard type conjectures

Let $\mathcal{A}$ (resp. $\mathcal{A}_s$) denote the set of points $x$ of $M$ such that every minimizing trajectory joining $x_0$ to $x$ is singular (resp. strictly singular). Obviously $\mathcal{A}_s \subset \mathcal{A}$. Th. 1.1 and 1.2 yield the following result.

**Corollary 4.9.** *The subset $\mathcal{A}_s$ has an empty interior in $M$. In the case of a corank-one distribution the subset $\mathcal{A}$ has Lebesgue measure zero in $M$.*

Let now $\mathcal{S}$ (resp. $\mathcal{S}_{min}$, resp. $\mathcal{S}_{min}^{strict}$) denote the set of points $x$ of $M$ such that there exists a singular trajectory (resp. a singular minimizing trajectory, resp. a strictly singular minimizing trajectory) steering $x_0$ to $x$. Notice that $\mathcal{S}$ is the set of critical values of the end-point-mapping $E_{x_0}$.

**Corollary 4.10.** *The set $\mathcal{S}_{min}^{strict}$ has an empty interior in $M$.*

We formulate the following conjecture.

**Conjecture 4.11.** The subset $N_3$ of Prop. 4.4 is of full Lebesgue measure in $M$. In particular, the set $\mathcal{S}_{min}$ has Lebesgue measure zero in $M$.

We end the paper with the following open question.

**Conjecture 4.12.** The end-point mapping satisfies Sard's property, *i.e.* the set $\mathcal{S}$ has Lebesgue measure zero in $M$.

This conjecture has been formulated and discussed, among others, in [7]. Up to now, it is still open, even in the case of a corank-one distribution.

**Acknowledgement.** The authors are very indebted to F. H. Clarke for useful discussions.

# References

[1] A. Agrachev. Compactness for sub-Riemannian length-minimizers and subanalyticity. *Rend. Sem. Mat. Univ. Politec. Torino*, 56(4):1–12, 1998.

[2] A. Agrachev, B. Bonnard, M. Chyba, and I. Kupka. Sub-Riemannian sphere in Martinet flat case. *ESAIM Control Optim. Calc. Var.*, 2:377–448 (electronic), 1997.

[3] A. Bellaïche. Tangent space in sub-Riemannian geometry. Sub-Riemannian geometry, Birkhäuser, 1996.

[4] J.-M. Bismut. Large deviations and the Malliavin calculus. Progress in Mathematics 45, Birkhäuser, 1984.

[5] J. M. Borwein and D. Preiss. A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions. *Trans. Amer. Math. Soc.*, 303(2):517–527, 1987.

[6] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski. *Nonsmooth Analysis and Control Theory.* Graduate Texts in Mathematics, vol. 178. Springer-Verlag, New York, 1998.

[7] R. Montgomery. A tour of subriemannian geometries, their geodesics and applications. Mathematical Surveys and Monographs, 91. American Mathematical Society, Providence, RI, 2002.

[8] L. Pontryagin et al.. Théorie mathématique des processus optimaux. Eds Mir, Moscou, 1974.

[9] E. Trélat. Some properties of the value function and its level sets for affine control systems with quadratic cost. *J. of Dyn. and Cont. Syst.*, 6(4):511–541, 2000.