Available online at www.sciencedirect.com
ScienceDirect
Applied Numerical Mathematics 58 (2008) 946–954
APPLIED NUMERICAL MATHEMATICS
www.elsevier.com/locate/apnum

A pseudo-penalization method for high Reynolds number unsteady flows

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Available online 29 April 2007

Abstract

A pseudo-penalization method, i.e., an implicit formulation of the standard volume penalization method, is introduced and validated by considering the 2D wake of a cylinder at Reynolds number $Re = 200$.

MSC: 76D05; 76D25; 76M22

Keywords: Penalization methods; Spectral methods; Wake flows

1. Introduction

Efficient numerical solvers of the Navier–Stokes equations are mainly available for simple geometries: when complex geometries are concerned, more sophisticated algorithms are required and this is at the price of a much higher computational cost. This is especially obvious for spectral methods, the cost of a spectral element computation being generally higher than the cost of a standard (Fourier, Chebyshev or Legendre) spectral one. As a result it is often necessary to use coarse grids yielding to under-resolved results. This problem is especially sensitive when turbulent flows are computed, using DNS (Direct Numerical Simulation) or LES (Large-Eddy Simulation), since in this case very small structures develop in the flow. Moreover, the inertial time-scale being much smaller than the momentum diffusion scale, these structures maintain a long time before disappearing. Thus, for wake flows the very small structures that develop in the very near wake may be recovered in the far wake and so using very fine grids close to the obstacle and a rough one a few length-scales beyond may be disappointing.

Although the methodology described in the paper may be applied to various flows, we focus on wake flows, since the modeling of the obstacle by using a penalization method is particularly attractive: The obstacle is embedded in a domain of simple shape for which an efficient solver is available and the computational domain is discretized everywhere with a fine grid. Such an approach, that we call the "pseudo-penalization method", is described here. Simulation results have already been reported in previous papers [2,8] but the method used has not yet been described in details. The departure point is the standard volume penalization method, see, e.g. [1,5–7,10,11] for wake flow problems, for which a force term of the form $\boldsymbol{f} = -C\chi\boldsymbol{u}$ is introduced in the Navier–Stokes equations to cancel the velocity inside

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doi:10.1016/j.apnum.2007.04.011

the obstacle ($\boldsymbol{u}$ is the velocity, C the penalization coefficient and χ the characteristic function of the obstacle, $\chi(\boldsymbol{x})=1$ ($\chi(\boldsymbol{x})=0$) for $\boldsymbol{x}$ inside (outside) the obstacle). However, the pseudo-penalization method described hereafter does not use an explicit penalization term so that no penalization coefficient appears in its formulation.

In Section 2 the pseudo-penalization method is introduced. Improvements in the modeling of the obstacle are proposed and justified in Section 3. In order to illustrate the capabilities of the method, in Section 4 we focus on the 2D wake of a cylinder at Reynolds number $Re=200$ and finally conclude in Section 5.

2. The pseudo-penalization method

In dimensionless form the Navier–Stokes (NS) system may write:

$$\partial_t \boldsymbol{u} + N(\boldsymbol{u}) = -\nabla p + \nu \Delta \boldsymbol{u} \quad \text{in } \Omega \times (0, t_F), \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2}$$

$$B(\boldsymbol{u}) = \boldsymbol{g} \quad \text{on } \Gamma \times (0, t_F), \tag{3}$$

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 \quad \text{in } \Omega, \tag{4}$$

where t (t_F) is the (final) time, Ω the computational domain and Γ its boundary, $\boldsymbol{u}$ the velocity, p a pressure term, ν the dimensionless viscosity (inverse of the Reynolds number) and where $N(\boldsymbol{u})$ stands for the non-linear transport term (in any form: convective, conservative, etc.). The boundary operator B and the function $\boldsymbol{g}$ are used to express the boundary condition, $\boldsymbol{u}_0$ stands for the initial velocity field and both the boundary and initial conditions are assumed such that the NS problem is well-posed. To solve numerically the NS system we assume that the non-linear term $N(\boldsymbol{u})$ is handled explicitly and the diffusion term implicitly. Then the following semi-discrete problem must be considered at each time-step:

$$\nu \Delta \boldsymbol{u}^{n+1} - \frac{\alpha}{\tau} \boldsymbol{u}^{n+1} - \nabla p^{n+1} = \boldsymbol{f}^{n+1} \quad \text{in } \Omega, \tag{5}$$

$$\nabla \cdot \boldsymbol{u}^{n+1} = 0, \tag{6}$$

$$B(\boldsymbol{u}^{n+1}) = \boldsymbol{g}^{n+1} \quad \text{on } \Gamma, \tag{7}$$

where τ is the time-step, $\boldsymbol{u}^k$ (p^k) the numerical approximation at time $k\tau$ of $\boldsymbol{u}$ (p), α a coefficient and $\boldsymbol{f}^{n+1}$ a force term.

The coefficient α and $\boldsymbol{f}^{n+1}$ depend on the time-scheme. Thus, if a backward Q-order finite difference approximation of the time derivative is used, i.e. if

$$\partial_t \boldsymbol{u} = \frac{1}{\tau} \sum_{q=0}^{Q} \alpha_q^Q \boldsymbol{u}^{n+1-q} + \mathrm{O}(\tau^Q), \tag{8}$$

where α_q^Q, $q=0,\ldots,Q$, are given coefficients, then

$$\alpha = \alpha_0^Q, \qquad \boldsymbol{f}^{n+1} = \left[N(\boldsymbol{u})\right]^{n+1} + \frac{1}{\tau} \sum_{q=1}^{Q} \alpha_q^Q \boldsymbol{u}^{n+1-q}, \tag{9}$$

with $[N(\boldsymbol{u})]^{n+1}$ for a $\mathrm{O}(\tau^Q)$ approximation of the non-linear term.

Let us assume now that an obstacle, defined by a domain ω, is embedded in the computational domain Ω, i.e. $\bar{\omega} \subset \Omega$, and solve:

$$\nu \Delta \boldsymbol{u}^{n+1} - \frac{\alpha}{\tau} \boldsymbol{u}^{n+1} - \nabla p^{n+1} = (1-\chi) \boldsymbol{f}^{n+1} \quad \text{in } \Omega, \tag{10}$$

$$\nabla \cdot \boldsymbol{u}^{n+1} = 0, \tag{11}$$

$$B(\boldsymbol{u}^{n+1}) = \boldsymbol{g}^{n+1} \quad \text{on } \Gamma, \tag{12}$$

where χ is the characteristic function of ω. Clearly, in $\Omega \backslash \omega$ we still have an approximation of the NS equations, whereas in ω we are left with the steady Stokes equations penalized by the term $\alpha \boldsymbol{u}/\tau$ and expressed at time t_{n+1}:

$$\nu\Delta \boldsymbol{u} - \frac{\alpha}{\tau}\boldsymbol{u} - \nabla p = 0 \quad \text{in } \omega \times \{t_{n+1}\}, \tag{13}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{14}$$

The pseudo-penalization method, as expressed by Eqs. (10)–(11), is thus equivalent to consider the Navier–Stokes equations in $\Omega\backslash\omega$ coupled to penalized steady Stokes equations in ω. The coupling occurs through the essential and natural transmission conditions:

$$[\![\boldsymbol{u}]\!] = 0, \qquad \nu[\![\nabla \boldsymbol{u}]\!] \cdot \boldsymbol{n} = [\![p]\!]\boldsymbol{n} \tag{15}$$

where we have denoted by $[\![\cdot]\!]$ the jump at the interface and where $\boldsymbol{n}$ is the unit outwards vector normal to this interface.

In the limit $\tau = 0$ the penalization coefficient $C = \alpha/\tau$ is infinite and $\boldsymbol{u}|_\omega = 0$. For $\tau \to 0$, one may expect that $\boldsymbol{u}|_\omega = \mathrm{O}(\tau)$, at least beyond a thin layer at the boundary of ω. With $\boldsymbol{v} = \nabla \times \boldsymbol{u}$ for the vorticity and $\epsilon = \tau/\alpha$, from the identities $\nabla \times \nabla p \equiv 0$ and $\nabla \times \nabla \times \boldsymbol{u} \equiv \nabla(\nabla \cdot \boldsymbol{u}) - \Delta \boldsymbol{u}$ we indeed have:

$$\nu\Delta \boldsymbol{v} - \frac{1}{\epsilon}\boldsymbol{v} = 0 \quad \text{in } \omega \times (0, t_F), \tag{16}$$

$$\Delta \boldsymbol{u} + \nabla \times \boldsymbol{v} = 0 \tag{17}$$

so that as soon as $\boldsymbol{v} \approx 0$, say such that $|\nabla \times \boldsymbol{v}| \leqslant \epsilon/\nu$, then $\boldsymbol{u} = -\epsilon\nabla p + \mathrm{O}(\epsilon^2)$. Assuming that $\lim_{\epsilon\to 0} p = p_0$, we then have the first order leading term $\boldsymbol{u} \approx -\epsilon\nabla p_0$. Now, from Eq. (16), one expects that with d for the distance to the boundary of ω, $\boldsymbol{v} \propto \exp(-d/\sqrt{\epsilon\nu})$, so that the thickness of the artificial boundary layer scales like $\sqrt{\epsilon\nu}$.

In comparison with the standard volume penalization method, the penalization coefficient of the pseudo-penalization method is thus not infinite but finite, with a $C = \mathrm{O}(1/\tau)$ value. Thus, for the first- and second-order backward finite difference approximations $C = 1/\tau$ and $C = 3/2\tau$, respectively. However in practice the penalization coefficient is always finite and sometimes with a rather low value. In [4] we thus used $C = 100$. This results from stability constraints if the penalization term is handled explicitly or from the ill-conditioning of the final algebraic system if it is handled implicitly. The explicit case is considered in Appendix A by focusing on a simple 1D problem. Moreover, using a large value of the penalization coefficient, say $C > 10^6$, as e.g. in [1,6], is certainly preferable in the native form of the penalization method and when very fine grids are involved, but this may be no-longer true with an improved modeling of the obstacle. Such an improved modeling is introduced in the next section and the link with the pseudo-penalization method is emphasized.

3. Improved modeling of the obstacle

3.1. Characteristic function

When using the volume penalization method, it may be of interest to smooth the characteristic function of the obstacle. This is especially true when spectral methods are concerned, to avoid or at least to weaken the Gibbs phenomenon. Thus in [4], to compute the wake of a sphere we proposed to regularize the characteristic function by applying the raised cosine filter. It is of interest to point out the result of the raised-cosine filtering in physical space. To this end, let us restrict ourselves to the 1D case with $\Omega = (0, 2\pi)$ and $\omega = (a, b)$, with $0 < a < b < 2\pi$. We assume to have a discretization $x_i = ih$, $i = 0, \dots, N-1$, with $h = 2\pi/N$. We assume also that $b - a > 2h$. In Fourier space the raised cosine filter reads:

$$\hat{g}_k = \frac{1}{2}\left(1 + \cos\left(\frac{2\pi k}{N}\right)\right), \quad |k| \leqslant \frac{N}{2}, \tag{18}$$

where k is the Fourier mode number. In physical space this corresponds to a sequence $g_i = g(ih)$, $i = 0, \dots, N-1$, such that:

$$g_i = \sum_{k=-N/2}^{N/2-1} \hat{g}_k \exp\left(\frac{\mathrm{i}ki}{2\pi/N}\right), \quad (\mathrm{i}^2 = -1). \tag{19}$$

Taking into account the discrete orthogonality property of the cosine and sine functions one obtains:

$$g_0 = \frac{N}{2}, \tag{20}$$

$$g_1 = g_{N-1}\ (= g_{-1}) = \sum_{k=-N/2}^{N/2-1} \cos^2\left(\frac{2\pi k}{N}\right) = \frac{N}{4}, \tag{21}$$

$$g_i = 0, \quad \text{otherwise.} \tag{22}$$

In physical space, applying the raised cosine filter consists of a discrete convolution with the sequence $\{g_i\}$ (up to the multiplicative coefficient N). For a function $u(x)$ such that $u(ih) = u_i$ we thus obtain the filtered values:

$$\bar{u}_i = \frac{1}{4}(u_{i-1} + 2u_i + u_{i+1}) \quad (u_{-1} = u_{N-1}). \tag{23}$$

For the characteristic function of $\omega = (a, b)$, the sole values which are changed by the raised cosine filter are thus those of the grid-points immediately close to $x = a$ and $x = b$. As a result the sequence $\{\dots 0, 0, 1, 1, \dots, 1, 1, 0, 0, \dots\}$ is simply changed in $\{\dots 0, 1/4, 3/4, 1, \dots, 1, 3/4, 1/4, 0, \dots\}$.

Let us now focus on the accuracy aspect. It is clear that the penalization method can only give a O(h) accuracy, except of course if $x = a$ and $x = b$ correspond to two grid-points. Such a result is not altered, but also not improved, if it is the raised-cosine filtered characteristic function which is used (but in any cases the approximation is then O(h)). Even if the O(h) approximation appears definitive, one can however try to improve the situation on the grounds of the two following remarks:

– The raised-cosine regularized characteristic function appears especially adapted if both $x = a$ and $x = b$ are exactly between two grid-points.
– Moving the end-points $x = a$ and $x = b$ between two grid-points changes nothing in the penalization method result.

It is then of interest to notice that the sequence $\{\dots 0, 1/4, 3/4, 1, \dots, 1, 3/4, 1/4, 0, \dots\}$ shown by the regularized characteristic function, say $\bar{\chi}$, may find a simple interpretation. Thus, assume that $x = a$ (and $x = b$) are exactly between two grid-points, then:

$$\bar{\chi}_i \equiv \bar{\chi}(ih) = \frac{1}{2h} \int_{(i-1)h}^{(i+1)h} \chi \, dx, \quad i = 0, \dots, N-1, \tag{24}$$

so that $\bar{\chi}$ can be obtained from χ by local averaging. If Eq. (24) is used as a new definition of the regularized characteristic function, one can obtain values of the $\bar{\chi}_i$ that take into account the exact location of the end-points, $x = a$ and $x = b$, with respect to the grid-points. The regularized characteristic function, as defined by (24), appears in any cases well adapted and moving the end-points between two grid-points has now an influence on the penalization method result. Note that such an approach appears similar to the one of the VOF (Volume Of Fluid) method to describe the unsteady free surface of a fluid using a fixed Cartesian grid.

The present 1D development can be easily extended to the 3D case: In the definition (24) it is then a 3D integral which should be considered, generally at the price of a numerical quadrature. To this end we suggest to simply use the method of rectangles, which is the best adapted to sum the characteristic function.

3.2. Penalization coefficient

When using the regularized characteristic function, for a given grid there exists an optimal value of the penalization coefficient C. This optimal parameter, say $\hat{C}$, depends on the discretization: $\hat{C} \equiv \hat{C}(N)$.

To make clear this point we present in Fig. 1 the numerical results of some tests carried out on a 1D steady advection–diffusion problem, for which the exact solution is known. Clearly, on the left-hand side, for a given grid the approximation error shows a minimum corresponding to an optimal value $C = \hat{C}$. One observes that $\hat{C}$ is an increasing function of N (a Chebyshev method is used, so that N is here the degree of the polynomial approximation and the number of grid-points is $N + 1$) and that for $C \gg \hat{C}$ the result becomes independent of C. On the right-hand side we have plotted the variations of $\hat{C}$ with respect to N. It can be checked that $C \sim N^2$.

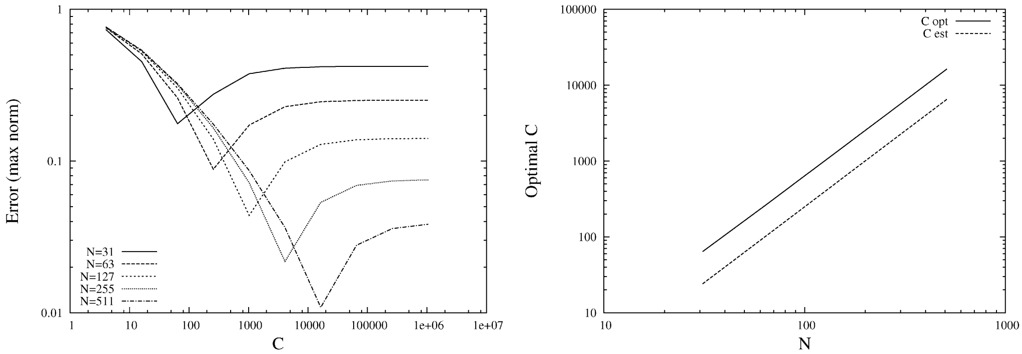

Fig. 1. The steady advection–diffusion equation $\partial_x u = 0.1\partial_x^2 u$ is considered in $(-1,0)\cup(0.5,1)$, with $u(-1)=1$, $u(0)=u(0.5)=u(1)=0$. The problem is solved by using a collocation Chebyshev method in $(-1,1)$ with penalization in $(0,0.5)$. The raised cosine filter is used to smooth the characteristic function. Left: Max-norm of the error vs C for different N (polynomial approximation degree); Right: Optimal value of C vs N and estimate of this value.

The quadratic increase of $\hat{C}$ with respect to the number of grid-points may be interpreted. Clearly, if the penalization coefficient is too large, then the numerical solution depends too strongly on the intermediate values of $\bar{\chi}$, i.e. so that $0<\bar{\chi}<1$, and one recovers the non-smoothed situation but with an enlarged characteristic function. At the optimum there should be a good balancing between the penalization term and the numerical approximations of the other terms, so that any (small) perturbation of the numerical solution should induce comparable variations. For the second-order PDE in which we are interested, it is the second-order term which is the most sensitive to such a perturbation, because its approximation is in any case proportional to $1/h^2$. When looking at Eq. (13), or simply at the penalized 1D advection–diffusion equation, one then expects that $\hat{C}\sim \nu/h^2$.

Coming back to the advection–diffusion numerical test this would mean that $\hat{C}\approx 0.1/(2/N)^2$. In Fig. 1 (right) we compare this estimated value, $\hat{C}_{\text{est}}$, with $\hat{C}$. We observe that $\hat{C}=\mathrm{O}(\hat{C}_{\text{est}})$, more precisely that $\hat{C}=\gamma\hat{C}_{\text{est}}$, with $\gamma\approx 2.5$. This value of γ is of course certainly not universal, depending on the test-problem, on the numerical method, etc. but one may expect that $\gamma=\mathrm{O}(1)$.

3.3. *Link with the pseudo-penalization method*

It remains to make the link with the $C=\alpha/\tau$ coefficient of the pseudo-penalization method. To this end let us come back to the NS equations and derive the $\mathrm{O}(1/\tau)$ value on the basis of the following assumptions:

- The CFL condition $\max|\boldsymbol{u}|\tau/h\leqslant 1$ holds and $\max|\boldsymbol{u}|\gtrsim 1$ (a correct choice of the reference velocity is assumed), so that $\tau\leqslant h$.
- The dimensionless viscosity is such that $\nu\leqslant h$. Such an hypothesis is generally fulfilled when high Reynolds flows are computed (remind that with $\nu\ll 1$, the boundary layer thickness is $\mathrm{O}(\nu^{1/2})$).

Then one has:

$$\hat{C}=\frac{\gamma\nu}{h^2}\leqslant\frac{\gamma}{h}\leqslant\frac{\gamma}{\tau} \tag{25}$$

so that it may be expected that the $C=\alpha/\tau$ implicit value of the penalization coefficient of the pseudo-penalization method will provide satisfactory results.

Moreover, in Section 2 it was found that the artificial boundary layer thickness involved by the pseudo-penalization method was $\mathrm{O}(\sqrt{\epsilon\nu})$, with $\epsilon=\tau/\alpha$. Under the previous assumptions it turns out that this thickness is then $\mathrm{O}(h)$. Thus, the pseudo-penalization method involves an artificial boundary layer coherent with the spatial discretization and so with the smoothing of the characteristic function that has been proposed.

4. 2D wake of a cylinder

Realistic 3D results using the pseudo-penalization method may be found elsewhere [2,8]. Here we focus on a more academic example: the 2D wake of a cylinder in a cross-flow confined geometry for a Reynolds number $Re = 200$. The computational domain is $\Omega = (-6.5, 17.5) \times (-4, 4)$ and the cylinder is centered at $x = y = 0$. At $y = \pm 4$ we have a free-slip boundary condition. At the inlet we enforce $\boldsymbol{u} = (1, 0)$ and at the outlet we use as soft outflow boundary condition an advection equation at the velocity $\boldsymbol{u} = (1, 0)$. At the initial time $t = 0$ the fluid is at rest. A multi-domain decomposition technique is used in the x-streamwise direction: we have 5 subdomains and the subdomain interfaces are located at $x = \{-0.5, 2.5, 6.5, 11.5\}$. In each subdomain a Chebyshev collocation method is used, with polynomials of degree 60 and 120 in the x and y directions, respectively. The time-step equals $\tau = 4 \times 10^{-3}$. The time-scheme is second-order accurate and makes use of three steps: (i) Transport step: the convective term is handled explicitly with an OIF (Operator Integration Factor) semi-Lagrangian method (see e.g. [3]); (ii) Diffusion step: the viscous term is handled implicitly; (iii) Projection step: a Darcy problem is solved to obtain a divergence free velocity field and the pressure is then updated. Details on the algorithm may be found in [2,8].

In Fig. 2 we show the isolines of a transported quantity (Péclet number $Pe = 1400$), in order to well visualize the spatial development of the wake. The Strouhal number, characteristic of the vortex shedding phenomenon equals $St = 0.194$. Such a value of the Strouhal number compares well with the $St = 0.198$ value given in [9].

In Fig. 3 are plotted some mean profiles of the stream-wise component of the velocity at different x-locations and the variations of u_x along the streamwise direction $y = 0$. The figure clearly points out the recirculation length, which equals $l_R = 0.906$, and the minimum of the streamwise velocity, $\min(u_x) \approx -0.18$. These quantities are again in very good agreement with [9] (note that in [9] the reference length is not the diameter but the radius of the cylinder).

To visualize the efficiency of the pseudo-penalization, in Fig. 4 (left) are plotted the components of the velocity along the diameter, both in the x-streamwise and in the y-crossflow directions. In order to estimate the amplitude of the residual velocity inside the cylinder, the absolute values are given in log-plot in Fig. 4 (right). These values appear less than 10^{-2}. The expected values were $O(2\tau/3)$, so that with $2\tau/3 = 2.67 \times 10^{-3}$ one may consider again the result to be satisfactory.

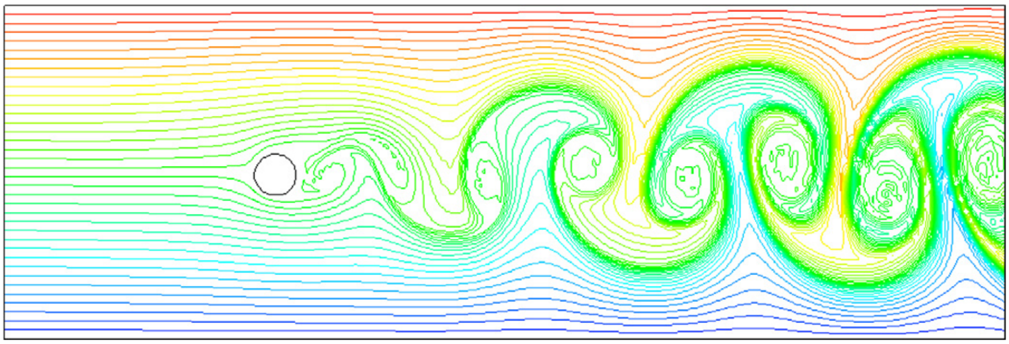

Fig. 2. Instantaneous passive scalar field.

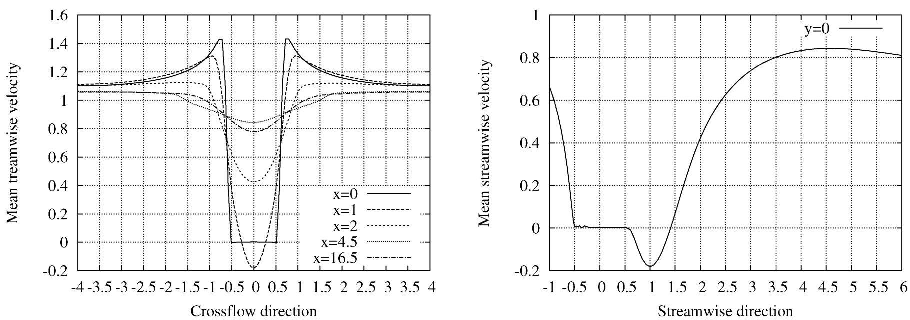

Fig. 3. Mean streamwise velocity at different x-locations (left) and in the $y = 0$ direction (right).

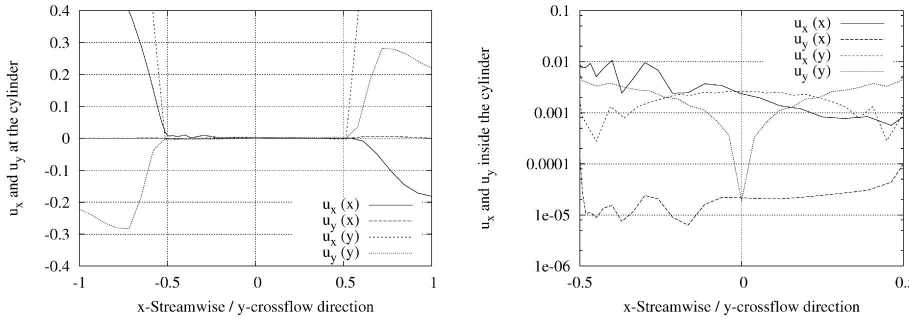

Fig. 4. Components of the velocity along $x = 0$ and $y = 0$ around the cylinder (left); Log plot of the absolute values inside the cylinder (right).

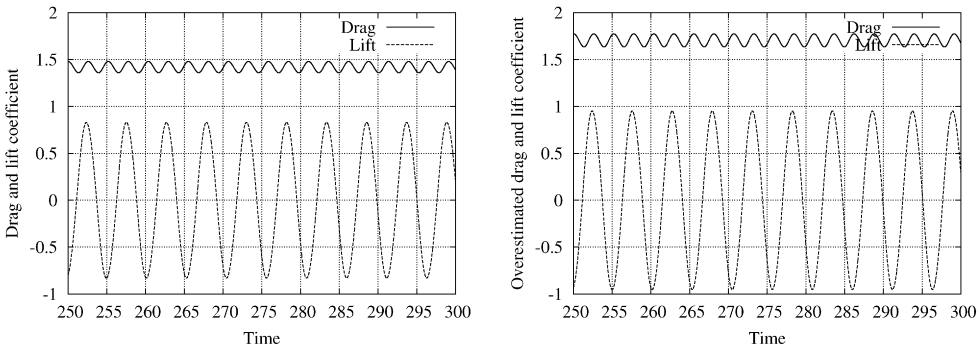

Fig. 5. Drag and lift coefficients from the formulations (28) (left) and (27) (right).

In the frame of penalization methods computing the drag and lift coefficients is simple, since it consists of the computation of the integral of the force term, say $-C\boldsymbol{u}$, inside the domain ω of the obstacle. Thus we find for the drag coefficient C_x:

$$C_x = 2 \int_{\omega} C u_x \, \mathrm{d}\omega \tag{26}$$

and a similar expression for the lift coefficient C_y. Note that the coefficient 2 comes from the fact that we use ρU^2 (ρ for density) and not $0.5\rho U^2$ for the reference pressure. For the pseudo-penalization described here we have a priori the two following possibilities:

– From the time approximation of the momentum equation, Eq. (10), and knowing that it is the smoothed characteristic function which is actually used (the time index is omitted):

$$C_x = -2 \int_{\Omega} \bar{\chi} f_x \, \mathrm{d}\Omega. \tag{27}$$

– From the penalized steady Stokes equations, Eq. (13):

$$C_x = 2 \int_{\Omega} \chi \frac{\alpha}{\tau} u_x \, \mathrm{d}\Omega. \tag{28}$$

These integrals have been computed with a high order quadrature rule (exact for polynomial of degree N in each variable) as well as with the trapezoidal rule, with coherent results. With respect to [9] it turns out that the formulation (27) overestimates the drag and lift coefficients whereas formulation (28) gives better results, although the drag coefficient remains a little high. Moreover, as could be expected the first formulation gives results very coherent with the second one as soon as it is χ and not $\bar{\chi}$ which is used in the computation.

In Fig. 5 (left) we show the variations of the drag and lift coefficients in the time interval $(250, 300)$. For the sake of comparison we also give the result obtained with Eq. (27) (right). The lift coefficient compares well with the results of [9] whereas the drag coefficient is a little overestimated. For $\bar{C}_x$ (mean value of C_x) we find $\bar{C}_x = 1.42$, whereas $\bar{C}_x = 1.32$ in [9]. However this difference could also result from the crossflow confinement of our flow, with a blocking factor of $1/8$ in the present study rather than $1/16.46$ in [9].

5. Conclusion

Penalization methods are very attractive for the modeling of bluff bodies, especially when numerical efficiency is required. Here we have introduced a pseudo-penalization method and got satisfactory results for an academic flow: the 2D wake of a cylinder at Reynolds number $Re = 200$. The pseudo-penalization method differs from standard volume penalization methods because no penalization term is explicitly introduced in the governing equations. The main advantage of the proposed approach is then that no additional stability constraints arise. We have particularly focused on the requirement of using a regularized characteristic function of the obstacle and proposed a simple way to achieve this goal. In any cases the space accuracy remains $\mathrm{O}(h)$, which may be disappointing in the frame of spectral methods, but it must be emphasized that this loss of accuracy is actually only local.

Appendix A

To illustrate the stability problem that can arise with an explicit treatment of the penalization term we consider the penalized 1D advection–diffusion equation:

$$\partial_t u + \partial_x u = \nu \partial_x^2 u - Cu. \tag{29}$$

In time we use a first order approximation with implicit treatment of the diffusion term and explicit treatment of the convective and penalization terms.

In space we use the Fourier expansion $u = \sum_{k\in\mathbb{Z}} \hat{u}_k \exp(\mathrm{i}kx)$ to carry out a von Neumann stability analysis. One obtains:

$$\hat{u}_k^{n+1} = g_k \hat{u}_k^n, \quad g_k = \frac{1 - C\tau - \mathrm{i}k\tau}{1 + \nu k^2 \tau}, \tag{30}$$

where τ is the time-step, n the time index and g_k the amplification factor of the mode k.

The stability condition writes: $\forall k \in \mathbb{Z}$, $|g_k| \leqslant 1$, which yields:

$$f(k) \equiv \nu^2\tau^2 k^4 + \tau(2\nu - \tau)k^2 + C\tau(2 - C\tau) \geqslant 0, \quad \forall k \in \mathbb{Z}. \tag{31}$$

One can easily check that:

– if $\tau/2\nu \leqslant 1$, then $f(k)$ shows one minimum at $k = 0$;
– if $\tau/2\nu > 1$, then $f(k)$ shows two minima, say at $k = \pm k_{\min}$ and one local maximum at $k = 0$.

One obtains:

$$f(0) = C\tau(2 - C\tau), \quad k_{\min}^2 = \frac{\tau - 2\nu}{2\nu^2\tau}, \quad f(k_{\min}) = f(0) - \left(\frac{\tau}{2\nu} - 1\right)^2. \tag{32}$$

As a result, with H for the Heaviside function the stability condition reads:

$$C\tau(2 - C\tau) \geqslant \left(\frac{\tau}{2\nu} - 1\right)^2 H\left(\frac{\tau}{2\nu} - 1\right) \tag{33}$$

and the scheme is thus unconditionally unstable if $C > 2/\tau$.

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