M2 IM - Machine Learning- 2018-2019
http://math.unice.fr/~rubentha/cours.html

Written Examination (1h)

Documents and calculators are not allowed. The grading will be function of your justifications. The exercises are independent. This sheet has to be returned with your written answers.

Exercise 1. Let $N \in \mathbb{N}^*$, $K \in \mathbb{N}^*$, $(z_{n,k})_{1\le n\le N, 1\le k\le K}$ in $\mathbb{R}$, $(x_n)_{1\le n\le N}$ in $\mathbb{R}^N$ and

$$\Phi \,:\, (\mu_1, \dots, \mu_K) \in (\mathbb{R}^N)^K \mapsto \sum_{n=1}^{N}\sum_{k=1}^{K} z_{n,k}(x_n - \mu_k)^T(x_n - \mu_k)\,.$$

(1) Compute the gradient of Φ.
(2) Find the absolute minimum of Φ. Write the answer in the frame below and the justification on the provided paper.

Exercise 2. Show that

$$\boldsymbol{w}^T\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{w} = w_0^2\left(\sum_{n=1}^{N} x_{n,1}^2\right) + 2w_0w_1\left(\sum_{n=1}^{N} x_{n,1}x_{n,2}\right) + w_1^2\left(\sum_{n=1}^{N} x_{n,2}^2\right)$$

where

$$\boldsymbol{w} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix},\ \boldsymbol{X} = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \\ \vdots & \vdots \\ x_{N,1} & x_{N,2} \end{bmatrix}.$$

Exercise 3. We suppose $X_1, X_2, \dots$ are i.i.d. (independant, identically distributed) variables of law $\mathcal{B}(\theta)$ (meaning $X_1 = 0$ with probability $1-\theta$ and 1 with probability θ). The parameter θ is in $(0,1)$ and is unknown. Let N in $\mathbb{N}^*$. We sample $X_1(\omega)$, $X_2(\omega)$, $\dots$, $X_N(\omega)$. We set N_1 to be the number of X_i in $\{X_1, \dots, X_N\}$such that $X_i = 1$. We set N_0 to be the number of X_i in $\{X_1, \dots, X_N\}$ such that $X_i = 0$.

(1) Compute the density of $(X_1, \dots, X_N)$ knowing θ. This is the likelihood of θ.
(2) Compute the max-likelihood estimator of θ.