Université Nice-Sophia Antipolis
SMEMP302 - ECUE Probabilitic Computational Methods, 2022-2023
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Answers for final exam (duration : 2h30)

Authorized documents: course notes only.

Exercise 1.

(1)

(a) We set $X^{(1)}, X^{(2)}, \dots$ to be independent copies of X. The law of large numbers tells us that:

$$\frac{1}{M}\sum_{i=1}^{M} \mathbb{1}_{[l,l+1]}(X^{(i)}) \underset{M\to+\infty}{\longrightarrow} p_l \text{ , } a.s.$$

(b) We have

$$\begin{aligned}\mathrm{Var}\left(\frac{1}{M}\sum_{i=1}^{M}\mathbb{1}_{[l,l+1]}(X^{(i)})\right) &= \frac{\mathrm{Var}(\mathbb{1}_{[l,l+1]}(X))}{M} \\ \text{(variance of a Bernoulli variable)} &= \frac{p_l(1-p_l)}{M}\,.\end{aligned}$$

(2)

(a) We introduce X_l of density $f_l : x \in \mathbb{R} \mapsto \mathbb{1}[l,+\infty)e^{-(x-l)}$. We have

$$\begin{aligned}p_l &= \int_0^{+\infty} \mathbb{1}_{[l,l+1]}(x)e^{-x}dx \\ &= \int_l^{+\infty} \frac{\mathbb{1}_{[l,l+1]}(x)e^{-x}}{f_l(x)} f_l(x)dx \\ &= \mathbb{E}\left(\frac{\mathbb{1}_{[l,l+1]}(X_l)e^{-X_l}}{f_l(X_l)}\right).\end{aligned}$$

If we set $X_l^{(1)}, X_l^{(2)}, \dots$ to be independent copies of X_l, then

$$\frac{1}{M}\sum_{k=1}^{M}\frac{\mathbb{1}_{[l,l+1]}(X_l^{(k)})e^{-X_l^{(k)}}}{f_l(X_l^{(k)})}$$

(b) We have

$$\begin{aligned}M\times\mathrm{Var}\left(\frac{1}{M}\sum_{k=1}^{M}\frac{\mathbb{1}_{[l,l+1]}(X_l^{(k)})e^{-X_l^{(k)}}}{f_l(X_l^{(k)})}\right) &= \mathbb{E}\left(\left(\frac{\mathbb{1}_{[l,l+1]}(X_l)e^{-X_l}}{f_l(X_l)}\right)^2\right) - p_l^2 \\ &= \int_l^{l+1}\left(\frac{e^{-x}}{e^{-(x-l)}}\right)^2 e^{-(x-l)}dx - p_l^2 \\ &= \int_l^{l+1} e^{-x-l}dx - p_l^2\,.\end{aligned}$$

(3) We have

$$\begin{aligned}
\frac{\int_l^{l+1} e^{-x-l}dx - p_l^2}{p_l - p_l^2} &= e^{-l} \times \frac{\int_l^{l+1} e^{-x}dx - e^l\left(\int_l^{l+1} e^{-x}dx\right)^2}{\int_l^{l+1} e^{-x}dx - \left(\int_l^{l+1} e^{-x}dx\right)^2} \\
&= e^{-l} \times \frac{1 - e^l\left(\int_l^{l+1} e^{-x}dx\right)}{1 - \left(\int_l^{l+1} e^{-x}dx\right)} \\
&= e^{-l} \times \frac{1 - e^l(e^{-l} - e^{-(l+1)})}{1 - (e^{-l} - e^{-(l+1)})} \\
&= e^{-l} \times \frac{e^{-1}}{1 - (e^{-l} - e^{-(l+1)})} \\
&\underset{l\to+\infty}{\sim} e^{-l-1}\,.
\end{aligned}$$

Exercise 2. We have for all $k \geq 1$,

$$\begin{aligned}
\overline{X}_{t_k^n} &= \overline{X}_{t_{k-1}^n} + \overline{X}_{t_{k-1}^n}(W_{t_k^n} - W_{t_{k-1}^n}) \\
&= \overline{X}_{t_{k-1}^n}(1 + \Delta W_{t_k^n})\,.
\end{aligned}$$

So, by recurrence:

$$\overline{X}_{t_k^n} = \prod_{l=1}^{k}(1 + \Delta W_{t_l^n})\,.$$

Exercise 3.

(1) We have

$$\begin{aligned}
\mathbb{E}((\mathbb{E}(f(X_T)) - Y)^2) &= \mathbb{E}((\mathbb{E}(f(X_T)) - \mathbb{E}(Y) + \mathbb{E}(Y) - Y)^2) \\
&= (\mathbb{E}(f(X_T)) - \mathbb{E}(Y))^2 + \mathbb{E}((\mathbb{E}(Y) - Y)^2) \\
&\quad +2\mathbb{E}((\mathbb{E}(f(X_T)) - \mathbb{E}(Y)) \times (\mathbb{E}(Y) - Y)^2 \\
&= (\mathbb{E}(f(X_T)) - \mathbb{E}(Y))^2 + \mathrm{Var}(Y)\,.
\end{aligned}$$

(2)

(a) We have

$$\begin{aligned}
(\mathbb{E}(f(X_T)) - \mathbb{E}(Y))^2 &= (\mathbb{E}(f(X_T)) - \mathbb{E}(f(\overline{X}_T^N)))^2 \\
&\leq \frac{C}{N^{2\beta}}\,.
\end{aligned}$$

So we want

$$N = \left(\frac{C}{\epsilon^2}\right)^{\frac{1}{2\beta}}\,.$$

(b) We set K to be the Lipschitz constant of f. We have

$$\begin{aligned}
\mathbb{E}((f(X_T) - f(\overline{X}_T^N))^2) &\leq K^2\mathbb{E}(|X_T - \overline{X}_T^N|^2) \\
&\leq \frac{K^2C}{N^\alpha} \underset{N\to0}{\longrightarrow} 0\,.
\end{aligned}$$

We already have that $\mathbb{E}(f(\overline{X}_T^N)) \underset{N\to0}{\longrightarrow} \mathbb{E}(f(X_T))$. The triangular inequality gives us

$$\begin{aligned}
\|f(\overline{X}_T^N)\|_2 &\leq \|f(X_T)\|_2 + \|f(\overline{X}_T^N) - f(X_T)\|_2\,, \\
\|f(X_T)\|_2 &\leq \|f(\overline{X}_T^N)\|_2 + \|f(X_T) - f(\overline{X}_T^N)\|_2\,.
\end{aligned}$$

So

$$\|f(X_T)\|_2 - \|f(X_T) - f(\overline{X}_T^N)\|_2 \leq \|f(\overline{X}_T^N)\|_2 \leq \|f(X_T)\|_2 + \|f(\overline{X}_T^N) - f(X_T)\|_2\,.$$

Which implies

$$\|f(\overline{X}_T^N)\|_2 \underset{N\to+\infty}{\longrightarrow} \|f(X_T)\|_2 \,.$$

In conclusion:

$$\mathrm{Var}(f(\overline{X}_T^N)) \underset{N\to+\infty}{\longrightarrow} \mathrm{Var}(f(X_T)) \,.$$

We have

$$\mathrm{Var}(Y) = \frac{\mathrm{Var}(f(\overline{X}_T^N))}{M} \,,$$

which we identify with

$$\frac{\mathrm{Var}(f(X_T))}{M}$$

(as N is very big). We want

$$\frac{\mathrm{Var}(f(X_T))}{M} = \epsilon^2 \,,$$

so we choose

$$M = \frac{\mathrm{Var}(f(X_T))}{\epsilon^2}$$

(c) The computational time needed to attain the quadratic error $2\epsilon^2$ is of order $M+N$, which is proportional to $\epsilon^{-(2+1/\beta)}$.