KAM for quasi-linear and fully nonlinear forced KdV

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1. Introduction: KAM and quasi-linear PDEs

KAM and Nash-Moser theory for PDEs (∞-dim dynamical systems)
Since late 80s - early 90s: Kuksin, Craig, Wayne, Bourgain, Pöschel, ...
find existence, regularity, stability,
for periodic/quasi-periodic/almost-periodic solutions.

Majority of KAM results: semilinear PDEs, i.e. ∞-dim dynamical systems

$$\partial_t u = F(u)$$

where the vector field F is

$$F(u) = Lu + \mathcal{N}(u),$$

$$L = \text{linear, and contains derivatives } (\partial_x, \Delta, \dots)$$

$$\mathcal{N} = \text{nonlinear, } \textit{bounded} \text{ (like } \mathcal{N}(u) = f(t,x,u))$$

Much less KAM results for PDEs where $\mathcal{N}(u)$ also contains derivatives of u,

$$\mathcal{N} = \text{unbounded nonlinear vector field}$$

(stronger perturbation effect)

Existing KAM results: for "semilinear-with-derivatives" PDEs (Tao's terminology), i.e.

$$0 < \text{order}(\mathcal{N}) < \text{order}(L).$$

For example: KdV

$$u_t = u_{xxx} + \partial_x(u^2),$$

$$L = \partial_x^3 = \text{order}\,3, \quad \mathcal{N}(u) = \partial_x(u^2) = \text{order}\,1.$$

Quasi-periodic solutions for "semilinear-with-derivatives" PDEs:

Kuksin (1998, 2000), Kappeler & Pöschel (2003) for KdV
Existence for Hamiltonian analytic perturbation

$$H(u) = H_0(u) + \varepsilon K(u), \quad H_0(u) = \int_{\mathbb{T}} \Big(\frac{u_x^2}{2} - \frac{u^3}{6} \Big)\, dx,$$

$$u_t + u_{xxx} + \partial_x \big(u^2 + \varepsilon \nabla K(u) \big) = 0, \quad \nabla K = \text{bounded nonlinear}$$

Bambusi & Graffi (2001) related linear problem

Liu & Yuan (2011), Zhang, Gao & Yuan (2011) for 1-dim DNLS

$$i u_t + u_{xx} + f(u, u_x) = 0$$

and Benjamin-Ono eq.

$$u_t + \mathcal{H} u_{xx} + f(u, u_x) = 0$$

(where order$(L) = 2$, order$(\mathcal{N}) = 1$), existence in class C^∞

Berti, Biasco & Procesi (2011, 2012) for 1-dim DNLW

$$u_{tt} - u_{xx} + f(u, u_x) = 0,$$

existence and linear stability in analytic class,
using Hamiltonian or reversible structure.

For periodic solutions: Craig (2000) for DNLW

$$u_{tt} - u_{xx} + f(|D_x|^{\beta} u) = 0, \quad 0 < \beta < 1.$$

When $\mathcal{N}$ contains derivatives of *the same* order as L,

$$\text{order}(\mathcal{N}) = \text{order}(L) :$$

quasi-linear or fully nonlinear PDEs.

Quasi-linear: if $\mathcal{N}(u)$ depends linearly on $\partial_x^m u$ ($L =$ order m)
Fully nonlinear: otherwise.

(no KAM results)

In general, it is not even clear if there could be quasi-periodic solutions:
(Kappeler & Pöschel (2003))

• Question (Kappeler & Pöschel (2003)):
is it possible to prove KAM for quasi-linear Hamiltonian perturbations of KdV

$$H(u) = H_0(u) + \varepsilon K(u), \quad K = \text{order } 1 \ ?$$

For example, if

$$K(u) = \int_{\mathbb{T}} F(u, u_x)\, dx,$$

then the KdV equation is

$$u_t + u_{xxx} + \partial_x\Big(f_0(u, u_x)\Big) - \partial_{xx}\Big(f_1(u, u_x)\Big) = 0,$$

$(f_0, f_1) = \nabla F$.

• More general question: is it possible to extend KAM theory to quasi-linear or fully nonlinear PDEs?

Related results for quasi-linear or fully nonlinear PDEs:

Cauchy problem Delort, Metivier, Alazard, Burq, Zuily, . . .
Quasi-linear Klein-Gordon eq., water waves (fully nonlinear)

Periodic solutions
Rabinowitz (1969) DNLW with dissipation $\alpha \neq 0$,

$$u_{tt} - u_{xx} + \varepsilon f(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) + \alpha u_t = 0.$$

• **Iooss, Plotnikov & Toland** (2002, 2005, 2006) Water Waves (Fully nonlinear PDEs)
Nash-Moser scheme, reduction to constant coefficients up to a regularizing rest

Baldi (2008, 2012) Benjamin-Ono eq.
(autonomous, reversible, quasi-linear or fully nonlinear)

$$u_t + \mathcal{H}u_{xx} + \partial_x(u^3) + f(x, u, \mathcal{H}u, u_x, \mathcal{H}u_x, \mathcal{H}u_{xx}) = 0$$

Kirchhoff eq. (forced, special quasi-linear DNLW)

$$u_{tt} - \Big(1 + \int_\Omega |\nabla u|^2 dx\Big)\Delta u = \varepsilon f(t, x).$$

Quasi-periodic solutions

Baldi, Berti & Montalto (preprint)

KAM for forced quasi-linear or fully nonlinear perturbations of KdV

$$u_t + u_{xxx} + \varepsilon f(\omega t, x, u, u_x, u_{xx}, u_{xxx}) = 0$$

prove existence and, assuming additional structure (Hamiltonian or reversible), linear stability.

This gives a positive answer to Kappeler & Pöschel's question, at least in the forced case, for solutions close to $u = 0$.

2. Main results

Consider the perturbed KdV

$$u_t + u_{xxx} + \varepsilon f(\omega t, x, u, u_x, u_{xx}, u_{xxx}) = 0 \qquad (1)$$

on the torus

$$x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z},$$

with external forcing frequency vector ω with ν Diophantine frequencies

$$\omega = \lambda\bar{\omega}, \quad \lambda \in \left[\frac{1}{2}, \frac{3}{2}\right], \quad \bar{\omega} \in \mathbb{R}^\nu, \quad |\bar{\omega} \cdot l| \geq \frac{\gamma}{|l|^\tau} \quad \forall l \in \mathbb{Z}^\nu \setminus \{0\}.$$

To find quasi-periodic solutions $u(t,x) = \tilde{u}(\omega t, x)$, study

$$\omega \cdot \partial_\varphi u + u_{xxx} + \varepsilon f(\varphi, x, u, u_x, u_{xx}, u_{xxx}) = 0 \qquad (2)$$

in the unknown

$$u = u(\varphi, x), \quad u : \mathbb{T}^\nu \times \mathbb{T} \to \mathbb{R}, \quad \varphi \in \mathbb{T}^\nu, \ x \in \mathbb{T}.$$

The linear part $(\omega \cdot \partial_\varphi + \partial_{xxx})$ has nonzero kernel

$$\mathrm{Ker}(\omega \cdot \partial_\varphi + \partial_{xxx}) = \mathbb{R}.$$

To deal with this degeneracy, we work in spaces of zero mean functions, assuming compatibility conditions on f:

- either f is of the form $f = \partial_x(g)$

$$f(\varphi, x, u, u_x, u_{xx}, u_{xxx}) = \partial_x\Big(g(\varphi, x, u, u_x, u_{xx})\Big)$$

- or f has the reversible structure

$$f(-\varphi, -x, z_0, -z_1, z_2, -z_3) = f(\varphi, x, z_0, z_1, z_2, z_3)$$

State the results

Theorem 1. (Existence for quasi-linear $f = \partial_x(g)$)

There are q, s with the following property.
Assume that $f \in C^q$ is of the form

$$f(\varphi, x, u, u_x, u_{xx}, u_{xxx}) = \partial_x \Big(g(\varphi, x, u, u_x, u_{xx}) \Big)$$

(quasi-linear case) and $f(\varphi, x, z_0, z_1, z_2, z_3)$ satisfies

$$\text{(Q)} \quad \partial_{z_2} f = \alpha(\varphi) \Big(\partial^2_{z_3 x} f + z_1 \partial^2_{z_3 z_0} f + z_2 \partial^2_{z_3 z_1} f + z_3 \partial^2_{z_3 z_2} f \Big)$$

for some function $\alpha(\varphi)$ (independent on x).

Then, $\forall \varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 = \varepsilon_0(f, \nu)$ is small enough, there exists a Cantor set $\mathcal{C}_\varepsilon \subset [\frac{1}{2}, \frac{3}{2}]$ of Lebesgue measure

$$|\mathcal{C}_\varepsilon| \to 1 \quad \text{as } \varepsilon \to 0,$$

such that $\forall \lambda \in \mathcal{C}_\varepsilon$ the KdV equation (2) with $\omega = \lambda \bar{\omega}$ has a solution $u(\varepsilon, \lambda) \in H^s(\mathbb{T}^{\nu+1})$, with

$$\|u(\varepsilon, \lambda)\|_{H^s(\mathbb{T}^{\nu+1})} \to 0 \quad \text{as } \varepsilon \to 0.$$

Linear stability requires more structure:

- (i) Hamiltonian

  or

- (ii) reversible.

(i) Hamiltonian KdV

$$u_t = \partial_x \nabla_{L^2(\mathbb{T})} H(u), \quad H(u) = \int_{\mathbb{T}} \Big(\frac{u_x^2}{2} + \varepsilon F(\omega t, x, u, u_x) \Big)\, dx$$

Phase space

$$H_0^1(\mathbb{T}) = \Big\{ u(x) \in H^1(\mathbb{T}, \mathbb{R}) : \int_{\mathbb{T}} u(x)\, dx = 0 \Big\}$$

Symplectic form

$$\Omega(u, v) = \int_{\mathbb{T}} (\partial_x^{-1} u) v \, dx, \qquad u, v \in H_0^1(\mathbb{T})$$

where $\partial_x^{-1} u =$ primitive of u with zero average.

Note: (Q) automatically satisfied, and $f = \partial_x(g)$
($\Rightarrow$ existence of solutions is a corollary of Theorem 1)

Theorem 2. (Hamiltonian)

For all Hamiltonian quasi-linear KdV, the quasi-periodic solution $u(\varepsilon, \lambda)$ found in Theorem 1 is linearly stable.

(More details follow).

(ii) Reversible KdV

A dynamical system

$$u_t = F(t)(u) \tag{3}$$

is reversible if there is a map R of the phase space (reflection) such that

$$R^2 = I, \qquad F(t) \circ R = -R \circ F(-t).$$

Consequence: if $u(t)$ solves (3), then

$$(Su)(t) := Ru(-t)$$

also solves (3). $\rightarrow$ Look for solutions in the invariant subspace

$$\{u : Su = u\} = \{\text{fixed points of } S\}.$$

Reversible KdV: consider the reflection

$$R : L^2(\mathbb{T}) \to L^2(\mathbb{T}), \quad u(x) \to u(-x), \quad R^2 = I.$$

The linear part

$$L = \partial_{xxx}$$

of the KdV vector field satisfies

$$L \circ R = -R \circ L$$

(true for $L = \partial_x^m$ for any odd m).
The nonlinear part $\mathcal{N}(\varphi)$

$$\mathcal{N}(\varphi)(u) := \varepsilon f(\varphi, x, u, u_x, u_{xx}, u_{xxx})$$

satisfies

$$\mathcal{N}(\varphi) \circ R = -R \circ \mathcal{N}(-\varphi)$$

provided $f(\varphi, x, z_0, z_1, z_2, z_3)$ satisfies the parity condition

$$\text{(REV)} \quad f(-\varphi, -x, z_0, -z_1, z_2, -z_3) = -f(\varphi, x, z_0, z_1, z_2, z_3).$$

Corresponding invariant subspace:

$$\begin{aligned} X &:= \{u : \text{even in the pair } (\varphi, x)\} \\ &= \{u : u(-\varphi, -x) = u(\varphi, x)\}. \end{aligned}$$

Set

$$\begin{aligned} Y &:= \{u : \text{odd in the pair } (\varphi, x)\} \\ &= \{u : u(-\varphi, -x) = u(\varphi, x)\}. \end{aligned}$$

All terms in KdV equation map even into odd functions:

$$\omega \cdot \partial_\varphi \, , \; \partial_{xxx} \, , \; \mathcal{N} \; : \; X \to Y.$$

In the reversible case: consider

- fully nonlinear KdV, with f satisfying

$$\text{(F)} \quad \partial_{z_2} f = 0,$$

i.e. $f = f(\varphi, x, u, u_x, u_{xxx})$ does not depend explicitly on u_{xx},

or

- quasi-linear KdV, with f satisfying

$$\text{(Q)} \quad \begin{cases} \partial^2_{z_3 z_3} f = 0, \\ \partial_{z_2} f = \alpha(\varphi)\Big(\partial^2_{z_3 x} f + z_1 \partial^2_{z_3 z_0} f + z_2 \partial^2_{z_3 z_1} f + z_3 \partial^2_{z_3 z_2} f\Big) \end{cases}$$

for some function $\alpha(\varphi)$ (independent on x),
i.e. f depends linearly on u_{xxx} (quasi-linear), and satisfies the same condition as Theorem 1.

Theorem 3. (Reversible)

There are q, s with the following property.
Assume that $f \in C^q$ satisfies

- *the parity condition (REV); and*
- *(F) (fully nonlinear) or (Q) (quasi-linear).*

Then, $\forall \varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 = \varepsilon_0(f, \nu)$ is small enough, there exists a Cantor set $\mathcal{C}_\varepsilon \subset [\frac{1}{2}, \frac{3}{2}]$ of Lebesgue measure

$$|\mathcal{C}_\varepsilon| \to 1 \quad \text{as } \varepsilon \to 0,$$

such that $\forall \lambda \in \mathcal{C}_\varepsilon$ the KdV equation (2) with $\omega = \lambda \bar{\omega}$ has a solution $u(\varepsilon, \lambda) \in H^s(\mathbb{T}^{\nu+1})$, $u(\varepsilon, \lambda)$ even in (φ, x), with

$$\|u(\varepsilon, \lambda)\|_{H^s(\mathbb{T}^{\nu+1})} \to 0 \quad \text{as } \varepsilon \to 0.$$

Moreover, $u(\varepsilon, \lambda)$ is linearly stable.

3. Ingredients of the proof

To construct the solution of the nonlinear problem:

- Nash-Moser scheme in spaces $H^s(\mathbb{T}^{\nu+1})$.

Main question: inversion of the linearized operator at u

$$\mathcal{L} = \mathcal{L}(u) = \omega \cdot \partial_\varphi + +(1 + a_3)\partial_{xxx} + a_2\partial_{xx} + a_1\partial_x + a_0$$

where $a_i = a_i(\varphi, x)$, $i = 0, 1, 2, 3$ are variable coefficients,

$$a_i(\varphi, x) := (\partial_{z_i} f)(\varphi, x, u(\varphi, x), u_x(\varphi, x), u_{xx}(\varphi, x), u_{xxx}(\varphi, x)).$$

- Reduction to constant coefficients
  - reduction in the order ∂_x^m
    use diffeomorphisms of $\mathbb{T}$, re-parametrization of time, multiplication operators, Fourier multipliers
    (inspired to Iooss, Plotnikov & Toland technique)
  - reduction in the size ε^m
    use KAM procedure, imposing 2nd order Melnikov conditions
    (like Kuksin & Eliasson linear reducibility)

Theorem 4. (Reducibility)

There are q, s, μ with the following property.
Assume that $f \in C^q$ satisfies (F) or (Q), and $\|u\|_{s+\mu} < 1$.

Then, $\forall \varepsilon$ small enough, there exists a Cantor set

$$\Lambda_\infty(u) \subset \left[\frac{1}{2}, \frac{3}{2}\right], \quad |\Lambda_\infty(u)| \to 1 \quad \text{as } \varepsilon \to 0,$$

such that $\forall \lambda \in \Lambda_\infty(u)$ there are bounded linear invertible operators

$$W_1, W_2 : H^s(\mathbb{T}^{\nu+1}) \to H^s(\mathbb{T}^{\nu+1})$$

that semi-conjugate the linearized operator $\mathcal{L} = \mathcal{L}(u)$ to the diagonal operator $\mathcal{L}_\infty$,

$$\mathcal{L} = W_1 \mathcal{L}_\infty W_2^{-1}, \quad \mathcal{L}_\infty = \omega \cdot \partial_\varphi + \mathcal{D}_\infty, \quad \mathcal{D}_\infty = \operatorname{diag}_{j \in \mathbb{Z}}\{\mu_j\},$$

with

$$\mu_j = i(-m_3 j^3 + m_1 j) + r_j,$$

$$m_3, m_1 \in \mathbb{R}, \quad r_j \in \mathbb{C}, \quad |r_j| < C\varepsilon \quad \forall j \in \mathbb{Z}.$$

Moreover, W_1, W_2 are also time-dependent transformations of the phase space:

$$W_i(\varphi) : H^s_x \to H^s_x, \quad i = 1, 2, \quad \forall \varphi \in \mathbb{T}^\nu.$$

A curve $h(t)$ in H^s_x is a solution of the linearized KdV

$$\partial_t h + \partial_{xxx} h + a_3(\omega t, x)\partial_{xxx} h + \ldots + a_0(\omega t, x) h = 0$$

iff the transformed curve in H^s_x

$$v(t) := W_2(\omega t) h(t)$$

solves

$$\partial_t v + \mathcal{D}_\infty v = 0,$$

namely the uncoupled system

$$v_j'(t) + \mu_j v_j(t) = 0, \quad j \in \mathbb{Z}.$$

In the Hamiltonian or reversible case, all $\mu_j \in i\mathbb{R}$.

Theorem 5. (Linear stability)

In the Hamiltonian or reversible case, the curve $h(t)$ in H^s_x that solves the linearized KdV

$$\partial_t h + \partial_{xxx} h + a_3(\omega t, x)\partial_{xxx} h + \ldots + a_0(\omega t, x) h = 0$$

satisfies

$$\|h(t)\|_{H^s_x} \leq C\|h(0)\|_{H^s_x} \quad \forall t \in \mathbb{R},$$

and

$$\begin{aligned}\|h(0)\|_{H^s_x} - \varepsilon^{\alpha} C\|h(0)\|_{H^{s+1}_x} &\leq \|h(t)\|_{H^s_x} \\ &\leq \|h(0)\|_{H^s_x} + \varepsilon^{\alpha} C\|h(0)\|_{H^{s+1}_x}\end{aligned}$$

where $\alpha \in (0,1)$.

Remark

Without Ham/rev structure, linear stability cannot be proved (in general, eigenvalues $\mu_j \notin i\mathbb{R}$).
However, in any case we find asymptotic for μ_j at any order of accuracy.

Reduction to constant coefficients:

- (i) regularization procedure to conjugates $\mathcal{L}$ to $\mathcal{L}_5$,

$$\mathcal{L}_5 = \omega \cdot \partial_\varphi + m_3 \partial_{xxx} + m_1 \partial_x + \mathcal{R},$$

  where $m_3, m_1 \in \mathbb{R}$, and $\mathcal{R}$ is bounded;
- (ii) KAM reduction scheme to complete the diagonalization.

(ii) KAM: ∞ many steps, imposing 2nd Melnikov conditions on the small divisors, and solving homological equation (with Fourier-truncations in time).
It works well starting from $\mathcal{L}_5$, as the off-diagonal part $\mathcal{R}$ is bounded.

Some details about (i). Starting point:

$$\mathcal{L} = \omega \cdot \partial_\varphi + (1 + a_3)\partial_{xxx} + a_2 \partial_{xx} + a_1 \partial_x + a_0,$$

$a_i = a_i(\varphi, x)$.

1. Change of the space coordinate

Any time-dependent diffeomorphism of $x \in \mathbb{T}$

$$\mathbb{T} \to \mathbb{T}, \quad x \to x + \beta(\varphi, x)$$

gives a transformation B of $H^s(\mathbb{T}^{\nu+1})$

$$\begin{aligned} B : H^s(\mathbb{T}^{\nu+1}) &\to H^s(\mathbb{T}^{\nu+1}), \\ u(\varphi, x) &\to Bu(\varphi, x) := u(\varphi, x + \beta(\varphi, x)), \end{aligned}$$

which is also a time-dependent transformation $B(\varphi)$ of the phase space H^s_x

$$\begin{aligned} B(\varphi) : H^s_x &\to H^s_x, \\ u(x) &\to B(\varphi)u(x) := u(x + \beta(\varphi, x)). \end{aligned}$$

- To preserve reversible structure: need $\beta(\varphi, x)$ odd in x, even in φ;
- to preserve Hamiltonian structure: use instead

$$Bu(\varphi, x) = \big(1 + \beta_x(\varphi, x)\big) u(\varphi, x + \beta(\varphi, x)),$$

which is symplectic: at each $\varphi \in \mathbb{T}^\nu$,

$$\Omega\big(B(\varphi)u, B(\varphi)v\big) = \Omega(u, v) \quad \forall u, v \in H_0^1.$$

In any case, choose suitable $\beta(\varphi, x) \in C^s(\mathbb{T}^{\nu+1})$:

$$(1 + a_3(\varphi, x))(1 + \beta_x(\varphi, x))^3 = b_3(\varphi) = \text{independent of } x$$

and get

$$B^{-1}\mathcal{L}B = \mathcal{L}_1 = \omega \cdot \partial_\varphi + b_3 \partial_{xxx} + b_2 \partial_{xx} + b_1 \partial_x + b_0,$$

where the coefficient $b_3 = b_3(\varphi)$ does not depend on x.
Moreover, in the Hamiltonian case, $b_2 = 2\partial_x b_3 = 0$.

2. Reparametrization of time

Any x-independent diffeomorphism of $\varphi \in \mathbb{T}^\nu$ of the type

$$\mathbb{T}^\nu \to \mathbb{T}^\nu, \quad \varphi \to \varphi + \omega\alpha(\varphi), \quad \alpha : \mathbb{T}^\nu \to \mathbb{R}$$

gives a transformation A of $H^s(\mathbb{T}^{\nu+1})$

$$\begin{aligned} A : H^s(\mathbb{T}^{\nu+1}) &\to H^s(\mathbb{T}^{\nu+1}), \\ u(\varphi, x) &\to Au(\varphi, x) := u(\varphi + \omega\alpha(\varphi), x). \end{aligned}$$

It corresponds to a ω-quasi-periodic reparametrization of time

$$\mathbb{R} \to \mathbb{R}, \quad t \to t + \alpha(\omega t).$$

Choose suitable $\alpha(\varphi) \in C^s(\mathbb{T}^\nu)$: leading coefficients becomes proportional,

$$A^{-1}\mathcal{L}_1 A = \rho(\varphi)\mathcal{L}_2, \quad \mathcal{L}_2 = \omega \cdot \partial_\varphi + m_3\partial_{xxx} + c_2\partial_{xx} + c_1\partial_x + c_0,$$

where $m_3 \in \mathbb{R}$.

Arrived to:

$$\mathcal{L}_2 = \omega \cdot \partial_\varphi + m_3 \partial_{xxx} + c_2(\varphi, x)\partial_{xx} + c_1(\varphi, x)\partial_x + c_0(\varphi, x),$$

where

$$m_3 = \frac{1}{(2\pi)^\nu} \int_{\mathbb{T}^\nu} \left(\frac{1}{2\pi} \int_{\mathbb{T}} \big(1 + a_3(\varphi, x)\big)^{-\frac{1}{3}} dx \right)^{-3} d\varphi$$

is a constant, $m_3 = 1 + O(\varepsilon)$.

In the Hamiltonian case, we already have $c_2(\varphi, x) = 0$.

In the general case, we want to eliminate $c_2(\varphi, x)\partial_{xx}$.

3. Multiplication operator

$$\mathcal{L}_2 = \omega \cdot \partial_\varphi + m_3 \partial_{xxx} + c_2(\varphi, x)\partial_{xx} + c_1(\varphi, x)\partial_x + c_0(\varphi, x).$$

A multiplication operator

$$\Phi h = v(\varphi, x) h$$

conjugates $\mathcal{L}_2$ to

$$\Phi^{-1} \mathcal{L}_2 \Phi = \mathcal{L}_3 = \omega \cdot \partial_\varphi + m_3 \partial_{xxx} + d_1(\varphi, x)\partial_x + d_0(\varphi, x)$$

provided the coefficient c_2 satisfy

$$\int_{\mathbb{T}} c_2(\varphi, x)\, dx = 0,$$

namely the coefficients a_2, a_3 of the linearized operator

$$\mathcal{L} = \omega \cdot \partial_\varphi + (1 + a_3)\partial_{xxx} + a_2 \partial_{xx} + a_1 \partial_x + a_0$$

satisfy

$$\int_{\mathbb{T}} \frac{a_2(\varphi, x)}{1 + a_3(\varphi, x)}\, dx = 0 \quad \forall \varphi. \tag{4}$$

Assumption (F) implies that $a_2 = 0$.
Assumption (Q) implies that $a_2 = \alpha(\varphi)\partial_x a_3$.
In both cases (Q) and (F), one has

$$\int_{\mathbb{T}} \frac{a_2(\varphi, x)}{1 + a_3(\varphi, x)}\, dx = 0 \quad \forall \varphi,$$

and the regularization procedure can go on.

Without assumptions (Q),(F), we can always reduce $c_2(\varphi, x)\partial_{xx}$ to a time dependent term $d_2(\varphi)\partial_{xx}$.

If d_2 is a constant, then $d_2\partial_{xx}$ is diagonal, it is a friction term and it even simplifies the analysis.

In the general case, instead, $d_2(\varphi)$ changes sign, and it is not clear if regular solutions could exists.

We assume (Q) or (F) to prevent that situation.

Other simple transformations (translation $x \to x + p(\varphi)$, one step of "descent method" $I + w\partial_x^{-1}$) lead to

$$\mathcal{L}_5 = \omega \cdot \partial_\varphi + m_3 \partial_{xxx} + m_1 \partial_x + \mathcal{R}, \quad \mathcal{R} = \text{bounded}.$$

This is sufficient to start the KAM reducibility algorithm.

Remark

One can go further with the regularization procedure, until any finite order

$$\mathcal{L}_k = \omega \cdot \partial_\varphi + m_3 \partial_{xxx} + \ldots + m_{-k} \partial_x^{-k} + \mathcal{R}_k, \quad \partial_x^k \circ \mathcal{R} = \text{bounded}.$$

For periodic solutions this is sufficient to invert $\mathcal{L}$ (Neumann series).

For quasi-periodic solutions it is not enough. Moreover, the precedure cannot be iterated infinitely many times, as this is not a quadratic scheme.

4. Autonomous problem (work in progress)

Hamiltonian and reversible quasi-linear KdV

$$H(u) = \int_{\mathbb{T}} \Big(\frac{u_x^2}{2} - \frac{u^4}{4} + f(u, u_x) \Big) dx,$$

PDE

$$u_t + u_{xxx} + \partial_x(u^3) + \mathcal{N}(u) = 0,$$

where

$$\mathcal{N}(u) := -\partial_x\{(\partial_1 f)(u, u_x)\} + \partial_{xx}\{(\partial_2 f)(u, u_x)\} = O(u^5).$$

Prime integral:

$$\int_{\mathbb{T}} u^2 \, dx.$$

Bifurcation analysis

Tangential sites $S = \{\pm\bar{j}_1, \ldots, \pm\bar{j}_\nu\}$, $\bar{j}_n \in \mathbb{N}$
Unperturbed frequency $\bar{\omega} = (\bar{j}_1^3, \ldots, \bar{j}_\nu^3)$.

Using formal power series in the amplitude parameter ε

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \ldots, \quad \omega = \bar{\omega} + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots$$

and formal Lyapunov-Schmidt decomposition, construct approximate solution

$$\bar{u} := \varepsilon\bar{u}_1 + \varepsilon^2\bar{u}_2$$

such that $F(\bar{u}) = O(\varepsilon^5)$.
$\bar{u}$ is a good starting point for the Nash-Moser iteration.

Frequency:

$$\omega = \bar{\omega} + \varepsilon^2\widetilde{\omega}.$$

Frequency-amplitude relation: $\widetilde{\omega}$ are ν free parameters, which control the amplitudes of each Fourier mode of $\bar{u}_1$.

Linearized operator

Main points:

- variable coefficients $\omega \cdot \partial_\varphi + (1 + a_3)\partial_{xxx} + \dots$ (as above);
- smallness question:

since $\omega = \bar{\omega} + \varepsilon^2 \widetilde{\omega}$, to impose 2nd Melnikov conditions

$$|\omega \cdot l + \lambda_j - \lambda_k| > \frac{\gamma}{|l|^\tau}$$

and get parameters $\widetilde{\omega}$ in a $|\text{Cantor}| > 0$ one needs $\gamma \sim \varepsilon^2$,
and for $|\text{Cantor}| \to$ full one needs

$$\frac{\gamma}{\varepsilon^2} \to 0 \qquad \text{as } \varepsilon \to 0.$$

KAM reducibility: $\mathcal{L} = \mathcal{D} + \varepsilon^\alpha \mathcal{R} \to \mathcal{L}^+ = \mathcal{D}^+ + \varepsilon^{2\alpha} \mathcal{R}^+$ provided

$$\frac{\varepsilon^\alpha}{\gamma} \quad \text{is sufficiently small.}$$

$\to$ One needs $\alpha > 2$, i.e. deal with terms $O(\varepsilon), O(\varepsilon^2)$ of $\mathcal{L}$ in a non perturbative way.

Normal form

Using prime integral, replace Hamiltonian $H(u)$ with

$$H(u) + a\Big(\int_{\mathbb{T}} u^2\,dx\Big)^2, \quad a \in \mathbb{R}.$$

One step of Birkhoff normal form

$$\omega \cdot \partial_\varphi + \partial_{xxx} + \varepsilon^3 \mathcal{R} \quad \text{outside } S \text{ (normal sites)}$$

and

$$\omega \cdot \partial_\varphi + \partial_{xxx} + \varepsilon^2 B + \varepsilon^3 \mathcal{R} \quad \text{in } S \text{ (tangential sites)}$$

where B is independent of time, and couples space-Fourier modes $j, -j \in S$

$$B \leftrightarrow ij\,\widetilde{\omega}_j \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

- Put all these ingredients together in a good way!