Analytic and Gevrey regularity for solutions to Hörmander operators

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I. Sum of Squares Operators

$$P = X_1^2 + \ldots + X_\nu^2$$

$X_1, \ldots, X_\nu$: analytic, real vector fields defined in $\Omega \subset \mathbb{R}^N$ open.

Hörmander's condition: $\mathcal{L}(X_1, \ldots, X_\nu)$ has rank N at each point of Ω.

- P is C^∞-hypoelliptic in Ω (Hörmander, 1967)
- $P = \partial_t^2 + \partial_y^2 + t^2\partial_x^2$ is not analytic hypoelliptic (ahe) in $\mathbb{R}^3$ (Baouendi–Goulaouic, 1972)

Positive results

$$P = X_1^2 + \dots X_\nu^2$$

Theorem (Tartakoff 1980), (Treves 1978)

If the characteristic set Σ of P is a symplectic manifold and if the principal symbol of P vanishes precisely of order 2 on Σ then P is (ahe).

Example:

- $P = \partial_t^2 + t^2 \partial_x^2$ is (ahe) in $\mathbb{R}^2$;

Further examples

- $P = \partial_t^2 + t^{2k}\partial_x^2$ is (ahe) in $\mathbb{R}^2$
  Matsuzawa, 1971.
- $P = \partial_t^2 + t^{2p}\partial_{x_1}^2 + t^{2q}\partial_{x_2}^2$ is (ahe) in $\mathbb{R}^3$ if and only if $p = q$.
  Oleinik, 1973.
- $P = \partial_y^2 + \left(\partial_x + my^{m-1}\partial_s\right)^2$ is (ahe) in $\mathbb{R}^3$ if and only if $m = 2$.
  Tartakoff–Treves ($m = 2$), Grigis-Sjostrand ($m = 3$),
  Hanges–Himonas ($m \geq 3$ odd), Christ ($m \geq 3$).
- $P = \partial_t^2 + t^2\partial_x^2 + (x\partial_x)^2$ is not (ahe) in $\mathbb{R}^2$
  Metivier, 1981.

Global analytic hypoellipticity

$\mathcal{M}$: real-analytic manifold, $X_1, \ldots, X_\nu$ defined on $\mathcal{M}$ and satisfying Hörmander condition.

$$P = X_1^2 + \ldots + X_\nu^2.$$

Definition

P is globally hypelliptic in $\mathcal{M}$ if given $u \in C^\infty(\mathcal{M})$ (or even $u \in \mathcal{D}'(\mathcal{M})$) then $Pu \in C^\omega(\mathcal{M})$ implies $u \in C^\omega(\mathcal{M})$.

Assume $\mathcal{M}$ compact and connected. By the Bony's maximum principle, every solution to $Pu = 0$ on $\mathcal{M}$ is constant and hence an equation $Pu = f$, with f given, has at most one solution modulo constants. So it makes sense to ask if a given P, which is not (ahe) in $\mathcal{M}$ in the usual sense, can be globally (ahe) in $\mathcal{M}$.

Examples

- In $\mathbf{T}^3 = S^1 \times S^1 \times S^1$, with coordinates written as (x, t, y) consider

$$P = \partial_t^2 + \partial_y^2 + a(t)^2 \partial_x^2,$$

where $a \in C^\omega(S^1)$ does not vanish identically. Then P is globally (ahe) in $\mathbf{T}^3$ (C-Himonas, 1993). There is a generalization to a larger class of operators due (Christ 1993).

- In $\mathbf{T}^2 = S^1 \times S^1$, with coordinates written as (x, t) consider

$$M = \partial_t^2 + \sin(t)^2 \partial_x^2 + (\sin(x)\partial_x)^2$$

Then M is not globally (ahe) in $\mathbf{T}^2$

Proof that M is not globally (ahe)

Argument due to J.-M. Trepréau

- There are an open neighbohood of the origin U in $\mathbf{T}^2$ and $u \in C^\infty(U) \setminus C^\omega(U)$ such that $Mu \in C^\omega(U)$ (Metivier).
- Since M is elliptic in $U \setminus \{(0,0),(0,\pi),(\pi,0),(\pi,\pi)\}$ we can assume that $u|_{U\setminus\{(0,0)\}}$ is real-analytic
- Since $H^1(\mathbf{T}^2, \mathcal{C}^\omega) = 0$ we can write

$$u|_{U\setminus\{(0,0)\}} = v|_{U\setminus\{(0,0)\}} - w|_{U\setminus\{(0,0)\}},$$

  where $v \in C^\omega(U)$, $w \in C^\omega(\mathbf{T}^2 \setminus \{(0,0)\})$.
- Define

$$\tilde{u} = \begin{cases} u - v & \text{in } U \\ -w & \text{in } \mathbf{T}^2 \setminus \{(0,0)\} \end{cases}$$

  Then $\tilde{u} \in C^\infty(\mathbf{T}^2) \setminus C^\omega(\mathbf{T}^2)$ but $M\tilde{u} \in C^\omega(\mathbf{T}^2)$. ■

Hörmander type

$$P = X_1^2 + \dots X_\nu^2$$

$$X_j = \sum_{k=1}^{N} a_{jk}(x)\partial/\partial x_k, \;\; \sigma_j(x,\xi) = \sum_{k=1}^{N} a_{jk}(x)\xi_k, \quad \sigma_j : T^*\Omega \to \mathbb{R}.$$

If $A \in T^*\Omega \setminus 0$ denote C^ω_A the space of germs of real-analytic functions at A by $I^k_A \subset$ the ideal of C^ω_A spanned by the germs at A of all Poisson brackets of length $\ell \leq k$:

$$\{f_{j_\ell}, \dots, \{f_{j_1}, f_{j_0}\} \dots\}, \quad f_j \doteq \sigma(X_j)\,.$$

Important remark: Hörmander condition says that for each $A \in T^*\Omega \setminus 0$ there is $k \in \mathbb{Z}_+$ such that $I^k_A = C^\omega_A$

We set

$$\rho(A) = \min\left\{k : \; I^k_A = C^\omega_A\right\}$$

Theorem (Treves, 2006)

$$P = X_1^2 + \ldots + X_\nu^2.$$

There exists a well defined partition of $\Sigma = \cup \Sigma_j$ *into connected, pairwise disjoint analytic submanifolds* Σ_j *satisfying:*

- *The union is locally finite.*
- *For each* j *the functions*

$$\Sigma_j \ni A \mapsto \dim \left(T_A \Sigma_j \cap T_A {\Sigma_j}^\perp \right)$$

$$\Sigma_j \ni A \mapsto \rho(A)$$

  are constant on Σ_j
- *Each* Σ_j *is maximal with respect to these properties.* ■

Treves' conjecture: P is (ahe) if each Σ_j is symplectic.

Back to Oleinik operator

$$P = \partial_t^2 + t^{2p}\partial_{x_1}^2 + t^{2q}\partial_{x_2}^2, \;\; 1 \leq p < q.$$

$$\Sigma = \{t = \tau = 0\} = \{(x, \xi) : \xi \neq 0\} \subset \mathbb{R}^4$$

$$\sigma_1 = \tau, \;\; \sigma_2 = t^p \xi_1, \;\; \sigma_3 = t^q \xi_2$$

$$\begin{cases} \Sigma_1 & : \quad t = \tau = 0, \;\; \xi_1 > 0 \\ \Sigma_2 & : \quad t = \tau = 0, \;\; \xi_1 < 0 \\ \Sigma_3 & : \quad t = \tau = \xi_1 = 0, \;\; \xi_2 > 0 \\ \Sigma_4 & : \quad t = \tau = \xi_1 = 0, \;\; \xi_2 < 0 \end{cases}$$

$$\rho = p \text{ on } \Sigma_1 \cup \Sigma_2; \;\; \rho = q \text{ on } \Sigma_3 \cup \Sigma_4.$$

Σ_1 and Σ_2 are symplectic; Σ_3 and Σ_4 are not.

Microlocal analytic singularities of of the solutions to $Pu = 0$ are contained in $\Sigma_3 \cup \Sigma_4$.

A particular case

Theorem (Okaji, 1985), (C-Hanges, 2009)

If, near A, Σ is a codimension 2 symplectic manifold such that $\rho|_\Sigma$ is constant and if u is a solution to $Pu = f$ with f real-analytic then u is real-analytc near A.

Example: If P is the Oleinik operator and if $Pu = f \in C^\omega$ then u is real-analytic in $\Sigma_1 \cup \Sigma_2$. ■

Gevrey hypoellipticity

$$P = X_1^2 + \ldots + X_\nu^2 \text{ under Hörmander condition}$$

Definition

P is G^s-hypoelliptic in Ω ($s \geq 1$) if given $U \subset \Omega$ open and $u \in \mathcal{D}'(U)$ then $Pu \in G^s(U)$ implies $u \in G^s(U)$. Here $G^s(U)$ denotes the space of gevrey functions of order s in U.

Theorem (Derridj–Zuilly, 1972)

$\exists s_0 = s_0(P) > 1$ such that P is G^s-hypoelliptic in Ω if $s > s_0$

Examples.

- $P = \partial_t^2 + \partial_y^2 + t^2\partial_x^2$ is G^s-hypoelliptic if and only if $s \geq 2$.
- $P = \partial_t^2 + t^2\partial_x^2 + (x\partial_x)^2$ is G^s-hypoelliptic if and only if $s \geq 2$.
- $P = \partial_t^2 + t^{2p}\partial_{x_1}^2 + t^{2q}\partial_{x_2}^2$, $1 \leq p \leq q$ if G^s hypoelliptic if and only if $s \geq (q+1)/(p+1)$ (Christ, 1997).

Hypoellipticity in the hyperfunction sense

Theorem (Schapira, 1969, cf. also Kawai)

Let $P(D)$ be a constant coefficients LPDO in $\mathbb{R}^N$. If the following property holds, for every $\Omega \subset \mathbb{R}^N$ open:

$$u \in B(\Omega),\ P(D)u \in C^\infty(\Omega) \implies u \in \mathcal{D}'(\Omega)$$

then $P(D)$ is elliptic.

Theorem

$P(x, D)$: analytic LPDO in $\Omega \subset \mathbb{R}^N$. $P(x, D)$ is said to be $\mathfrak{H}$-hypoelliptic in Ω if, given any $U \subset \Omega$ open, and any $u \in B(U)$ then $P(x, D)u \in C^\infty(U)$ implies $u \in C^\infty(U)$.

Example. $P(D)$ is $\mathfrak{H}$-hypoelliptic if and only if it is elliptic. More precisely, if $P(D)$ is hypoelliptic but not elliptic and if

$$s_0 = \min\{s : P(D) \text{ is } G^s\text{-hypoelliptic}\}$$

then $\forall U \subset \mathbb{R}^N$ open and for every $1 < s < s_0$ $\exists u \in \mathcal{D}^{(s)\prime}(U) \setminus \mathcal{D}'(U)$: $P(D)u = 0$ (C-Hanges, 2009).

Natural question (proposed by J.M.Bony)

$P(x, D)$ analytic LPDO in $\Omega \subset \mathbb{R}^N$

$$P(x, D) \text{ (ahe) } \overset{?}{\Longrightarrow} {}^tP(x, D) \ \ \mathfrak{H}\text{-hypoelliptic}$$

An abstract result

Theorem (C-Hanges, 2009)

$P(x, D)$ analytic LPDO in $\Omega \subset \mathbb{R}^N$. If $P(x, D)$ is (ahe) and L^2-solvable on any $U \subset\subset \Omega$ open then the following holds, for any $U \subset\subset \Omega$ open

$$u \in B(U), \quad {}^tP(x, D)u \in L^2(U) \implies u \in L^2(U).$$

Corollary

Let P be a sum of squares operator satisfying Hörmander's condition. If P is (ahe) then P^* is $\mathfrak{H}$-hypoelliptic.

Okaji's example

$$P = \left(\partial_t + it^k \,\partial_x\right)^2 + c\,\partial_x$$
$$k \text{ even}, \;\; c \in \mathbb{C}, \;\; c \neq 0\,.$$

Theorem (Okaji, 1988)

P is (ahe) near the origin and ${}^tP$ is not solvable in any neighborhood of the origin. In particular P is not (he) in any neighborhood of the origin.

C.-Trépreau (1998):

$$P \text{ (ahe)} \implies {}^tP : B_0 \to B_0 \text{ surjective.}$$

Thus $\exists f \in C_0^\infty$:

$$\nexists\, u \in \mathcal{D}_0' \;:\;\; {}^tPu = f$$
$$\exists\, v \in B_0 \;:\;\; {}^tPv = f\,.$$

Hence ${}^tP$ is not $\mathfrak{H}$-hypoelliptic near 0. □

Classical methods for disproving (ahe)

$P(x, D)$ analytic LPDO in Ω.

- If $P(x, D)$ is (ahe) then given $x_0 \in \Omega$ there is $C > 0$ such that for every solution of $P(x, D)f = 0$ on Ω it holds that

$$|(\partial^\alpha f)(x_0)| \leq C^{|\alpha|+1}\alpha! \left(\sup_\Omega |f|\right) .$$

- In general it is only possible to construct asymptotic solutions to $P(x, D)f_\lambda \sim 0$.
- This requires non homogeneous a priori inequalities. (cf. Metivier, 1980)

A new method (C-Hanges, 2012)

$P(x, D_x)$ analytic LPDO in Ω
$P(z, D_z)$ defined in $\Omega_\bullet \subset \mathbb{C}^N$, $\Omega_\bullet \cap \mathbb{R}^N = \Omega$
$U \subset\subset \Omega$, $\Gamma \subset \mathbb{R}^N \setminus 0$ open, convex cone, $\delta > 0$

$$\mathcal{W}_\delta(U, \Gamma) = \{x + iy : \; x \in U, \; y \in \Gamma, \; |y| < \delta\} \subset \Omega_\bullet$$

Theorem

Assume that, for some $s > 1$, $u \in \mathcal{D}^{(s)\prime}(U)$, $P(x, D)u \in C^\infty(U) \Rightarrow u \in C^\infty(U)$. Given $K_0 \subset\subset U$ there are compact sets $K \subset \mathcal{W}_\delta(U; \Gamma) \cup (U + i\{0\})$, $K' \subset U$, $M \in \mathbb{Z}_+$ and $C > 0$ such that

$$\sup_{K_0} |F| \leq C \left(\sup_{\mathcal{W}_\delta(U;\Gamma) \cap K} |F(x + iy)| e^{-1/|y|^{1/(s-1)}} + \right.$$

$$\left. + \| \; P(x, D_x)F \|_{C^M(K')} \right), \qquad F \in \mathcal{O}(\Omega_\bullet).$$

A perturbation of the Baouendi-Goulaouici operator

$$P = P^* = \partial_{x_1}^2 + \partial_{x_2}^2 + x_1^2 g(x_2)^2 \partial_{x_3}^2.$$

$$g \in \mathcal{O}(D(r_0)), \text{ real on }]-r_0, r_0[, \quad g(0) = 1$$

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$$F_\lambda(z) = e^{i\lambda z_3} f_\lambda(\lambda^{1/2} z_1, z_2), \quad \lambda \geq 1,$$

with $f_\lambda = f_\lambda(\zeta, z_2)$ holomorphic. Then

$$PF_\lambda(z) = e^{i\lambda z_3}(Q_\lambda f_\lambda)(\lambda^{1/2} z_1, z_2),$$

$$Q_\lambda = \partial_{z_2}^2 - \lambda \left\{ \zeta^2 g(z_2)^2 - \partial_\zeta^2 \right\}$$

We interpret the equation $Q_\lambda f_\lambda \sim 0$ as a second order ODE in $z_2 \in D(r_0)$ valued in a convenient scale of Banach spaces of entire functions in the variable ζ (Ovcyannikov method).

For each $\beta > 1/2$ we can prove the existence of $f_\lambda(\zeta, z_2) \in \mathcal{O}(\mathbb{C} \times D(r_0))$:

- $f_\lambda(0,0) = 1$;
- *There are* $a > 0$, $C > 0$ *such that*

$$|f_\lambda(\zeta, z_2)| \leq C\, e^{a(|\Im\zeta|^2 + \lambda^\beta)}, \ \ (\zeta, z_2) \in \mathbb{C} \times D(r_0).$$

- $\forall M \in \mathbb{Z}_+$,

$$\lambda^M \sum_{p+q \leq M} \sup_{\mathbb{R}\times]-r_0, r_0[} |(\partial_\xi^p \partial_x^q Q_\lambda f_\lambda)(\xi, x)| \stackrel{\lambda\to\infty}{\longrightarrow} 0$$

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With $\Gamma \subset \mathbb{R}^3 \setminus 0$ to be chosen, applying our a priori estimate with $K = \{0\}$ and $\delta \leq 1$ gives

$$c \leq \sup_{y\in\Gamma,\, |y_1|\leq 1\,, |z_2|<r_0} |e^{i\lambda z_3} f_\lambda(\sqrt{\lambda} z_1, z_2)| e^{-1/|y|^{1/(s-1)}}$$

$$+R(\lambda)$$

where $c > 0$ and $R(\lambda) \to 0$ when $\lambda \to \infty$.

$$c \leq \sup_{y\in\Gamma,\, |y_1|\leq 1\,, |z_2|\leq r_0} |e^{i\lambda z_3} f_\lambda(\sqrt{\lambda} z_1, z_2)| e^{-1/|y|^{1/(s-1)}}$$

$$+R(\lambda)$$

$$\leq \sup_{y\in\Gamma} \left\{ e^{-\lambda(y_3 - a|y_1|) + a\lambda^\beta - |y|^{-1/(s-1)}} \right\} + R(\lambda).$$

Take $\Gamma \subset\subset \{y_3 > a|y_1|\}$ and $1 < s < 2$.
Let $\beta > 1/2$ with $s < 1/\beta$. If $\beta(s-1) < \theta < 1-\beta$ we estimate the exponent as

- If $y \in \Gamma$ and $|y| \leq \lambda^{-\theta}$ then $|y|^{-1/(s-1)} \geq \lambda^{\theta/(s-1)}$ and the exponent is $\leq -\lambda^{\theta/(s-1)} + a\lambda^\beta$.
- $\exists \epsilon > 0$ such that $y_3 - a|y_1| \geq \epsilon|y|$ on Γ. Hence if $y \in \Gamma$ and $|y| > \lambda^{-\theta}$ then $(y_3 - a|y_1|) \geq \epsilon\lambda^{-\theta}$ and the exponent is now $\leq -\epsilon\lambda^{1-\theta} + a\lambda^\beta$.

Hence P is not (ahe) and for each $1 < s < 2$ there exists $u \in \mathcal{D}^{(s)\prime}(U) \setminus \mathcal{D}'(U)$ such that $Pu \in C^\infty(U)$. ■