Growth of Sobolev norms for the cubic defocusing NLS
Lecture 1

Marcel Guardia

February 8, 2013

The equations

We consider two equations:

- The cubic defocusing nonlinear Schrödinger equation (NLS):

$$\begin{cases} -i\partial_t u + \Delta u = |u|^2 u \\ u(0,x) = u_0(x) \end{cases}$$

- The cubic defocusing nonlinear Schrödinger equation with a convolution potential:

$$\begin{cases} -i\partial_t u + \Delta u + V(x) * u = |u|^2 u \\ u(0,x) = u_0(x) \end{cases}$$

- For both equations: $x \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, $t \in \mathbb{R}$ and $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$.

Outline of lectures

Tuesday:

- Introduction: transfer of energy and growth of Sobolev norms
- Previous results
- Results about NLS
- Cascade of energy
- Sketch of the proof of the results about NLS

Friday:

- The dynamics of the toy model and end of the sketch of the proof of the results about NLS
- Results to NLS with convolution potential
- Open questions

Transfer of energy and growth of Sobolev norms

Properties of NLS

- Defocusing implies well posed globally in time in Sobolev spaces $H^s(\mathbb{T}^2)$, $s \geq 1$.
- From the Dynamical System point of view, it defines a complete flow in $H^s(\mathbb{T}^2)$, $s \geq 1$.
- Solutions of NLS conserve two quantities:
  - The Hamiltonian

$$E[u](t) = \int_{\mathbb{T}^2} \left(\frac{1}{2} |\nabla u|^2 + \frac{1}{4} |u|^4 \right) dx(t)$$

  - The mass

$$\mathcal{M}[u](t) = \int_{\mathbb{T}^2} |u|^2 dx(t) = \int_{\mathbb{T}^2} |u|^2 dx(0),$$

the square of the L^2-norm.

Transfer of energy

- Fourier series of u,

$$u(t,x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{inx}$$

- Can we have transfer of energy to higher and higher modes as $t \to +\infty$?
- It is possible that a solution u starts oscillating only on scales comparable to the spatial period and eventually oscillates on arbitrarily small scale?

Sobolev norms

- Sobolev norms

$$\|u(t)\|_{H^s(\mathbb{T}^2)} := \|u(t,\cdot)\|_{H^s(\mathbb{T}^2)} := \left(\sum_{n\in\mathbb{Z}^2} \langle n\rangle^{2s}|a_n(t)|^2\right)^{1/2},$$

  where $\langle n\rangle = (1+|n|^2)^{1/2}$.
- Thanks to mass and energy conservation,

$$\|u(t)\|_{H^1(\mathbb{T}^2)} \leq C\|u(0)\|_{H^1(\mathbb{T}^2)} \text{ for all } t \geq 0.$$

- The L^2 norm is conserved.
- The energy transfer can be measured with the growth of the Sobolev norms with $s > 1$.
- Only possibility for H^s to grow indefinitely: the energy of u moves to higher and higher Fourier modes.

How fast the energy transfer can be?

Polynomial upper bounds for the growth of Sobolev norms were first obtained by Bourgain (1993),

Theorem

Let us consider a solution u of the cubic defocusing NLS on $\mathbb{T}^2$, then

$$\|u(t)\|_{H^s} \le t^{2(s-1)+}\|u(0)\|_{H^s} \quad \textit{for} \quad t \to +\infty.$$

- Question by Bourgain (2000): Are there solutions u such that for $s > 1$,

$$\|u\|_{H^s} \to +\infty \qquad \text{as } t \to +\infty$$

- Moreover, he conjectured that if such solutions exists, the growth should be subpolynomial in time. That is,

$$\|u(t)\|_{H^s} \ll t^{\varepsilon}\|u(0)\|_{H^s} \qquad \text{for } t \to +\infty, \text{ for all } \varepsilon > 0.$$

Previous results

Previous results for NLS

- Kuksin (1997) studies the equation

$$-i\dot{w} = -\delta\Delta w + |w|^{2p}w, \ \delta \ll 1, p \geq 1.$$

  and obtains orbits whose s-Sobolev norms grows by an inverse power of δ.
- The rescaling $u_\delta(t,x) = \delta^{-1}u(\delta^{-1}t,x)$ is a solution of NLS.
- This orbit undergoes arbitrarily large finite growth of Sobolev norms but it has large initial condition.
- For large initial condition, dispersion is much weaker than the nonlinearity.

Small initial data: stability of $u = 0$

- We are interested in growth of Sobolev norms from small initial data.
- As a first step, omit the cubic terms and consider the linear equation.

$$-i\partial_t u + \Delta u = 0$$

- Its solutions are

$$u(t,x) = \sum_{n\in\mathbb{Z}^2} u^{[k]} e^{i(nx+|n|^2 t)}$$

- Therefore, their Sobolev norms are first integrals for all s.

Stability of $u = 0$

- From the point of view of dynamical systems, $u = 0$ is an elliptic critical point.
- For the linearized equation, $u = 0$ is stable in any H^s topology.
- It is stable for the nonlinear equation?
- Stable in L^2 and H^1.
- Growth of Sobolev norms for small solutions implies its instability in H^s, $s > 1$.
- How long does it take such instability to be noticeable?

Previous results for small initial data

Colliander, Keel, Staffilani, Takaoka, Tao (2010) proved the following deep result:

Theorem

Fix $s > 1$, $\mathcal{C} \gg 1$ and $\mu \ll 1$. Then there exists a global solution $u(t,x)$ of NLS on $\mathbb{T}^2$ and T satisfying that

$$\|u(0)\|_{H^s} \leq \mu, \qquad \|u(T)\|_{H^s} \geq \mathcal{C}.$$

- The solutions they obtain have small initial mass and energy (smaller than μ).
- They remain small as time evolves whereas the s-Sobolev norm grows considerably.

Other results for NLS

- Hani (2011)
  - Obtains arbitrarily large growth of Sobolev norms for $s \in (0,1)$ for solutions of NLS which are very close to plain waves
    $$u(t,x) = Ae^{i\left(nx+|n|^2t+|A|^2t\right)}.$$
  - Obtains unbounded growth for a familiy of pseudo PDEs which are simplifications of NLS.
- Carles and Faou (2011) obtain orbits for NLS with transfer of energy among modes.
  Nevertheless, this transfer of energy does not lead to growth of Sobolev norms.

Results for NLS (joint work with Vadim Kaloshin)

Refining the I-team results, we estimate the speed of the growth of Sobolev norms.

Theorem (V. Kaloshin-M. G.)

Let $s > 1$. Then, there exists $c > 0$ with the following property: for any large $\mathcal{K} \gg 1$ there exists a a global solution $u(t,x)$ of NLS on $\mathbb{T}^2$ and a time T satisfying

$$0 < T \leq \mathcal{K}^c,$$

such that

$$\|u(T)\|_{H^s} \geq \mathcal{K}\, \|u(0)\|_{H^s}.$$

Moreover, this solution can be chosen to satisfy

$$\|u(0)\|_{L^2} \leq \mathcal{K}^{-\alpha}, \ \alpha > 0.$$

We can impose to start with small Sobolev norm but we get a slower growth.

Theorem (V. Kaloshin-M. G.)

Fix $s > 1$. Then, there exists $c > 0$ with the following property: for any small $\mu \ll 1$ and large $\mathcal{C} \gg 1$ there exists a global solution $u(t, x)$ of NLS on $\mathbb{T}^2$ and a time T satisfying that

$$0 < T \leq \left(\frac{\mathcal{C}}{\mu}\right)^{c \ln(\mathcal{C}/\mu)}$$

such that

$$\|u(0)\|_{H^s} \leq \mu, \qquad \|u(T)\|_{H^s} \geq \mathcal{C}.$$

- In a neighborhood of size μ of $u = 0$ in H^s, instabilities are noticeable after time $T \sim \mu^{-c \ln|\mu|}$.

Comparison with Bourgain conjecture

- Bourgain conjecture:

$$\|u(t)\|_{H^s} \ll t^{\varepsilon}\|u(0)\|_{H^s} \qquad \text{for } t \to +\infty, \ \text{for all } \varepsilon > 0.$$

- Our result

$$\|u(T)\|_{H^s} \geq T^{\frac{1}{c}}\|u(0)\|_{H^s}. \qquad \text{for } T \gg 1.$$

- Our result does not contradict Bourgain conjecture about the subpolynomial growth:
  - The theorem deals with arbitrarily large but finite growth in the Sobolev norms.
  - Bourgain conjecture refers to unbounded growth.
- Growth of the s-Sobolev norm may slow down as time grows.

Comments on the theorem

- Other dimensions:
  - NLS in $\mathbb{T}^1$ is integrable (Zakharov-Shabat equation) and growth of Sobolev norms is not possible.
  - Our result is valid in any $\mathbb{T}^d$, $d \geq 2$ taking solutions which only depend on two spatial variables.
- One would like to obtain solutions supported at low modes at $t = 0$ which transfer energy to high modes.
- We do not obtain such solutions: our solutions start being supported at high modes and transfer energy to much higher modes.

More comments on the theorem

- We can obtain more detailed information about the distribution of the Sobolev norm of the solution u, among its Fourier modes when $t = T$: we can ensure that there exist $n_1, n_2 \in \mathbb{Z}^2$ such that

$$\|u(T)\|_{H^s}^2 \geq |n_1|^{2s}|a_{n_1}(T)|^2 + |n_2|^{2s}|a_{n_2}(T)|^2 \geq \mathcal{K}^2\|u(0)\|_{H^s}.$$

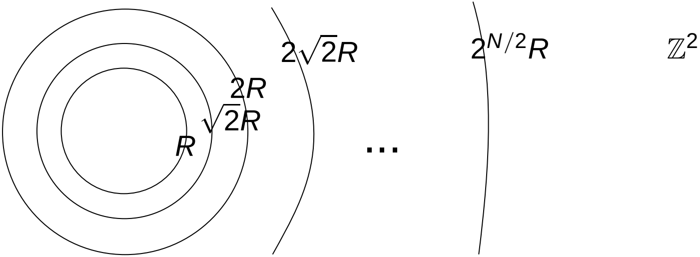
  That is, the Sobolev norm is essentially localized in two modes.
- This is a consequence of the energy cascade that we construct.
- Recall that we want growth of the Sobolev norm

$$\|u(t)\|_{H^s(\mathbb{T}^2)} := \|u(t,\cdot)\|_{H^s(\mathbb{T}^2)} := \left(\sum_{n\in\mathbb{Z}^2} \langle n\rangle^{2s}|a_n(t)|^2\right)^{1/2},$$

  for $s > 1$ whereas the H^1 norm is almost constant.

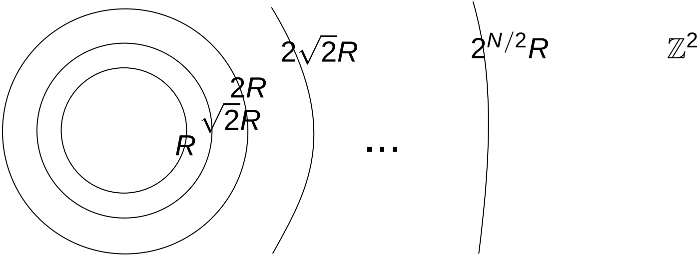
The energy cascade

Heuristic picture of the energy cascade



- We start with a set of modes contained in a disk of radius R in Fourier space.
- At each step we *activate* modes in a disk of bigger radius while we put to sleep the previously activated modes.
- We want modes at the *boundary* of the new disk to be activated.

Heuristic picture of cascade



- Since the H^1 norm is almost constant, at each step half of the modes we activate are close to the boundary and the other half closer to zero.
- Mass spreads equally among the activated modes, which balances the H^1 norm.
- The s-Sobolev norm is essentially given by the bigger harmonics.

Finite time growth only!

- We want to make N jumps in the growth of Sobolev norms.
- Here N depends on the growth $\mathcal{K}$: $N \sim \ln(\mathcal{K})$.
- At each step only half of the modes cause growth of Sobolev norms.
- This implies that we need $\sim 2^N$ at each disk.
- We need N sets of 2^N modes such that we can construct an energy cascade which at each step has one of these sets activated and the others at rest.
- Then,
  - At the begining, the Sobolev norm is localized in a large number of Fourier modes.
  - At the end, the Sobolev norm is localized in only two Fourier modes.

Sketch of the proof

Outline of the proof

The I-team strategy:

- Birkhoff normal form and the resonant truncation
- The finite set of modes Λ and the derivation of the toy model
- The dynamics of the toy model
- Final approximation argument

First step: eliminate some nonlinear terms

- NLS as an ode for the Fourier coefficients of u:

$$-i\dot{a}_n = |n|^2 a_n + \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2\\ n_1-n_2+n_3=n}} a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n\in\mathbb{Z}^2.$$

- We are dealing with small data.
- Thanks to
  - One step of Birkhoff normal form
  - The gauge invariance

  we eliminate some terms in the cubic nonlinearity.

One step of normal form

More rigorously, there exists a symplectic change of coordinates $a = \Gamma(\beta)$ which transforms

$$-i\dot{a}_n = |n|^2 a_n + \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2\\ n_1-n_2+n_3=n}} a_{n_1}\overline{a_{n_2}}a_{n_3}. \qquad n\in\mathbb{Z}^2.$$

into

$$-i\dot{\beta}_n = |n|^2\beta_n - \beta_n|\beta_n|^2 + \sum_{n_1,n_2,n_3\in\mathcal{A}(n)} \beta_{n_1}\overline{\beta_{n_2}}\beta_{n_3} + \mathcal{R}$$

where $\|\mathcal{R}\|_{\ell^1} \leq \mathcal{O}\left(\|\beta\|_{\ell^1}^5\right)$ and

$$\mathcal{A}(n) = \Big\{(n_1,n_2,n_3)\in\left(\mathbb{Z}^2\right)^3 : n_1 - n_2 + n_3 = n \\ |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 = 0, n_1\neq n, n_3\neq n\Big\}.$$

The resonant truncation equation

- New equation

$$-i\dot{\beta}_n = |n|^2\beta_n - \beta_n|\beta_n|^2 + \sum_{n_1,n_2,n_3\in\mathcal{A}(n)} \beta_{n_1}\overline{\beta_{n_2}}\beta_{n_3} + \mathcal{R}$$

- Change to rotating coordinates: $\beta_n = r_n e^{i|n|^2 t}$
- New equation

$$-i\dot{r}_n = -r_n|r_n|^2 + \sum_{n_1,n_2,n_3\in\mathcal{A}(n)} r_{n_1}\overline{r_{n_2}}r_{n_3} + \mathcal{R}$$

- Since $\|r\|_{\ell^1}$ is small, $\mathcal{R}$ can be neglected.
- We drop $\mathcal{R}$ and we work with the resonant truncated equation

$$-i\dot{r}_n = -r_n|r_n|^2 + \sum_{n_1,n_2,n_3\in\mathcal{A}(n)} r_{n_1}\overline{r_{n_2}}r_{n_3}$$

- Later we will see that $\mathcal{R}$ indeed does a small influence.

The resonant interaction

$$\mathcal{A}(n) = \Big\{ (n_1, n_2, n_3) \in \left(\mathbb{Z}^2\right)^3 : n_1 - n_2 + n_3 = n \\ |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 = 0, n_1 \neq n, n_3 \neq n \Big\}.$$

- $(n_1, n_2, n_3) \in \mathcal{A}(n)$ if and only if (n_1, n_2, n_3, n) form a rectangle in $\mathbb{Z}^2$.
- The I-team used these rectangles to construct a finite set of modes such that some solutions of the resonant truncated equation localized in these modes undergo the energy cascade.

The I-team finite set of modes

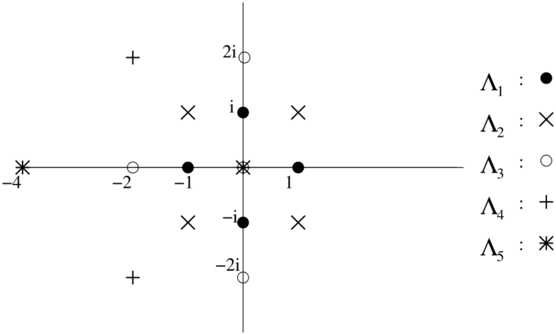
- We want to obtain a finite set of modes Λ,
  - which is closed under resonant interactions:

$$(n_1, n_2, n_3) \in \mathcal{A}(n), n_1, n_2, n_3 \in \Lambda \Longrightarrow n \in \Lambda.$$

  - whose modes are distributed in a nested set of disks to have energy cascade.
- For any closed under resonant interactions finite set, the resonant truncated PDE becomes and ODE.
- Split $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_N$: each generation Λ_j will be in one disk.
- To push mass from one generation to the next one, we want resonant rectangles with two vertices in one generation and the other two in the next one.
- The I-team chooses very cleverly the modes so that they interact in a very particular way.

The properties of the set Λ

- Define a nuclear family to be a rectangle (n_1, n_2, n_3, n_4) where the frequencies n_1, n_3, the parents belong to the generation Λ_j and n_2, n_4, the children belong to the generation Λ_{j+1}.
- The I-team imposes conditions in terms of the nuclear families:
  - I For all $n_1 \in \Lambda_j$ there exists a unique family (n_1, n_2, n_3, n_4) such that $n_1, n_3 \in \Lambda_j$ are the parents and $n_2, n_4 \in \Lambda_{j+1}$ are the children.
  - II For all $n_2 \in \Lambda_{j+1}$ there exists a unique family (n_1, n_2, n_3, n_4) such that $n_1, n_3 \in \Lambda_j$ are the parents and $n_2, n_4 \in \Lambda_{j+1}$ are the children.
  - III The sibling of a frequency is never equal to its spouse.
  - IV Besides nuclear families, Λ does not contain other rectangles.
- We need to add an additional condition:
  - V Let us consider $n \notin \Lambda$. Then, n is vertex of at most two rectangles having two vertices in Λ and two vertices out of Λ.



- This set does not satisfy any of the wanted conditions.
- Important property of this set:

$$\frac{\sum_{n\in\Lambda_N} |n|^{2s}}{\sum_{n\in\Lambda_1} |n|^{2s}} \geq \frac{1}{2} 2^{(s-1)(N-1)} \geq \mathcal{K}^2.$$

- Slightly perturb the previous set in $\mathbb{Q}^2$.
- Then
  - We can make it satisfy the required conditions.
  - It still satisfies

$$\frac{\sum_{n\in\Lambda_N} |n|^{2s}}{\sum_{n\in\Lambda_1} |n|^{2s}} \geq \frac{1}{2} 2^{(s-1)(N-1)} \geq \mathcal{K}^2.$$

- Then, blow it up to make it belong to $\mathbb{Z}^2$.
- Now, the modes in Λ satisfy $n \sim B^{N^2}$, $B > 0$.

Invariant subspaces

- Conclusion: $\mathcal{M} = \{r_n = 0 \quad \text{for } n \notin \Lambda\}$ is invariant.
- It has dimension $N2^N$.
- The resonant truncated equation in $\mathcal{M}$ can be written as

$$-i\dot{r}_n = -r_n|r_n|^2 + 2r_{n_{\text{child}_1}} r_{n_{\text{child}_2}} \overline{r_{n_{\text{spouse}}}} + 2r_{n_{\text{parent}_1}} r_{n_{\text{parent}_2}} \overline{r_{n_{\text{sibling}}}}.$$

- $\mathcal{M}$ has an invariant submanifold of dimension N:

$$\widetilde{\mathcal{M}} = \left\{ r_n = r_{n'} \quad \text{if } n, n' \notin \Lambda_j \quad \text{for some } j \right\}.$$

- That is, if we take the same initial condition for all the modes in the same generation, they remain equal for all time.

The finite dimensional toy model

- Define $b_j = r_n$ for any $n \in \Lambda_j$.
- In other words,

$$r_n = b_j \quad \text{if } n \in \Lambda_j$$
$$r_n = 0 \quad \text{if } n \notin \Lambda$$

  is a solution of the resonant truncated equation.
- Then

$$-i\dot{r}_n = -r_n|r_n|^2 + 2r_{n_{\text{child}_1}} r_{n_{\text{child}_2}} \overline{r_{n_{\text{spouse}}}} + 2r_{n_{\text{parent}_1}} r_{n_{\text{parent}_2}} \overline{r_{n_{\text{sibling}}}}.$$

  becomes

$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right),\ j = 0, \dots N.$$

- Following the I-team, we call this system the toy model.
- It can be seen as a Hamiltonian system on a lattice $\mathbb{Z}$ with nearest neighbor interactions.

The finite dimensional toy model

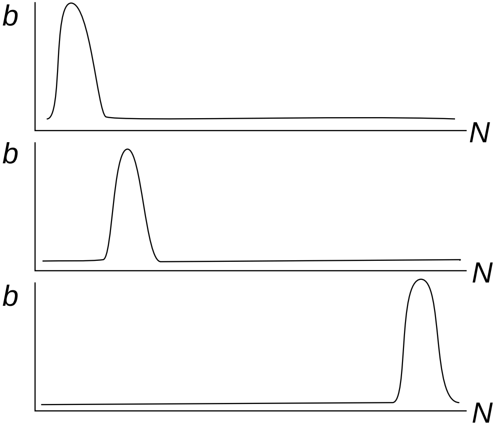
- The toy model is Hamiltonian with respect to

$$h(b) := \frac{1}{4}\sum_{j=1}^{N} |b_j|^4 - \frac{1}{2}\sum_{j=1}^{N} \left(\overline{b}_j^2 b_{j-1}^2 + b_j^2 \overline{b}_{j-1}^2\right)$$

  and $\Omega = \frac{i}{2} db_j \wedge d\overline{b}_j$.
- The mass conservation now becomes conservation of

$$\mathcal{M}(b) = \sum_{j=1}^{N} |b_j|^2.$$

- Note that the toy model is invariant by certain rescaling.
- We restrict to $\mathcal{M}(b) = 1$.



- We wanted an orbit of the truncated resonant model localized in Λ_1 at $t = 0$ and localized at Λ_N at $t = T \gg 1$.
- Equivalently, now we want an orbit $b(t)$ of the toy model localized in b_1 at $t = 0$ and localized in b_N at $t = T \gg 1$.

- Take an orbit in $\mathcal{M}(b) = 1$ such that for some $T \gg 1$,

$$\begin{aligned} b_1(0) &\sim 1 \\ b_j(0) &\ll 1,\ j \neq 1 \end{aligned} \quad \text{and} \quad \begin{aligned} b_N(T) &\sim 1 \\ b_j(T) &\ll 1,\ j \neq N \end{aligned}$$

- Then, the growth of the Sobolev norm for

$$\begin{aligned} r_n &= b_j \quad \text{if } n \in \Lambda_j \\ r_n &= 0 \quad \text{if } n \notin \Lambda \end{aligned}$$

is

$$\frac{\|r(T)\|_{H^s}^2}{\|r(0)\|_{H^s}^2} \sim \frac{\sum_{n\in\Lambda_N} |r_n(T)|^2 |n|^{2s}}{\sum_{n\in\Lambda_1} |r_n(0)|^2 |n|^{2s}} \sim \frac{\sum_{n\in\Lambda_N} |b_N(T)|^2 |n|^{2s}}{\sum_{n\in\Lambda_1} |b_1(0)|^2 |n|^{2s}}$$
$$\sim \frac{\sum_{n\in\Lambda_N} |n|^{2s}}{\sum_{n\in\Lambda_1} |n|^{2s}} \geq \frac{1}{2} 2^{(s-1)(N-1)} \geq \mathcal{K}^2.$$

- To estimate the time of instability, we want to obtain this orbit with quantitative estimates with respect to the number of modes N.

The toy model theorem

Theorem

Let $\gamma \gg 1$ and $N \gg 1$ be large. Then for $\delta = e^{-\gamma s N}$, there exists an orbit of the toy model and $T_0 > 0$ such that

$$\begin{array}{c} |b_1(0)| > 1-\delta \\ |b_j(0)| < \delta \quad \text{for } j \neq 1 \end{array} \quad \textit{and} \quad \begin{array}{c} |b_N(T_0)| > 1-\delta \\ |b_j(T_0)| < \delta \quad \text{for } j \neq N. \end{array}$$

Moreover, there exists a constant $C > 0$ independent of δ and N such that T_0 satisfies

$$0 < T_0 < CN \ln\left(\frac{1}{\delta}\right) \sim N^2.$$

Growth of Sobolev norms for the cubic defocusing NLS
Lecture 2

Marcel Guardia

February 8, 2013

Outline of today's lecture

- End of the proof of NLS results
  - The dynamics of the toy model
  - Final Approximation argument
- Growth of Sobolev norms for NLS with convolution potential
- Open problems

- Cubic defocusing NLS:

$$\begin{cases} -i\partial_t u + \Delta u = |u|^2 u \\ u(0,x) = u_0(x) \end{cases}, \quad x \in \mathbb{T}^2$$

- We wanted to prove the existence of orbit with growth of Sobolev norms: $\|u(T)\|_{H^s} \geq \mathcal{K}\|u(0)\|_{H^s}$ with $\mathcal{K}^{c_0} \leq T \leq \mathcal{K}^{c_1}$.
- NLS in Fourier space

$$-i\dot{a}_n = |n|^2 a_n + \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n \in \mathbb{Z}^2.$$

- After some changes of variables, our equation was transformed

$$-i\dot{r}_n = -r_n|r_n|^2 + \sum_{n_1,n_2,n_3 \in \mathcal{A}(n)} r_{n_1}\overline{r_{n_2}}r_{n_3} + \mathcal{R}$$

where $\mathcal{R} = \mathcal{O}\left(\|r\|_{\ell^1}^5\right)$.

- At first step, we omitted $\mathcal{R}$ and considered

$$-i\dot{r}_n = -r_n|r_n|^2 + \sum_{n_1,n_2,n_3\in\mathcal{A}(n)} r_{n_1}\overline{r_{n_2}}r_{n_3}.$$

- Choosing carefully a finite set of modes, obtaining orbits with growth of Sobolev norms was equivalent to obtain orbits of the toy model

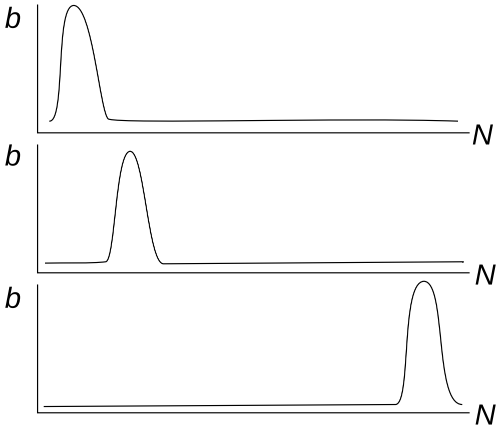
$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right),\ j = 0,\dots N.$$

  such that

$$\begin{matrix} b_1(0) \sim 1 \\ b_j(0) \ll 1,\ j \neq 1 \end{matrix} \quad \text{and} \quad \begin{matrix} b_N(T) \sim 1 \\ b_j(T) \ll 1,\ j \neq N \end{matrix}$$

- Recall that $N \sim \ln \mathcal{K}$, where $\mathcal{K}$ is the desired growth.

We wanted orbits undergoing such behavior:



The toy model theorem

Theorem

Let $\gamma \gg 1$ and $N \gg 1$ be large. Then for $\delta = e^{-\gamma s N}$, there exists an orbit of the toy model and $T_0 > 0$ such that

$$\begin{array}{l} |b_1(0)| > 1-\delta \\ |b_j(0)| < \delta \ \ \text{for } j \neq 1 \end{array} \quad \text{and} \quad \begin{array}{l} |b_N(T_0)| > 1-\delta \\ |b_j(T_0)| < \delta \ \ \text{for } j \neq N. \end{array}$$

Moreover, there exists a constant $C > 0$ independent of δ and N such that T_0 satisfies

$$0 < T_0 < CN \ln\left(\frac{1}{\delta}\right) \sim N^2.$$

Comments on the theorem

- The I-team studied certain solutions of this toy model using Gronwall-like estimates.
- Their methods would lead to bad time estimates

$$T > C^{\mathcal{K}^{\alpha}}, \;\; C > 0, \alpha \geq 2.$$

- Our main contribution: analysis of the toy model model using
  - Dynamical systems tools (normal forms, Shilnikov boundary problem).
  - A careful choice of the initial conditions

Dynamics of the toy model

Features of the toy model

$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right),\ j = 0, \dots N.$$

- It is a system on a lattice $\mathbb{Z}$ with nearest neighbor interactions.
- The toy model is Hamiltonian with respect to

$$h(b) := \frac{1}{4}\sum_{j=1}^{N} |b_j|^4 - \frac{1}{2}\sum_{j=1}^{N}\left(\overline{b}_j^2 b_{j-1}^2 + b_j^2 \overline{b}_{j-1}^2\right)$$

  and $\Omega = \frac{i}{2} db_j \wedge d\overline{b}_j$.
- Conservation of mass

$$\mathcal{M}(b) = \sum_{j=1}^{N} |b_j|^2.$$

- We restrict our study to $\mathcal{M}(b) = 1$ due to the scaling symmetry.

Dynamics of the toy model

- Toy model:

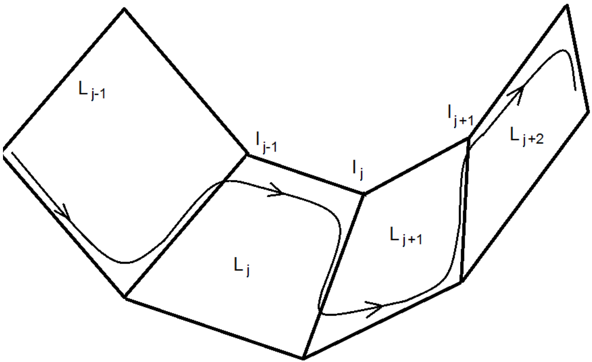
$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right),\ j = 0, \dots N,$$

- Each 4-dimensional plane

$$L_j = \{b_1 = \cdots = b_{j-1} = b_{j+2} = \cdots = b_N = 0\}$$

  is invariant.
- The dynamics in L_j is given by a simple Hamiltonian

$$h_j(b_j, b_{j+1}) = \frac{1}{4}\left(|b_j|^4 + |b_{j+1}|^4\right) - \frac{1}{2}\left(b_j^2\overline{b}_{j+1}^2 + \overline{b}_j^2 b_{j+1}^2\right).$$



- We construct solutions that stay close to the planes $\{L_j\}_{j=2}^{N-1}$ and go from one intersection $l_j = L_j \cap L_{j+1}$ to the next one $l_{j+1} = L_{j+1} \cap L_{j+2}$ consequently for $j = 3, \ldots, N-2$.
- In the intersections l_j only b_j is nonzero.

Dynamics in L_j

- To construct such orbits, we need to understand the dynamics in each L_j.
- The Hamiltonian h_j and $\mathcal{M}_j(b_j, b_{j+1}) = |b_j|^2 + |b_{j+1}|^2$ are first integrals in involution in L_j.
- This implies that the system in L_j is integrable.

- In L_j there are two periodic orbits

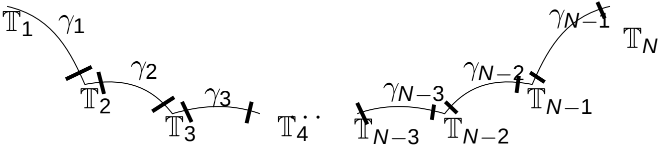
$$\mathbb{T}_j = \{(b_j(t), b_{j+1}(t)) = (e^{-i(t-t_0)}, 0)\}$$

and

$$\mathbb{T}_{j+1} = \{(b_j(t), b_{j+1}(t)) = (0, e^{-i(t-t_0)})\}.$$

- These periodic orbits in L_j are hyperbolic.
- The stable and unstable invariant manifolds of $\mathbb{T}_j$ of $\mathbb{T}_{j+1}$ coincide.
- Call γ_j to the two dimensional manifold asymptotic to $\mathbb{T}_j$ as $t \to -\infty$ and asymptotic to $\mathbb{T}_{j+1}$ as $t \to \infty$.
- Then, we construct an orbit that:
  - Starts close to the periodic orbit $\mathbb{T}_2$.
  - Travels close to γ_2 until it reaches a neighborhood of $\mathbb{T}_3$.
  - Travels close to γ_3 until it reaches a neighborhood of $\mathbb{T}_4$.
  - ... and so on until it reaches a neighborhood of $\mathbb{T}_{N-1}$.

The shadowing



- We want an orbit that shadows the sequence of periodic orbits and heteroclinic connections
- Our orbits need to be close to γ_j's to follow the instability path.
- However, if they are too close to the γ_j's, we need an extremely long time for the shadowing.
- We put sections transversal to the flow.
- We study:
    - Local maps: study the dynamics close to the periodic orbits $\mathbb{T}_j$.
    - Global maps: study the dynamics close to the heteroclinic connections γ_j.

The local and global maps

- The shadowing for the global map is basically applying (refined) Gronwall estimates.
- The local map is more delicate:
  - The periodic orbits $\mathbb{T}_j$ are of mixed type: hyperbolic and elliptic eigenvalues.
  - The hyperbolic eigenvalues are

$$\lambda, \lambda, -\lambda, -\lambda, \quad \text{for certain } \lambda > 0.$$

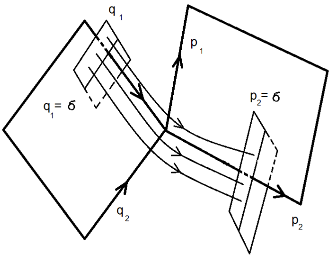
  - This resonance complicates the analysis of the local map.
  - To simplify exposition take $\lambda = 1$.

The model problem for the local map

- The periodic orbit is partially elliptic and partially hyperbolic.
- The elliptic modes:
  - Remain almost constant.
  - They do not cause much influence on the hyperbolic modes.
- For this explanation, we set the elliptic modes to zero.
- After some reductions, we have a Hamiltonian of the form

$$H(p,q) = p_1q_1 + p_2q_2 + H_4(p,q)$$

  where
  - H_4 is a degree 4 homogeneous polynomial.
  - The variables (p_1, q_1) correspond to the variable b_{j-1} after diagonalizing the saddle and the variables (q_2, p_2) correspond to b_{j+1}.
- The periodic orbit has become the critical point $q = p = 0$.



- Fix a small $\sigma > 0$.
- To study the local dynamics, it suffices to analyze a map from a section

$$\Sigma_- = \{q_1 = \sigma,\ |p_1|, |q_2|, |p_2| \ll \sigma\},$$

to a section

$$\Sigma_+ = \{p_2 = \sigma,\ |p_1|, |q_1|, |q_2| \ll \sigma\}.$$

Dynamics of the linear saddle

- Since σ is small, one would expect that the dynamics is well approximated by the linear system

$$p_1(t) = p_1^0 e^t, \;\; q_1(t) = q_1^0 e^{-t}$$
$$p_2(t) = p_2^0 e^t, \;\; q_2(t) = q_2^0 e^{-t}$$

- Then, we want to apply the local map to points in a rectangle of different widths.
- Namely we consider points

$$\left(p_1^0, q_1^0, p_2^0, q_2^0\right) \sim \left(\delta, \sigma, \delta^{1/2}, \delta^{1/2}\right)$$

which are send to

$$\left(p_1^f, q_1^f, p_2^f, q_2^f\right) \sim \left(\delta^{1/2}, \delta^{1/2}, \sigma, \delta\right),$$

where $0 < \delta \ll \sigma$ measures the closeness to the heteroclinic connections.
- The needed time is $T = \ln \frac{\sigma}{\sqrt{\delta}}$

Dynamics of the resonant saddle

- Nevertheless, the system is not well approximated by its linear part due to the resonance.
- For typical initial conditions, it sends points

$$\left(p_1^0, q_1^0, p_2^0, q_2^0\right) \sim \left(\delta, \sigma, \delta^{1/2}, \delta^{1/2}\right)$$

  to

$$\left(p_1^f, q_1^f, p_2^f, q_2^f\right) \sim \left(\delta^{1/2} \ln(1/\delta), \delta^{1/2}, \sigma, \delta \ln^2(1/\delta)\right).$$

- We need $\sim N$ transitions.
- If we compose N local maps, we get points

$$\left(p_1^f, q_1^f, p_2^f, q_2^f\right) \sim \left(\sqrt{\delta} \ln^{2^{N-1}}(1/\delta), \sqrt{\delta}, 1, \delta \ln^{2^N}(1/\delta)\right).$$

- To control shadowing we need to stay close to the heteroclinics.
- We would need $\sqrt{\delta} \ln^{2^{N-1}}(1/\delta) \ll 1$

The cancellation trick

- Analyzing carefully the first order of the local map, we obtain that it sends

$$\left(p_1^0, q_1^0, p_2^0, q_2^0\right) \sim \left(\delta \ln(1/\delta), \sigma, \delta^{1/2}, \delta^{1/2}\right) \quad \text{to}$$

$$\left(p_1^f, q_1^f, p_2^f, q_2^f\right) \sim \left(\delta^{1/2}(g(p^0,q^0)\ln(1/\delta)+1), \delta^{1/2}, \sigma, \delta(g(p^0,q^0)\ln^2(1/\delta)+\ln(1/\delta))\right).$$

  for some function $g(p,q)$.
- If $g(p,q)=0$, $\quad \left(p_1^f, q_1^f, p_2^f, q_2^f\right) \sim \left(\delta^{1/2}, \delta^{1/2}, \sigma, \delta \ln(1/\delta)\right).$
- Exchange of sizes of (p_1, q_1) and (p_2, q_2) happens for the nonlinear system if $g(p,q)=0$.

Composing the local and the global maps

- We need to compose the local and global maps $\mathcal{B}^j$.
- We define sets $\mathcal{U}_j$ in the transversal sections and we check

$$\mathcal{B}^j(\mathcal{U}_j) \subset \mathcal{U}_{j+1}$$

- To avoid deviations at each local map, we need to impose a restriction at every step to achieve the cancellation.
- To prove that the restrictions are compatible, we consider sets with a product-like structure.

Product-like structure sets

Roughly speaking:

- We start with a product set

$$\mathcal{U}_1 = B(r_1) \times \dots B(r_N)$$

  where

$$B(r) = \{|z| < r\}$$

- At each step, we impose a condition on the mode b_{j-1}; $g(\mathrm{Re}\, b_{j-1}, \mathrm{Im}\, b_{j-1}) = 0$.
- Inductively, we restrict the domain to

$$\mathcal{U}_j = N(r_1) \times \dots \times N(r_{j-2}) \times N(r_{j-1}) \times B(r_j) \dots B(r_N)$$

  with

$$N(r_{j-1}) = B(r_{j-1}) \cap \{g(\mathrm{Re}\, b_{j-1}, \mathrm{Im}\, b_{j-1}) = 0\}$$

- Since the restrictions involve a different mode at each step, the conditions are compatible.

Composing the local and the global maps, we obtain the already stated result.

Theorem

Let $\gamma \gg 1$ and $N \gg 1$ be large. Then for $\delta = e^{-\gamma s N}$, there exists an orbit of the toy model and $T_0 > 0$ such that

$$\begin{array}{c} |b_1(0)| > 1-\delta \\ |b_j(0)| < \delta \ \text{ for } j \neq 1 \end{array} \quad \text{and} \quad \begin{array}{c} |b_N(T_0)| > 1-\delta \\ |b_j(T_0)| < \delta \ \text{ for } j \neq N. \end{array}$$

Moreover, there exists a constant $C > 0$ independent of δ and N such that T_0 satisfies

$$0 < T_0 < CN \ln\left(\frac{1}{\delta}\right) = C\gamma s N^2.$$

End of the proof of growth of Sobolev norms for NLS

- From the orbit $b(t)$ we obtain a solution of the resonant truncated system

$$-i\dot{r}_n = -r_n|r_n|^2 + \sum_{n_1,n_2,n_3 \in \mathcal{A}(n)} r_{n_1}\overline{r_{n_2}}r_{n_3},$$

which is given by

$$\begin{aligned} r_n &= b_j \quad \text{if } n \in \Lambda_j \\ r_n &= 0 \quad \text{if } n \notin \Lambda \end{aligned}$$

and has the desired growth in the Sobolev norm.

- We want to use this solution, to obtain a solution of the full system

$$-i\dot{r}_n = -r_n|r_n|^2 + \sum_{n_1,n_2,n_3 \in \mathcal{A}(n)} r_{n_1}\overline{r_{n_2}}r_{n_3} + \mathcal{R}$$

where $\|\mathcal{R}\|_{\ell^1} \leq \mathcal{O}\left(r^5\right)$, having the same growth in the Sobolev norm.

The scaling symmetry

- The approximate system

$$-i\dot{r}_n = -r_n|r_n|^2 + \sum_{n_1,n_2,n_3 \in \mathcal{A}(n)} r_{n_1}\overline{r_{n_2}}r_{n_3}$$

has the scaling symmetry

$$r^\lambda(t) = \lambda^{-1} r\left(\lambda^{-2} t\right)$$

- For any $\lambda > 0$, r^λ undergoes the same growth of the Sobolev norms as r.
- The growth is undergone for a time $T \sim \lambda^2 N^2$.

Approximating a solution of NLS by a solution of the toy model

- We want the quintic terms in the remainder of

$$-i\dot{r}_n = -r_n|r_n|^2 + \sum_{n_1,n_2,n_3\in\mathcal{A}(n)} r_{n_1}\overline{r_{n_2}}r_{n_3} + \mathcal{R}$$

  to be smaller than the cubic ones
- Thus, we look for a solution of the full system close to

$$r^\lambda(t) = \lambda^{-1} r\left(\lambda^{-2}t\right)$$

  with $\lambda \gg 1$.
- However, the bigger λ is, the slower the growth of the Sobolev norm is since $T \sim \lambda^2 N^2$.
- We choose $\lambda = e^{\kappa s N}$ with $\kappa > 0$ big enough.

Approximating a solution of NLS by a solution of the toy model

Main ideas:

- We linearize around the solution of the resonant truncated equation and we use refined Gronwall estimates for the difference between the two solutions.
- To obtain good estimates, we exploit the good knowledge of the solution of the toy model.
- It can only spread mass to modes out of Λ for a short time.
- We added an additional condition to Λ which also slows down the spreading.

Conclusion:

- Choosing $\lambda = e^{\kappa s N}$ with $\kappa > 0$, we obtain that there is an orbit of the original system close to the rescaled orbit of the truncated system for $t \in [0, T]$ with

$$T \sim \lambda^2 N^2 \sim e^{2\kappa s N} N^2.$$

- This orbit undergoes the same growth of the Sobolev norm as the orbit of the truncated system.
- Recall that N has been taken such that

$$2^{sN} \sim \mathcal{K}^2$$

- Therefore, there exists $c > 0$ such that

$$T \sim \mathcal{K}^c.$$

- This finishes the proof.

Results for NLS with a convolution potential

Robustness of the I-team mechanism

- The results we have presented so far deal with a concrete equation.
- Is this instability mechanism still valid if we modify the equation?
- If one adds higher order terms, one obtains the same results since we are dealing with small solutions.
- What happens if one modifies the linear part of the equation?

The cubic defocusing NLS equation with a convolution potential

- Equation
$$\begin{cases} -i\partial_t u + \Delta u + V(x) * u = |u|^2 u \\ u(0,x) = u_0(x) \end{cases}$$
where
  - $x \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, $t \in \mathbb{R}$ and $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$.
  - $V \in H^{s_0}(\mathbb{T}^2)$, $s_0 > 0$, has real Fourier coefficients.
- It still has conservation of energy and mass.
- Simplified model of NLS with a multiplicative potential.

NLS with a convolution potential as an infinite ode

- Potential in Fourier series

$$V(x) = \sum_{n\in\mathbb{Z}^2} v_n e^{inx}.$$

- Equation

$$-i\dot{a}_n = \left(|n|^2 + v_n\right) a_n + \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2 \\ n_1-n_2+n_3=n}} a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n\in\mathbb{Z}^2.$$

- Eigenvalues of $a=0$ are

$$\lambda_n^{\pm} = \pm i\left(|n|^2 + v_n\right).$$

- The potential might kill the resonances.
- One would expect stronger stability properties of the critical elliptic point $a=0$.

Stability results

Theorem (Bambusi-Grebert 2003)

Consider $M \geq 2$ and take a typical (in certain measure sense) potential V. Then, there exists $s_0, \mu_0 > 0$ such that for any $s \geq s_0$ and any solution $u(t)$ of NLS with potential V with initial condition $u(0) = u_0 \in H^s$ satisfying $\mu := \|u_0\|_{H^s} < \mu_0$, one has that

$$\|u(t)\|_{H^s} \leq 2\mu \quad \text{for} \quad |t| \lesssim \frac{1}{\mu^M}.$$

Also results by Glauckner and Lubich (2010)

Instability results

Theorem (M. G.)

Fix $s_0 > 0$ *and* $s > 1$ *and take* $V \in H^{s_0}(\mathbb{T}^2)$ *with real Fourier coefficients. Then, there exists* $c > 0$ *with the following property: for any small* $\mu \ll 1$ *and large* $\mathcal{C} \gg 1$ *there exists a global solution* $u(t,x)$ *of NLS with convolution potential* V *and a time* T *satisfying that*

$$0 < T \leq \left(\frac{\mathcal{C}}{\mu}\right)^{c \ln(\mathcal{C}/\mu)}$$

such that

$$\|u(0)\|_{H^s} \leq \mu, \qquad \|u(T)\|_{H^s} \geq \mathcal{C}.$$

- Exactly the same result as for cubic NLS with no potential.

Some remarks

- We can deal with any potential V.
- The obtained orbit u only depends on $\|V\|_{H^{s_0}}$ and not on V itself.
- Stability versus instability: In a μ-neighborhood of $u = 0$ in H^s, $s \geq 1$:
  - For almost any potential, stability for times $t \lesssim \mu^{-M}$, $M \geq 2$.
  - For any potential, instabilities for times $t \sim \mu^{-c \ln |\mu|}$, $c > 0$.
- Faou-Grebert: stability in analytic norms for times $t \sim \mu^{-\widetilde{c} \ln |\mu|}$, $\widetilde{c} \in (0, 1)$.

As for cubic NLS without potential, if we do not impose initial small Sobolev norm but only large growth, we obtain a growth polynomial in time.

Theorem (M. G.)

Fix $s_0 > 0$ *and* $s > 1$ *and take* $V \in H^{s_0}(\mathbb{T}^2)$ *with real Fourier coefficients. Then, there exists* $c > 0$ *with the following property: for any large* $\mathcal{K} \gg 1$ *there exists a a global solution* $u(t, x)$ *of NLS with convolution potential* V *in* $\mathbb{T}^2$ *and a time* T *satisfying*

$$0 < T \leq \mathcal{K}^c,$$

such that

$$\|u(T)\|_{H^s} \geq \mathcal{K} \, \|u(0)\|_{H^s}.$$

Moreover, this solution can be chosen to satisfy

$$\|u(0)\|_{L^2} \leq \mathcal{K}^{-\alpha}, \ \alpha > 0.$$

Drift along almost resonance terms

- Equation

$$-i\dot{a}_n = \left(|n|^2 + v_n\right) a_n + \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2 \\ n_1-n_2+n_3=n}} a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n \in \mathbb{Z}^2.$$

- Eigenvalues of $a = 0$ are

$$\lambda_n^{\pm} = \pm i\left(|n|^2 + v_n\right).$$

- The potential might kill the resonances,
- However, $V \in H^{s_0}(\mathbb{T}^2)$ implies

$$v_n \sim \frac{1}{|n|^{s_0}}$$

Drift along almost resonance terms

- Cubic resonant terms for NLS with no potential

$$|n_1|^2 - |n_2|^3 + |n_3|^2 - |n|^2 = 0$$

- Cubic resonant terms for NLS with potential,

$$|n_1|^2 - |n_2|^3 + |n_3|^2 - |n|^2 + v_{n_1} - v_{n_2} + v_{n_3} - v_{n_4} = 0$$

- Resonant terms for NLS with no potential involving high enough modes are almost resonant for NLS with potential:

$$|n_1|^2 - |n_2|^3 + |n_3|^2 - |n|^2 \ll 1$$

- With some modifications, one can apply the results previously obtained for NLS.

Open questions

- NLS with a multiplicative potential

$$-i\partial_t u + \Delta u + V(x)u = |u|^2 u.$$

- NLS with other nonlinearities

$$-i\partial_t u + \Delta u = |u|^{2p} u, \ \ p \geq 2.$$

- Cubic defocusing NLS in general flat tori $\mathbb{R}^2/(\lambda_1\mathbb{Z}^2 \times \lambda_2\mathbb{Z}^2)$ instead of $\mathbb{R}^2/(2\pi\mathbb{Z})^2$.

Open questions

- Growth of Sobolev norms for other Hamiltonian PDEs such as
  - Nonlinear wave equation
  - Quantum harmonic oscillator
- For the cubic nonlinear wave equation, Bourgain conjectured that the unbounded growth should be faster (polynomial in time).
- The I-team construction is strongly 2-dimensional. How can one prove analogous results in the one dimensional case for NLW or NLS with a different nonlinearity?
- For instance, perturbed Zakharov-Shabat equation

$$-i\partial_t u + \Delta u = |u|^2 u + f(u, x, t), \;\; f = \mathcal{O}(u^4), \, x \in \mathbb{T}.$$

- More related to Arnol'd Diffusion.
- How can one prove growth of Sobolev norms for $x \in \mathbb{R}^n$?
- We essentially obtain one orbit. How typical is this behavior?
- Bourgain conjecture is still open.