Hypoelliptic random walks

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Outline

Let M be a compact connected m-dimensional manifold equipped with a volume form dx.
Let $X_1, X_2, ..., X_N$ be a finite collection of smooth vectors fields on M such that

$$\forall i \quad div(X_i) = 0$$

Let $\mathcal{G}$ be the lie algebra generated by the X_i. We assume

$$\text{For any } x \in M, \ \mathcal{G}_x = T_x M$$

i.e the vectors fields X_i satisfy the hypoelliptic condition of Hörmander

Hypoelliptic Random Walk

Let $h \in]0, h_0]$ be a small parameter. Let us consider the following random walk on M, $x_0, x_1, ..., x_n, ...$ starting at $x_0 \in M$:

At step n, choose $j \in \{1, ..., N\}$ at random and $t \in [-h, h]$ at random (uniform), and define $x_{n+1} = \Phi_j(t, x_n)$ where $\Phi_j(t, x)$ is the flow of X_j starting at x.
Due to the condition $div(X_j) = 0$, this random walk is reversible for the probability p on M

$$dp = \frac{dx}{Vol(M)}$$

The Markov kernel

For any j, let $T_{j,h}$ be the self adjoint operator on $L^2(M, dp)$

$$T_{j,h}f(x) = \frac{1}{2h}\int_{-h}^{h} f(\Phi_j(t,x))dt \tag{1.1}$$

Then $T_{j,h}f(x) = \int f(y)K_{j,h}(x,dy)$ where $K_{j,h}$ is a Markov Kernel, and

$$K_h(x,dy) = \frac{1}{N}\sum_{j=1}^{N} K_{j,h}(x,dy),\ T_h(f)(x) = \int_M f(y)K_h(x,dy) \tag{1.2}$$

are the Markov kernel and the Markov operator associated to our random walk, i.e

$$P(x_{n+1} \in A | x_n = x) = \int_A K_h(x,dy) \tag{1.3}$$

Let $K_h^n(x,dy)$ be the kernel of the iterate operator T_h^n. Then $\int_A K_h^n(x,dy)$ is the probability to be in the set A after n steps of the walk starting at $x \in M$. Our goal is

1. To get estimates on the rate of convergence of the probability $K_h^n(x,dy)$ towards the stationary probability p

$$\|K_h^n(x,dy) - p\|_{TV} \quad \text{as } n \to \infty \quad \forall x$$

where

$$\|p_1 - p_2\|_{TV} = sup_{A\in\mathcal{B}(\Omega)}|p_1(A) - p_2(A)|$$

is the total variation distance

2. To describe some aspects of the spectral theory of the operator T_h acting as a self adjoint contraction on $L^2(M,dp)$.

Outline

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Spectral gap

Since T_h is Markov and self adjoint, its spectrum is a subset of $[-1, 1]$.

We shall denote by $g(h)$ the spectral gap of the operator T_h. It is defined as the best constant such that the following inequality holds true for all $u \in L^2 = L^2(M, dp)$

$$\|u\|_{L^2}^2 - (u|1)_{L^2}^2 \leq \frac{1}{g(h)}(u - T_h u|u)_{L^2} \tag{2.1}$$

The existence of a non zero spectral gap means that :
1 is a simple eigenvalue of T_h, and the distance between 1 and $Spec(T_h \setminus \{1\})$ is equal to $g(h)$.

Theorem

There exists $h_0 > 0$, $\delta_0 \in]0, 1/2[$, $m > 0$, *and constants* $C_i > 0$ *such that for any* $h \in]0, h_0]$, *the following holds true.*
i) The spectrum of T_h *is a subset of* $[-1 + \delta_0, 1]$, 1 *is a simple eigenvalue of* T_h, *and* $Spec(T_h) \cap [1 - \delta_0, 1]$ *is discrete. Moreover, for any* $0 \leq \lambda \leq \delta_0 h^{-2}$, *the number of eigenvalues of* T_h *in* $[1 - h^2\lambda, 1]$ *(with multiplicity) is bounded by* $C_1(1 + \lambda)^m$.
ii) The spectral gap satisfies

$$C_2 h^2 \leq g(h) \leq C_3 h^2 \tag{2.2}$$

and the following estimate holds true for all integer n

$$sup_{x \in M} \| K_h^n(x, dy) - \frac{dy}{Vol(M)} \|_{TV} \leq C_4 e^{-ng(h)} \tag{2.3}$$

The limit diffusion operator

Let $\mathcal{H}^1((X_i))$ be the Hilbert space

$$\mathcal{H}^1((X_i)) = \{u \in L^2(M),\ \forall j,\ X_j u \in L^2(M)\}$$

Let ν be the best constant such that the following inequality holds true for all $u \in \mathcal{H}^1((X_i))$

$$\|u\|_{L^2}^2 - (u|1)_{L^2}^2 \leq \frac{\mathcal{E}(u)}{\nu}, \quad \mathcal{E}(u) = \frac{1}{6Vol(M)} \int_M \sum_j |X_j u|^2(x) dx \tag{2.4}$$

By the hypoelliptic theorem of Hörmander, one has $\mathcal{H}^1((X_i)) \subset H^\mu(M)$, for some $\mu > 0$. For any fixed smooth function $g \in C^\infty(M)$, one has

$$\lim_{h\to 0} \frac{1 - T_h}{h^2} g = L(g), \quad L = -\frac{1}{6} \sum_j X_j^2 \tag{2.5}$$

L is the positive Laplacian associated to the Dirichlet form $\mathcal{E}(u)$. It has a compact resolvant and spectrum $\nu_0 = 0 < \nu_1 = \nu < \nu_2 < \ldots$. Let m_j be the multiplicity of ν_j. One has $m_0 = 1$ since $Ker(L)$ is spaned by the constant function 1 .

The spectrum of T_h near 1

Theorem

One has

$$lim_{h\to 0} h^{-2} g(h) = \nu \tag{2.6}$$

Moreover, for any $R > 0$ and $\varepsilon > 0$, there exists $h_1 > 0$ such that one has for all $h \in]0, h_1]$

$$Spec(\frac{1 - T_h}{h^2}) \cap]0, R] \subset \cup_{j\geq 1}[\nu_j - \varepsilon, \nu_j + \varepsilon] \tag{2.7}$$

and the number of eigenvalues of $\frac{1-T_h}{h^2}$ with multiplicities, in the interval $[\nu_j - \varepsilon, \nu_j + \varepsilon]$, is equal to m_j.

Dirichlet forms

Let

$$\mathcal{E}_h(u) = ((\frac{1-T_h}{h^2}u|u))_{L^2}$$

Lemma

There exists $h_0 > 0$, $C > 0$, such that for all $h \in]0, h_0]$ and any $u_h \in L^2(M)$ such that

$$\|u_h\|_{L^2}^2 + \mathcal{E}_h(u_h) \leq 1$$

one has

$$\begin{aligned} & u_h = v_h + w_h \\ & \forall j,\ \|X_j v_h\|_{L^2} \leq C \\ & \|w_h\|_{L^2} \leq Ch \end{aligned} \tag{2.8}$$

As a direct byproduct, using also $\sum_j \|X_j v\|^2 \leq Cte \liminf_{h\to 0} \mathcal{E}_h(v)$ for $v \in \mathcal{H}^1((X_i))$, we get

$$C_2 h^2 \leq g(h) \leq C_3 h^2$$

Basic bounds

Lemma

1 *Spec*$(T_h) \cap [1-\delta_0, 1]$ *is discrete, and there exists* $m > 0$ *such that for any* $0 \leq \lambda \leq \delta_0 h^{-2}$, *the number of eigenvalues of* T_h *in* $[1 - h^2\lambda, 1]$ *(with multiplicity) is bounded by* $C_1(1+\lambda)^m$.
2 *There exists* $A > 0$ *such that any eigenfuntion* $T_h(u) = \mu u$ *with* $\mu \in [1-\delta_0, 1]$ *satisfies the bound*

$$\|u\|_{L^\infty} \leq C_2 h^{-A} \|u\|_{L^2} \tag{2.9}$$

The first item is an abstract consequence of the preceeding lemma and of the injection $\mathcal{H}^1((X_i)) \subset H^\mu(M)$.
For the second item , one uses with p large enough the equation

$$u(x) = \mu^{-p} T_h^p(u)(x)$$

Total variation

Let Π_0 be the orthogonal projector in L^2 on the space of constant functions

$$\Pi_0(u)(x) = \frac{1}{Vol(M)} \int_M u(y)dy \tag{2.10}$$

Then

$$2sup_{x\in M} \| T^n_{h,x} - dp\|_{TV} = \| T^n_h - \Pi_0\|_{L^\infty \to L^\infty} \tag{2.11}$$

Thus, we have to prove that there exist C_0, h_0, such that for any n and any $h \in]0, h_0]$, one has

$$\| T^n_h - \Pi_0\|_{L^\infty \to L^\infty} \leq C_0 e^{-ng(h)} \tag{2.12}$$

Total variation

Observe that since $g_h \simeq h^2$, we may assume $n \geq Ch^{-2}$. In order to prove 2.12, we split T_h in 3 pieces, according to the spectral theory.
Let $0 < \lambda_{1,h} \leq ... \leq \lambda_{j,h} \leq \lambda_{j+1,h} \leq ... \leq h^{-2}\delta_0$ be such that the eigenvalues of T_h in the interval $[1-\delta_0, 1[$ are the $1 - h^2\lambda_{j,h}$, with associated orthonormals eigenfunctions $e_{j,h}$

$$T_h(e_{j,h}) = (1 - h^2\lambda_{j,h})e_{j,h}, \quad (e_{j,h}|e_{k,h})_{L^2} = \delta_{j,k} \tag{2.13}$$

Then we write $T_h - \Pi_0 = T_{h,1} + T_{h,2} + T_{h,3}$ with

$$\begin{aligned} T_{h,1}(x,y) &= \sum_{\lambda_{1,h} \leq \lambda_{j,h} \leq h^{-\alpha}} (1 - h^2\lambda_{j,h})e_{j,h}(x)e_{j,h}(y) \\ T_{h,2}(x,y) &= \sum_{h^{-\alpha} < \lambda_{j,h} \leq h^{-2}\delta_0} (1 - h^2\lambda_{j,h})e_{j,h}(x)e_{j,h}(y) \\ T_{h,3} &= T_h - \Pi_0 - T_{h,1} - T_{h,2} \end{aligned} \tag{2.14}$$

$T_{h,1}$

One has $T_h^n - \Pi_0 = (T_h - \Pi_0)^n = T_{h,1}^n + T_{h,2}^n + T_{h,3}^n$, and the contribution of $T_{h,2}^n + T_{h,3}^n$ is easily seen to be negligible.

Let E_α be the (finite dimensional) subspace of L^2 span by the eigenvectors $e_{j,h}, \lambda_{j,h} \leq h^{-\alpha}$. One has $dim(E_\alpha) \leq Ch^{-m\alpha}$.

Lemma

There exist $\alpha > 0$, $p > 2$ and C independent of h such that for all $u \in E_\alpha$, the following inequality holds true

$$\|u\|_{L^p(M)}^2 \leq C(\mathcal{E}_h(u) + \|u\|_{L^2}^2) \tag{2.15}$$

Nash inequality

From the previous lemma, using the interpolation inequality $\|u\|_{L^2}^2 \le \|u\|_{L^p}^{\frac{p}{p-1}} \|u\|_{L^1}^{\frac{p-2}{p-1}}$, we deduce the Nash inequality, with $1/D = 2 - 4/p > 0$

$$\|u\|_{L^2}^{2+1/D} \le Ch^{-2}((\mathcal{E}_h(u) + h^2\|u\|_{L^2}^2)\|u\|_{L^1}^{1/D}, \quad \forall u \in E_\alpha \tag{2.16}$$

For $\lambda_{j,h} \le h^{-\alpha}$, one has $h^2\lambda_{j,h} \le 1$, and thus for any $u \in E_\alpha$, one gets $\mathcal{E}_h(u) \le \|u\|_{L^2}^2 - \|T_h u\|_{L^2}^2$, thus we get from 2.16

$$\|u\|_{L^2}^{2+1/D} \le Ch^{-2}((\|u\|_{L^2}^2 - \|T_h u\|_{L^2}^2 + h^2\|u\|_{L^2}^2)\|u\|_{L^1}^{1/D}, \quad \forall u \in E_\alpha \tag{2.17}$$

Nash inequality

There exists C_2 such that $\forall h, \quad \forall n \geq h^{-2+\alpha/2}$ one has $\| T_{1,h}^n \|_{L^\infty \to L^\infty} \leq C_2$ and thus since $T_{1,h}$ is self adjoint on L^2 $\| T_{1,h}^n \|_{L^1 \to L^1} \leq C_2$. Fix $p \simeq h^{-2+\alpha/2}$. Take $g \in L^2$ such that $\|g\|_{L^1} \leq 1$ and consider the sequence $c_n, n \geq 0$

$$c_n = \| T_{1,h}^{n+p} g \|_{L^2}^2 \tag{2.18}$$

Then, $0 \leq c_{n+1} \leq c_n$ and from 2.17, we get

$$\begin{aligned} c_n^{1+\frac{1}{2D}} &\leq Ch^{-2}(c_n - c_{n+1} + h^2 c_n) \| T_{1,h}^{n+p} g \|_{L^1}^{1/D} \\ &\leq C C_2^{1/D} h^{-2}(c_n - c_{n+1} + h^2 c_n) \end{aligned} \tag{2.19}$$

Thus there exist A which depends only on C, C_2, D, such that for all $0 \leq n \leq h^{-2}$, one has $c_n \leq (\frac{Ah^{-2}}{1+n})^{2D}$

Thus there exist C_0 , such that for $N \simeq h^{-2}$, one has $c_N \leq C_0$. This implies

$$\| T_{1,h}^{N+p} g \|_{L^2} \leq C_0 \|g\|_{L^1} \tag{2.20}$$

and thus taking adjoints

$$\| T_{1,h}^{N+p} g \|_{L^\infty} \leq C_0 \|g\|_{L^2} \tag{2.21}$$

and so we get for any n and with $N + p \simeq h^{-2}$

$$\| T_{1,h}^{N+p+n} g \|_{L^\infty} \leq C_0 (1 - h^2 \lambda_{1,h})^n \|g\|_{L^2} \tag{2.22}$$

And thus for $n \geq h^{-2}$

$$\| T_{1,h}^{n} \|_{L^\infty \to L^\infty} \leq C_0 e^{-(n-h^{-2}) h^2 \lambda_{1,h}} = C_0 e^{\lambda_{1,h}} e^{-ngap}, \quad \forall h, \quad \forall n \geq h^{-2} \tag{2.23}$$

Outline

Main Lemma

Lemma

Under the hypoelliptic hypothesis, the following holds true for $h \in]0, h_0]$ with h_0 small enough:
There exists C such that for any u_h with support in L^2 such that

$$\|u_h\|_{L^2}^2 + |(\frac{1 - T_h}{h^2} u_h | u_h)_{L^2}| \leq 1 \tag{3.1}$$

one has

$$\begin{aligned} & u_h = v_h + w_h \\ & \forall j, \ \|X_j v_h\|_{L^2} \leq C \\ & \|w_h\|_{L^2} \leq Ch \end{aligned} \tag{3.2}$$

Main Lemma : sketch of proof

It is easy to see that 3.1 implies for some $C_0 > 0$

$$\begin{aligned} &\|u_h\|_{L^2} \leq C_0 \\ &\forall j \quad u_h = v_{h,j} + w_{h,j} \\ &\|X_j v_{h,j}\|_{L^2} \leq C_0 \\ &\|w_{h,j}\|_{L^2} \leq C_0 h \end{aligned} \tag{3.3}$$

and we want to prove that there exists $C > 0$ such that

$$\begin{aligned} &u_h = v_h + w_h \\ &\forall j, \ \|X_j v_h\|_{L^2} \leq C \\ &\|w_h\|_{L^2} \leq Ch \end{aligned} \tag{3.4}$$

Main Lemma : sketch of proof

In order to prove the implication (3.3) $\to$ (3.4) we will construct operators depending on h, $0 \leq j \leq N$, $1 \leq k \leq N$

$$\Phi, \quad C_j, \quad B_{k,j}$$

$$\text{such that} \quad \Phi, \quad C_j, \quad B_{k,j}, \quad C_j hX_j \ (j \geq 1), \quad B_{k,j} hX_j \ (j \geq 1)$$

are uniformly in h bounded on L^2 and

$$\begin{aligned} 1 - \Phi &= \sum_{j=1}^{N} C_j hX_j + hC_0 \\ X_k \Phi &= \sum_{j=1}^{N} B_{k,j} X_j + B_{k,0} \end{aligned} \tag{3.5}$$

and then we set

$$v_h = \Phi(u_h), \quad w_h = (1 - \Phi)(u_h)$$

Main Lemma : sketch of proof

The construction of the operators $C_j, B_{k,j}$ is easy if the vectors fields $X_1, ..., X_N$ are the derivatives coordinates $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_N}$ on $M = \mathbb{R}^N$, just using classical h-pseudodifferential operators.

Under the hypoelliptic hypothesis, the Hörmander-Weyl calculus does not work as soon as one needs ≥ 3 brackets to span the tangent space (after discussion with Jean-Michel Bony).

So we use the Rothschild-Stein method (Acta-Math 137, 1977) to reduce the problem to a construction on a free (up to rank r) nilpotent Lie group.

Main Lemma : sketch of proof

Let $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus ... \oplus \mathcal{G}_r$ be the free (up to rank r) Lie algebra with generators $Y_1, ..., Y_N$. One has $span(Y_1, ..., Y_N) = \mathcal{G}_1$ and $\mathcal{G}_j$ is spanned by the commutators of order j, $[Y_{k_1}[Y_{k_2}, ...[Y_{k_{j-1}}, Y_{k_j}]]...]$.

The exponential map identifies $\mathcal{G}$ with the Lie group G, and the $Y_1, ..., Y_N$ with left invariant vectors fields on G by

$$Y_j f(x) = \frac{d}{ds}(f(x.exp(sY_j))|_{s=0}$$

The action of $\mathbb{R}_+$ on $\mathcal{G}$ is given by

$$t.(v_1, v_2, ..., v_r) = (tv_1, t^2 v_2, ..., t^r v_r)$$

and

$$Q = \sum j \dim(\mathcal{G}_j)$$

is the quasi homogeneous dimension of $\mathcal{G}$.

Main Lemma : sketch of proof

Let $f * u$ be the convolution on G

$$f * u(x) = \int_G f(xy^{-1})u(y)dy$$

Then $Y_j f = f * Y_j\delta$. We will use operators of the form, with $\varphi \in \mathcal{S}(G)$, the Schwartz space on G

$$\Phi(f) = f * \varphi_h, \quad \varphi_h(x) = h^{-Q}\varphi(h^{-1}x) \tag{3.6}$$

Then the equation

$$Y_k\Phi = \sum B_{k,j} Y_j$$

is equivalent to find $\varphi_{k,j} \in \mathcal{S}(G)$ such that

$$Y_k\varphi = \sum_j Y_j\delta * \varphi_{k,j} \tag{3.7}$$

Main Lemma : sketch of proof

Also, the equation $1 - \Phi = \sum_j C_j h Y_j$ reduces to solve

$$\delta_0 - \varphi = \sum_j Y_j \delta * c_j \tag{3.8}$$

with $c_j \in C^\infty(G \setminus 0)$, Schwartz for $|x| \geq 1$, and quasi homogeneous of degree $-Q + 1$ near 0.

Both 3.7 and 3.8 are consequence of the following cohomological lemma :

Let $Z_j(f) = Y_j \delta * f$, which is a right invariant vector field.

Lemma

Let $\varphi \in \mathcal{S}$ be such that $\int_G \varphi dx = 0$. Then there exists $\varphi_k \in \mathcal{S}$ such that

$$\varphi = \sum_k Z_k(\varphi_k) \tag{3.9}$$

This lemma is proved by induction on a family (Z_k) of r.i vectors fields such that the $(Z_k(0))$ spanned a graded Lie algebra.