Tunnel Effect for Semiclassical Random Walk

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General framework

Let

- (M,g) be a Riemanian manifold, $d_g x$ be the volume form and $d_g(x,y)$ the associated distance.
- $\rho(x)$ be a measurable, bounded, strictly positive function such that $d\pi(x)=\rho(x)d_g x$ is a probability measure on M.

Let us define the semiclassical random walk associated to ρ: assume that at step n the walk is in x_n, then choose x_{n+1} in the ball $B(x_n,h)$ uniformly with respect to $d\pi$ where $h>0$ is a small parameter and $B_h(x)$ is the geodesic ball centred in x and with radius h.

The kernel associated to this random walk is

$$t_h(x,dy)=\frac{\mathbf{1}_{\{d_g(x,y)\leq h\}}}{\pi(B_h(x))}d\pi(y),\ \forall x\in\Omega.$$

You can associate to this kernel an operator acting on continuous bounded functions and defined by

$$T_h f(x) = \frac{1}{\pi(B_h(x))} \int_M \mathbf{1}_{B_h(x)}(y) f(y) d\pi(y)$$

This is a makov operator ($T_h(1) = 1$). It transpose operator T_h^t acts on Borel measures.

Definition

Let ν_h be a probability measure on M. We say that ν_h is stationnary for $t_h(x, dy)$ if $T_h^t(\nu_h) = \nu_h$, where T_h^t denotes the transpose operator of T_h acting on Borel measure.

One can see easily that $t_h(x, dy)$ admits the following stationnary measure

$$d\nu_h = \frac{\pi(B_h(x))}{Z_h} d\pi(x)$$

where Z_h is a normalizing constant.

Convergence to stationnary measure

Given a Markov kernel $k(x, dy)$ on a metric space (X, d) and K the associated operator, we denote $k^n(x, dy)$ the kernel of the operator K^n.

Theorem (cf Feller)

Assume that $k(x, dy)$ is a *strictly positive and regular* Markov kernel and that π is a stationnary measure for k. Then,

$$\forall x \in X, \forall B \in \mathcal{B}, \lim_{n \to \infty} k^n(x, B) = \pi(B)$$

k *strictly positive* means that there exists $p \in \mathbb{N}$ such that $k^p(x, A) > 0$ for all open subset A. Think the regularity condition as, $k(x, dy)$ having a continuous density.

Question

What can we say about the convergence speed?

The answer is closely related to precise study of the spectral theory of T_h.

This problem was adressed by Diaconis-Lebeau (2009) in the case of a segment, Lebeau-Michel (2010) in the case of a compact manifold, and Diaconis-Lebeau-Michel (2011) in the case of a bounded domain of $\mathbb{R}^d$.

General properties of Random Walk operators

The following properties are easy to prove

- T_h is self-adjoint on $L^2(M, d\nu_h)$.
- For all $p \in [1, \infty]$, $\|T_h\|_{L^p \to L^p} = 1$.
- 1 is an eigenvalue of T_h (Markov property)
- In all the situations that we have studied, $1 \notin \sigma_{ess}(T_h)$.

Hence, the spectrum of T_h near 1 is made of eigenvalues. We denote

$$1 = \mu_0(h) \geq \mu_1(h) \geq \mu_2(h) \geq ... \geq \mu_k(h) ... > 0$$

the positive eigenvalues, $(e_k^h)_k$ the associated L^2-normalized eigenfunctions.

An explicit example: the flat torus

- Suppose that $M = (\mathbb{R}/2\pi\mathbb{Z})^d$ is the flat d-dimensional torus endowed with the Euclidean metric. Then

$$T_h = \Gamma_d(-h^2\Delta_g).$$

Indeed, using Fourier expansion , it suffices to show that $T_h f_k = \Gamma_d(-h^2\Delta_g) f_k$ with $f_k(x) = e^{i\langle k,x\rangle}$, $k \in \mathbb{Z}^d$. Using the flatness of the metric, it comes

$$T_h f_k(x) = \frac{1}{c_d h^d}\int_{B(x,h)} e^{i\langle k,y\rangle} dy = \frac{e^{i\langle k,x\rangle}}{c_d}\int_{B(0,1)} e^{i\langle hk,u\rangle} du$$
$$= \Gamma_d(h^2|k|^2)e^{i\langle k,x\rangle} = \Gamma_d(-h^2\Delta_g) f_k(x)$$

- Using the Taylor expansion of Γ_d in 0, this implies for all $k \in \mathbb{N}$:

$$\mu_k(h) = 1 - \frac{\lambda_k}{2(d+2)}h^2 + O(h^4)$$

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Semiclassical random walk on Euclidean space

Let $\phi \in C^\infty(\mathbb{R}^d)$ be a function such that $d\pi = c_0 e^{-\phi(x)} dx$ is a probability measure for some $c_0 > 0$. Consider the random-walk operator defined by

$$T_h f(x) = \frac{1}{\pi(B_h(x))} \int_{B_h(x)} f(x') d\pi(x').$$

and its stationnary measure

$$d\nu_h = \frac{\pi(B_h(x)) e^{-\phi(x)}}{Z_h} dx$$

where Z_h is chosen so that $d\nu_h$ is a probability on $\mathbb{R}^d$.

Definition

We say that ϕ is *smooth tempered of linear type (STL)* if ϕ is smooth and if there are some positive numbers $(C_\alpha)_{\alpha\in\mathbb{N}^d}$, $R>0, \kappa_0>0$, such that

$$\forall |x|\geq R,\ |\partial_x^\alpha \phi|\leq C_\alpha$$

and

$$\forall |x|\geq R,\ |\nabla\phi|^2-\Delta\phi\geq \kappa_0$$

In order to describe the eigenvalues of T_h, let us introduce the operator

$$-\Delta_\phi = -\Delta + |\nabla\phi|^2 - \Delta\phi$$

which is the Witten Laplacian (on functions) associated to ϕ.
Observe that :

- $-\Delta_\phi$ is non-negative on $L^2(\mathbb{R}^d)$ and 0 is a simple eigenvalue associated to $e^{-\phi} \in L^1 \cap L^\infty \subset L^2$.
- ϕ STL $\Longrightarrow \sigma_{ess}(-\Delta_\phi) = [\kappa, +\infty[$ with $\kappa = \liminf_{|x|\to\infty}(|\nabla\phi|^2 - \Delta\phi)$.

In the following, we will denote $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \ldots$ the $L^2(\mathbb{R}^d, dx)$ eigenvalues of $-\Delta_\phi$ and
$1 = \mu_0(h) > \mu_1(h) \geq \ldots \mu_k(h) \geq \ldots$ those of T_h.

Spectral analysis

Theorem (Guillarmou-Michel, Math. Res. Letters, 2011)

Suppose that ϕ is STL, then:

- the essential spectrum of T_h on $L^2(\mathbb{R}^d, d\nu_h)$ is contained in $[M, A_h]$ where $M > -1$ and $A_h = 1 - \frac{\kappa}{2(d+2)}h^2 + O(h^4)$.
- for all $\alpha \in]0, 1[$, if $\lambda_k \in [0, \alpha\kappa]$, then

$$\mu_k(h) = 1 - \frac{\lambda_k}{2(d+2)}h^2 + \mathcal{O}_{k,\alpha}(h^4).$$

Total variation estimates

The total variation distance between two probability measures μ, ν is defined by

$$\|\mu-\nu\|_{TV} := \sup_{A \text{ measurable}} |\mu(A)-\nu(A)| = \frac{1}{2} \sup_{f\in L^\infty, \|f\|\leq 1} |\int f d\mu - \int f d\nu|$$

In particular,

$$\sup_{x\in M} \|t_h^n(x,dy) - d\nu_h(y)\|_{TV} = \frac{1}{2}\|T_h^n - \Pi_0\|_{L^\infty \to L^\infty}$$

where Π_0 denotes the othogonal projection on constant functions in $L^2(M, d\nu_h)$.

Theorem (Guillarmou-Michel, Math. Res. Letters, 2011)

There exist $C>0$ and $h_0>0$ such that for all $n\in\mathbb{N}$, $h\in]0,h_0]$ and $\tau>0$,

$$\sup_{|x|<\tau}\|t_h^n(x,dy)-d\nu_h\|_{TV}\leq Cq(\tau,h)e^{-ng(h)}$$

where $q(\tau,h)=h^{-\frac{d}{2}}\sup_{|x|<\tau}e^{\phi(x)}$ if ρ is STE.

- Suppose that
$$\pi = \pi_h := c_h e^{-\phi(x)/h} dx$$
is a probability measure on $\mathbb{R}^d$ for some $c_h > 0$, and assume additionnaly that ϕ is a Morse function (i.e. it has a finite number of critical points which are non degenerate). Observe that the density $e^{-\phi(x)/h}$ concentrates at minima of ϕ.
- Consider the corresponding random walk operator:
$$T_h f(x) = \frac{1}{\pi_h(B_h(x))} \int_{B_h(x)} f(x') d\pi_h(x').$$
with stationnary distribution $d\nu_h = \frac{\pi_h(B_h(x))e^{-\phi(x)/h}}{Z_h} dx$.

We want to analyze the spectrum of this operator.

Heuristic analysis

If we replace ϕ by ϕ/h in the preceding analysis, the limiting operator becomes

$$-\Delta_{\phi,h} = -h^2\Delta + |\nabla\phi|^2 - h\Delta\phi$$

Let us denote $0 = \lambda_{0,h} < \lambda_{1,h} \leq \ldots \leq \lambda_{k,h} \leq \ldots$ the low lying eigenvalues of $-\Delta_{\phi,h}$. The preceding theorem should say that the eigenvalues of T_h close to 1 are $\mu_{k,h} = 1 - h^2\frac{\lambda_{k,h}}{2(d+2)} + \mathcal{O}(h^4)$

Remarks

The preceding heuristic can not be true for the following reasons:

- We can not apply the preceding theorem since the density depends on h
- The eigenvalues of $-\Delta_{\phi,h}$ are known to be exponentially small: $\lambda_{k,h} \simeq e^{-S/h}$ for some $S > 0$. The error therm h^4 is not an error term anymore.

Assumptions on ϕ

We make the following assumptions on ϕ:

- there exists $c, R > 0$ and some constants $C_\alpha > 0$, $\alpha \in \mathbb{N}^d$ such that:

$$\forall \alpha \in \mathbb{N}^d, \forall x \in \mathbb{R}^d\ |\partial_x^\alpha \phi(x)| \leq C_\alpha$$

  and

$$\forall x \in \mathbb{R}^d,\ |\nabla \phi(x)| \geq c \text{ and } |\phi(x)| \geq c|x|.$$

- ϕ is a Morse function
- Denoting $\mathcal{U}^{(k)}$ the set of critical points of ϕ of index k, the values $\phi(U_j^{(1)}) - \phi(U_k^{(0)})$, $U_j^{(1)} \in \mathcal{U}^{(1)}$, $U_k^{(0)} \in \mathcal{U}^{(0)}$ are distincts.

Our result

Theorem [Bony-Hérau-Michel]

Suppose that the previous assumptions are fullfilled. Then

- There exists $\kappa_0 > 0$ such that the essential spectrum of T_h is contained in $[-1+\kappa_0, 1-\kappa_0]$.
- There are m_0 eigenvalues of T_h in the interval $[1-h^{3/2}, 1]$ and these eigenvalues enjoy the following asymptotics

$$\mu_{k,h} = 1 - h\theta_k e^{-S_k/h}(1 + \mathcal{O}(h))$$

where the coefficient θ_k, S_k are defined below.

Recall on semiclassical Witten Laplacian (I)

- Under the above assumptions, the spectrum of semiclassical Witten has been analyzed by many authors: Cycon-Froese-Kirch-Simon 87, Helffer-Sjöstrand 85, Bovier-Gayrard-Klein 04, Helffer-Klein-Nier 04.
- It is well known that $-\Delta_{\phi,h}$ has $m_0 := \sharp\mathcal{U}^{(0)}$ eigenvalues $0 = \lambda_1 \leq \ldots \leq \lambda_{m_0}$, in the interval $[0, h^{3/2}]$.
- The most accurate result in [HKN04] gives an approximation of these eigenvalues:

$$\lambda_k = (2d+2) b_k e^{-S_k/h}(1 + \mathcal{O}(e^{-\epsilon/h}))$$

with $b_k(x; h) = \sum_{j\geq 0} h^j \beta_{k,j}(x)$, $\beta_{k,1} = \theta_k$.

Recall on semiclassical Witten Laplacian (II)

The quantities, S_k, θ_k can be comptuted: there exists a labelling $\mathcal{U}^{(0)} = \{U_1^{(0)}, \dots, U_{m_0}^{(0)}\}$ and $j : \{1, \dots, m_0\} \to \{1, \dots, m_1\}$ such that

$$S_k = 2(\phi(U_{j(k)}^{(1)}) - \phi(U_k^{(0)}))$$

and

$$\theta_k = \frac{|\hat{\lambda}_1(U_{j(k)}^{(1)})|}{\pi} \sqrt{\frac{\det(\mathrm{Hess}\,\phi(U_k^{(0)}))}{\det(\mathrm{Hess}\,\phi(U_{j(k)}^{(1)}))}}$$

where $\hat{\lambda}_1(U_{j(k)}^{(1)})$ is the negative eigenvalue of $\mathrm{Hess}\,\phi(U_{j(k)}^{(1)})$.
If $m_0 = 2$ then the above labelling and the function j are such that

$$S_2 = \min_{U^{(0)} \in \mathcal{U}^{(0)}, U^{(1)} \in \mathcal{U}^{(1)}} \phi(U^{(1)}) - \phi(U^{(0)}).$$

Recall on semiclassical Witten Laplacian (II)

The strategy of Helffer-Klein-Nier is the following:

- Introduce $F^{(0)}$ the eigenspace associated to low lying eigenvalues on 0-forms. Let M denote the matrix of the operator $\Delta_{\phi,h}$ on $F^{(0)}$, then $M = (\langle \Delta_{\phi,h} e_j, e_k \rangle)_{j,k}$
- In order to compute the eigenvalues of M we want to replace the unknown eigenfunctions e_k by suitable BKW approximations $\psi_k^{(0)}$.
- Doing that leads to error terms which are too big.
- In order to overcome this difficulty, use the super symmetric structure.

Recall on semiclassical Witten Laplacian (IV)

- The supersummetric structure of the Witten Laplacian is the following
$$\Delta_{\phi,h} = (d_{\phi,h} + d^*_{\phi,h})^2 = d^*_{\phi,h} d_{\phi,h}$$
where $d_{\phi,h} = e^{-\phi/h}(hd)e^{\phi/h}$ and $d : \Lambda^0\mathbb{R}^d \to \Lambda^1\mathbb{R}^d$ is the exterior derivative.
- They use also the following remark:
$$\Delta^{(p+1)}_{\phi,h} d^{(p)}_{\phi,h} = d^{(p+1)}_{\phi,h} \Delta^{(p)}_{\phi,h}$$
where $d^{(p)}_{\phi,h}$ denotes the twisted exterior derivative on p-form and $\Delta^{(p)}_{\phi,h}$ the corresponding Laplacian on p forms.
- Denote $F^{(1)}$ the eigenspace associated to low lying eigenvalues on 1 forms, then
$$M = L^* L$$
where L is the matrix of $d^{(0)}_{\phi,h} : F^{(0)} \to F^{(1)}$.

Recall on semiclassical Witten Laplacian (V)

- The matrix L is well approximated by

$$L \simeq \left(\langle d_{\phi,h}^{(0)} \psi_j^{(0)}, \psi_k^{(1)} \rangle\right)_{j=1,\dots,m_0,k=1,\dots,m_1}$$

where $\psi_k^{(1)}$ denotes BKW approximations of eigenfunctions on 1-form.
- Conclude by computing the singular values of L.

First Reduction (I)

The operator T_h is self-adjoint on $L^2(\mathbb{R}^d, d\nu_h)$.

- Using a unitary transformation, we are reduce to analyze on operator $\tilde{T}_h$ on $L^2(\mathbb{R}^d)$ which is given by

$$\tilde{T}_h f(x) = a_h(x) \frac{1}{\alpha_d h^d} \int_{|x-y|<h} a_h(y) f(y) dy$$

  where $a_h(x)^{-2} = \frac{1}{\alpha_d h^d} \int_{|x-y|<h} e^{(\phi(x)-\phi(y))/h} dy$.
- Observe that the operator $f \mapsto \frac{1}{\alpha_d h^d} \int_{|x-y|<h} f(y) dy$ is a fourier multiplier $G(hD_x)$ with

$$G(\xi) = \frac{1}{\alpha_d} \int_{|x|<1} e^{ix\cdot\xi} dx$$

  Here we use the notation $D_x = \frac{1}{i}\nabla_x$.

First Reduction (II)

- From the preceding observations we deduce:

$$\tilde{T}_h = a_h G(hD_x) a_h \quad \text{and} \quad a_h^{-2} = e^{\phi/h} G(hD_x)(e^{-\phi/h})$$

- Since we study the spectrum of $\tilde{T}_h$ near 1, we introduce

$$P_h := 1 - \tilde{T}_h = a_h(b_h(x) - G(hD_x))a_h$$

where $b_h(x) = a_h^{-2}(x) = e^{\phi/h} G(hD_x)(e^{-\phi/h})$.

Rough analysis of P_h (I)

P_h is a h-pseudodifferential operator: $P_h = p(x, hD_x)$. Its symbol has an expansion $p = \sum_{j\geq 0} h^j p_j$ and the principal and subprincipal symbols are given by

$$p_0(x,\xi) = 1 - \frac{G(\xi)}{G(i\nabla_x\phi)}$$

and

$$p_1(x,\xi) = -\frac{G(\xi)\theta_1(x)}{G(i\nabla_x\phi)^2} + \frac{1}{2G(i\nabla_x\phi)}\langle Hess(\phi)\nabla G(\xi), \nabla G(i\nabla\phi(x))\rangle_{\mathbb{R}^d}$$

with $\theta_1(x) = \frac{1}{2\alpha_d}\sum_{i,j}\frac{\partial^2\phi}{\partial x_i\partial x_j}\int_{|z|<1} e^{-z\cdot\nabla_x\phi(x)} z_i z_j dz$.

Rough analysis of P_h (II)

Near any critical point x_c and near $\xi = 0$, these symbols enjoy the expansion:

$$p_0(x,\xi) = 1 - \frac{G(\xi)}{G(i\nabla_x\phi)} = \frac{1}{2(d+2)}(|\xi|^2 + |\nabla_x\phi|^2) + \mathcal{O}(|x-x_c|^3 + |\xi|^3)$$

and

$$p_1(x,\xi) = -\frac{1}{2(d+2)}\Delta\phi(x) + \mathcal{O}(|x-x_c| + |\xi|^2)$$

Supersymmetry for random walk

Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$. Let us denote $D_\phi = hD_x - i\nabla\phi(x) : \mathcal{S} \to \mathcal{S}^d$, where $D_x = -i\nabla_x$. For $k \in \mathbb{N}^*$, denote S_k^0 the space of bounded symbol from $T^*\mathbb{R}^d$ into $\mathcal{M}_k(\mathbb{R})$. The key point in our approach is the following Lemma.

Lemma

There exists some symbols $q_\phi, b_\phi \in S_d^0$, and $r \in S_1^0$ such that:

- $P_h = L_\phi^* L_\phi$ with $L_\phi = Q_\phi D_\phi a_h$ and $Q_\phi = Op_h(q_\phi)$.
- $L_\phi L_\phi^* = Q_\phi a_h \Delta_{\phi,h}^{(1)} a_h Q_\phi^* + hOp_h(b_\phi) + h^2 Op_h(r)$
- For all $U \in \mathcal{U}$,

$$q_\phi(U,0) = Id \text{ and } b_\phi(U,0) = 0.$$

Proof. Observe that $P_h = a_h P_h^\sharp a_h$ with

$$P_h^\sharp = G(hD) - e^{\phi/h} G(hD)(e^{-\phi/h})$$

Hence it suffices to find Q_ϕ such that $P_h^\sharp = D_\phi^* Q_\phi^* Q_\phi D_\phi$

- **Step 1: Show that there exists $\widehat{Q}_\phi$ s.t. $P_h^\sharp = D_\phi^* \widehat{Q}_\phi D_\phi$** Let $P_{\phi,h}^\sharp = e^{\phi/h} P_h^\sharp e^{-\phi/h}$. Since $P_h^\sharp$ has a symbol which is analytic wrt ξ, $P_{\phi,h}^\sharp$ is a pseudo. Moreover, $P_{\phi,h}^\sharp(1) = 0$. Hence, we can factorize

$$P_{\phi,h}^\sharp = \tilde{Q}_\phi h D_x.$$

Going back to $P_h^\sharp$, we get $P_h^\sharp = \overline{Q}_\phi D_\phi$.

It remains to factorize $\overline{Q}_\phi$ by D_ϕ^* on the left.

- This is equivalent to show that

$$\check{Q}_\phi := e^{-\phi/h}\overline{Q}_\phi e^{\phi/h} = e^{-2\phi/h}\tilde{Q}_\phi e^{2\phi/h}$$

  can be factorized by *div* on the left.
- We introduce the symbol $\check{q}_\phi$ of the left-quantization of $\check{Q}_\phi$. Since $G(\xi)$ is the fourier transform of an odd function, exact computations shows that $\check{q}_\phi(y,0) = 0$ for all y.
- Going back to the original operator by conjugation by $e^{\phi/h}$, we get the first step.

Remark

All this works for general pseudodifferential operators $P_h = Op_h^w(p)$ such that $P_h(e^{-\phi/h}) = 0$, $\xi \mapsto p(x,\xi)$ is analytic and $z \mapsto \mathcal{F}_{\xi\to z}(p)(x,z)$ is an odd function.

- **Second Step: Show that you can choose $\widehat{Q}_\phi$ non negative and construct its squareroot**
  To simplify, assume we work on $\mathbb{R}^2$. Then

$$P^\sharp = \begin{pmatrix} D^*_{1,\phi} \\ D^*_{2,\phi} \end{pmatrix} \cdot \begin{pmatrix} \widehat{Q}_{11} & \widehat{Q}_{12} \\ \widehat{Q}^*_{12} & \widehat{Q}_{22} \end{pmatrix} \begin{pmatrix} D_{1,\phi} \\ D_{2,\phi} \end{pmatrix}$$

  The key point is that $[D_{1,\phi}, D_{2,\phi}] = 0$ so that for any bounded operators A, B, we can rewrite $P^\sharp$ as

$$P^\sharp = \begin{pmatrix} D^*_{1,\phi} \\ D^*_{2,\phi} \end{pmatrix} \cdot \begin{pmatrix} \widehat{Q}_{11} + BD_{2,\phi} + D^*_{2,\phi}B^* & \widehat{Q}_{12} - BD_{1,\phi} \\ \widehat{Q}^*_{12} - D^*_{1,\phi}B^* & \widehat{Q}_{22} \end{pmatrix} \begin{pmatrix} D_{1,\phi} \\ D_{2,\phi} \end{pmatrix}$$

  or

$$P^\sharp = \begin{pmatrix} D^*_{1,\phi} \\ D^*_{2,\phi} \end{pmatrix} \cdot \begin{pmatrix} \widehat{Q}_{11} + D^*_{2,\phi}AD_{2,\phi} & \widehat{Q}_{12} \\ \widehat{Q}^*_{12} & \widehat{Q}_{22} - D^*_{1,\phi}AD_{1,\phi} \end{pmatrix} \begin{pmatrix} D_{1,\phi} \\ D_{2,\phi} \end{pmatrix}$$

To simplify, asume that ϕ has only one critical point in $x = 0$.
Denote $p^\sharp(x,\xi) = p^\sharp(x_1, x_2, \xi_1, \xi_2)$ the symbol of $P^\sharp$.
Given $\delta > 0$, we have to deal with 3 microlocal regions:
$\Omega_0 = \{|\xi|^2 + |x|^2 \leq 2\delta\}$, $\Omega_1 = \{|\xi_1|^2 + |x_1|^2 \geq \delta\}$,
$\Omega_2 = \{|\xi_2|^2 + |x_2|^2 \geq \delta\}$.

- On Ω_0, since

$$p^\sharp(x,\xi) = |\xi|^2 + |\nabla\phi(x)|^2 + \mathcal{O}(|(x,\xi)|^3),$$

  it is easy to prove that $\widehat{Q}_{ij} = \delta_{ij} + \mathcal{O}(h+\epsilon)$.
- Ω_1 and Ω_2 are treated in a similar way, using the preceding remark. Let us study Ω_1.
  The idea is to chose A and B in order to kill the antidiagonal terms and get a positive lower bound for diagonal terms.

- Killing $\widehat{Q}_{12}$ is done by choosing $B = \widehat{Q}_{12}/D_{1,\phi}$. This is possible since on Ω_1, $D_{1,\phi}^* D_{1,\phi} \geq \epsilon > 0$.
- Assume now that $\widehat{Q}_{12} \simeq 0$. We want to insure that $\widehat{Q}_{11}$ and $\widehat{Q}_{22}$ are positive. The fondamental point is that there exists $\alpha > 0$ such that

$$\forall (x,\xi) \in \Omega_0,\ p^\sharp(x,\xi) \geq 2\alpha.$$

On the other hand,

$$p^\sharp(x,\xi) = (|\xi_1|^2 + |\partial_1\phi|^2)\widehat{q}_{11}(x,\xi) + (|\xi_2|^2 + |\partial_2\phi|^2)\widehat{q}_{22}(x,\xi)$$

As a consequence

$$\widehat{q}_{11}(x,\xi) + (|\xi_2|^2 + |\partial_2\phi|^2)\frac{\widehat{q}_{22}(x,\xi) - \frac{\alpha}{(1+|\xi_2|^2+|\partial_2\phi|^2)}}{(|\xi_1|^2 + |\partial_1\phi|^2)} \geq \frac{\alpha}{|\xi_1|^2 + |\partial_1\phi|^2}$$

and we can take $A = Op_h\left(\frac{\widehat{q}_{22}(x,\xi) - \frac{\alpha}{1+(|\xi_2|^2+|\partial_2\phi|^2)}}{(|\xi_1|^2+|\partial_1\phi|^2)}\right)$.

- Doing that we get a new factorisation on Ω_1:

$$P^\sharp = \begin{pmatrix} D_{1,\phi}^* \\ D_{2,\phi}^* \end{pmatrix} \cdot \begin{pmatrix} Op_h(q_{11}) & O(h) \\ O(h) & Op_h(\frac{\alpha}{1+(|\xi_2|^2+|\partial_2\phi|^2)}) \end{pmatrix} \begin{pmatrix} D_{1,\phi} \\ D_{2,\phi} \end{pmatrix}$$

with $q_{11} \geq \frac{\alpha}{|\xi_1|^2+|\partial_1\phi|^2}$ on Ω_1.

- Gluing all microlocal region we get a final prefactorisation:

$$P^\sharp = \begin{pmatrix} D_{1,\phi}^* \\ D_{2,\phi}^* \end{pmatrix} \cdot \begin{pmatrix} Op_h(q_{11}) & O(h) \\ O(h) & Op_h(q_{22}) \end{pmatrix} \begin{pmatrix} D_{1,\phi} \\ D_{2,\phi} \end{pmatrix}$$

with $q_{11}, q_{22} \geq \frac{\alpha}{|\xi|^2+|\nabla\phi|^2}$

- Finally, operators such that $\begin{pmatrix} Op_h(q_{11}) & O(h) \\ O(h) & Op_h(q_{22}) \end{pmatrix}$ can be written as square of pseudo by standard arguments.

Outline of proof (I)

The main step fo the proof of the theorem are

- Prove that there are exactly m_0 eigenvalues of T_h in $[1-h^{3/2},1]$ and that these eigenvalues belong actually to $[1-e^{-\alpha/h},1]$. For this purpose, use the quasimode

$$\psi_k^{(0)}(x) = a_h^{-1}(x)\chi(x-U_k^{(0)})e^{-(\phi(x)-\phi(U_k^{(0)})/h}$$

where χ is a cut-off function near 0.

- Prove Agmon-type estimates for our operator, e.g.

$$\forall\epsilon>0,\ \|e^{(\phi(x)-\phi(U_k^{(0)})/h}e_k(x)\|_{L^2} = \mathcal{O}(e^{\epsilon/h})$$

This is possible since our symbol is analytic w.r.t. to ξ.

- Using Agmon estimates, prove that the true eigenfunction $e_1,\dots,e_{m_0}$ of P_h satisfy $e_k \simeq \psi_k^{(0)}$.

Outline of proof (II)

- Construct good quasimode $\psi_k^{(1)}$ for the operator $\tilde{P}_h = L_\phi L_\phi^*$ acting on 1-form. Introduce the corresponding eigenfunctions $f_1, \dots, f_{m_1}$ and prove that $f_k \simeq \psi_k^{(1)}$ near critical point of index 1.
- Introduce the matrix $\mathcal{L}_\phi$ of L_ϕ acting from F^0 onto F^1, where $F^0 = span(e_1, \dots, e_{m_0})$ and $F^1 = span(f_1, \dots, f_{m_1})$.
- Compute accurate approximation of the coefficient of $\mathcal{L}_\phi$, by use of the preceding quasimodes, and Graam-Schmidt procedure.
- Use the preceding results to compute the singular values of $\mathcal{L}_\phi$ and conclude.