Hamiltonian partial differential equations and Painlevé transcendents

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• Cauchy problem for evolutionary PDEs with slow varying initial data

• Introduction into Hamiltonian PDEs

• On perturbative approach to integrability

• Phase transitions from regular to oscillatory behavior. Universality conjecture and Painlevé transcendents
System

\[ u_t = F(u; u_x, u_{xx}, \ldots) \quad (1) \]

R.h.s. analytic in jets at \((u; 0, 0, \ldots)\)

and \(F(u; 0, 0, \ldots) \equiv 0\)

\[ n \text{-dimensional family of constant solutions } (n = \text{dim } u) \]

Slow-varying solutions \(u(x) = f(\epsilon x), \quad u_x = O(\epsilon), \quad u_{xx} = O(\epsilon^2)\)

Rescale \(x \mapsto \epsilon x, \quad t \mapsto \epsilon t\) and expand

\[ u_t = \frac{1}{\epsilon} F(u; \epsilon u_x, \epsilon^2 u_{xx}, \ldots) \]

\[ = A(u)u_x + \epsilon \left[ B_1(u)u_{xx} + B_2(u)u_x^2 \right] + \epsilon^2 \left[ C_1(u)u_{xxx} + C_2(u)u_{xx}u_x + C_3(u)u_x^3 \right] + \ldots \]

\(\epsilon\) -independent initial data \(u(x, t) = u_0(x)\)
Question 1: existence of solutions to (1) with slow varying initial data

Idea: for small times solutions to (1)

\[ u_t = A(u)u_x + \epsilon [B_1(u)u_{xx} + B_2(u)u_x^2] + \epsilon^2 [C_1(u)u_{xxx} + C_2(u)u_{xx}u_x + C_3(u)u_x^3] \ldots (1) \]

and

\[ u_t = A(u)u_x \] \hspace{1cm} (2)

with the same initial date are close

before the time of gradient catastrophe of (2)
Question 2: comparison of solutions to (1) ("perturbed") and (2) ("unperturbed") near the point of gradient catastrophe of (2) = point of phase transition of (1)

Two main classes: dissipative perturbations, e.g., Burgers equation

\[ u_t + uu_x = \epsilon u_{xx} \]

shock waves;

or Hamiltonian (i.e., conservative) perturbations, e.g., Korteweg - de Vries equation

\[ u_t + uu_x + \epsilon^2 u_{xxx} = 0 \]

oscillatory behaviour
Examples of Hamiltonian PDEs

1) KdV

\[ u_t + u u_x + \frac{\epsilon^2}{12} u_{xxx} = 0 \]

\( \Leftrightarrow \) \[ u_t + \partial_x \frac{\delta H}{\delta u(x)} = 0 \]

In the zero dispersion limit \( \epsilon = 0 \) \( \Rightarrow \) Hopf equation

\[ u_t + u u_x = 0 \]

2) Toda lattice

\[ \begin{align*}
\epsilon u_t &= v(x) - v(x - \epsilon) \\
\epsilon v_t &= e^{u(x+\epsilon)} - e^u(x)
\end{align*} \]

\( n = 2 \)

Long wave limit

\[ \begin{align*}
  u_t &= v_x \\
  v_t &= e^u u_x
\end{align*} \]
More general class of systems of the Fermi-Pasta-Ulam type

\[ H = \sum_{n=1}^{N} \frac{p_{n}^2}{2} + V(q_{n} - q_{n-1}) \]

For large \( N \) the equations of motion can be replaced by

\[
\begin{align*}
    u_t &= \frac{1}{\epsilon} [v(x) - v(x - \epsilon)] \\
    v_t &= \frac{1}{\epsilon} [V'(u(x + \epsilon)) - V'(u(x))] \\
\end{align*}
\]

\[ p_n = v(n\epsilon), \quad q_n - q_{n-1} = u(n\epsilon), \quad \epsilon = \frac{1}{N} \]

In the leading term one obtains an integrable PDE

\[
\begin{align*}
    u_t &= v_x \\
    v_t &= V''(u) u_x \\
\end{align*}
\]
3) Nonlinear Schrödinger equation

\[ i\epsilon \psi_t + \frac{\epsilon^2}{2} \psi_{xx} + |\psi|^2 \psi = 0 \]

In the real-valued variables

\[ u = |\psi|^2, \quad v = \frac{\epsilon}{2i} \left( \frac{\psi_x}{\psi} - \frac{\bar{\psi}_x}{\bar{\psi}} \right) \]

can be recast into the form

\[ u_t + (u v)_x = 0 \]

\[ v_t + v v_x - u_x + \frac{\epsilon^2}{4} \left( \frac{1}{2} \frac{u_x^2}{u^2} - \frac{u_{xx}}{u} \right) \bigg|_x = 0 \]
The Hamiltonian formulation

\[ u_t + \frac{\partial}{\partial x} \frac{\delta H}{\delta v(x)} = 0 \]

\[ v_t + \frac{\partial}{\partial x} \frac{\delta H}{\delta u(x)} = 0 \]

\[ H = \int \left[ \frac{1}{2} (u v^2 - u^2) + \frac{\epsilon^2}{8u} u_x^2 \right] \, dx \]

Hamiltonian operator

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\frac{\partial}{\partial x}
\]
General setting

Class of systems of PDEs depending on a small parameter $\epsilon$

$$u_t = A(u)u_x + \epsilon A_2(u; u_x, u_{xx}) + \epsilon^2 A_3(u; u_x, u_{xx}, u_{xxx}) + \ldots$$

$$u = (u^1, \ldots, u^n)$$

Terms of order $\epsilon^k$ are differential polynomials

of degree $k + 1$

$$\deg u^{(m)} = m, \quad m = 1, 2, \ldots$$
$\epsilon$-dependent dynamical systems

on the space of vector-functions $u(x)$

$u(x, t; \epsilon)$ \hspace{1cm} solution of the Cauchy problem

initial point $u_0(x)$
Hamiltonian formulation

\[ u_t = F(u; u_x, u_{xx}, \ldots) = P \frac{\delta H}{\delta u(x)} \]

Hamiltonian local functional

\[ H = \frac{1}{2\pi} \int_0^{2\pi} h(u; u_x, u_{xx}, \ldots) \, dx \]

(the case of 2\(\pi\)-periodic function)

\[ \frac{\delta H}{\delta u(x)} = \text{Euler-Lagrange operator of } h \]

\[ \frac{\delta H}{\delta u^i(x)} = \frac{\partial h}{\partial u^i} - \partial_x \frac{\partial h}{\partial u^i_x} + \partial_x^2 \frac{\partial h}{\partial u^i_{xx}} - \ldots \]
Finally, in the representation
\[
    u_t = F (u; u_x, u_{xx}, \ldots) = P \frac{\delta H}{\delta u(x)}
\]

\( P = (P^{i j}) \) is the matrix-valued operator of Poisson bracket
\[
\{ F, G \} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta F}{\delta u^i(x)} P^{i j} \frac{\delta G}{\delta u^j(x)} \, dx
\]

• bilinearity

• skew symmetry \( \{ G, F \} = - \{ F, G \} \)

• Jacobi identity
\[
\{\{ F, G \} , H \} + \{\{ H, F \} , G \} + \{\{ G, H \} , F \} = 0
\]
For example, any linear skew-symmetric matrix operator with **constant** coefficients

\[ P^{ij} = \sum_k P^{ij}_k \partial_x^k \]

\[ P^{ji}_k = (-1)^{k+1} P^{ij}_k \]

defines a Poisson bracket

For KdV  \( P = \partial_x \), for NLS \( P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x \)

More general class

\[ P^{ij} = \sum_k P^{ij}_k (u; u_x, \ldots) \partial_x^k \]
After rescaling \( x \mapsto \epsilon x \) obtain

\[
P^{ij} = \sum_{k \geq 0} \epsilon^k \sum_{s=0}^{k+1} P_{k,s}^{ij}(u; u_x, \ldots, u^{(s)}) \partial_x^{k-s+1}
\]

\[
\deg P_{k,s}^{ij}(u; u_x, \ldots, u^{(s)}) = s
\]
This class is invariant with respect of the group of Miura-type transformations of the form

\[ u \mapsto \tilde{u} = F_0(u) + \epsilon F_1(u, u_x) + \epsilon^2 F_2(u, u_x, u_{xx}) + \ldots \]

\[ \deg F_k(u, u_x, \ldots, u^{(k)}) = k \]

\[ \det \left( \frac{DF_0(u)}{Du} \right) \neq 0 \]
Thm. Assuming \( \det \left( P_{0,0}^{ij}(u) \right) \neq 0 \), any Hamiltonian operator is equivalent to \( P = \text{constant symmetric matrix} \cdot \partial_x \).

Proof uses the theory of Poisson brackets of hydrodynamic type (B.D., S. Novikov, 1983) and triviality of Poisson cohomology (E. Getzler; F. Magri et al., 2001).

So, any Hamiltonian PDEs can be written in the form

\[
\begin{aligned}
  u_t^i &= \eta^{ij} \frac{\partial}{\partial x} \frac{\delta H}{\delta u^j(x)}, \quad i = 1, \ldots, n, \quad \det(\eta^{ij}) \neq 0, \quad \eta^{ji} = \eta^{ij}
  
  H &= H_0 + \epsilon H_1 + \epsilon^2 H_2 + \cdots = \int [h_0(u) + \epsilon h_1(u; u_x) + \epsilon^2 h_2(u; u_x, u_{xx}) + \cdots] dx
\end{aligned}
\]
Perturbative approach to integrability of PDEs
(B.D., Youjin Zhang)

Integrable Hamiltonian system \( \dot{u} = \{u, H_0\} \)

Complete family of commuting first integrals

\[
\{H_0, F_0^{[i]}\} = 0, \quad i = 1, 2, \ldots
\]

\[
\{F_0^{[i]}, F_0^{[j]}\} = 0
\]

Definition. Perturbed Hamiltonian \( H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \ldots \) is integrable if every \( F_0^{[i]} \) can be deformed

\[
F^{[i]} = F_0^{[i]} + \epsilon F_1^{[i]} + \epsilon^2 F_2^{[i]} + \ldots, \quad i = 1, 2, \ldots
\]

to a first integral of \( H \), \( \{H, F^{[i]}\} = 0 \), and \( \{F^{[i]}, F^{[j]}\} = 0 \)
Homological equation \{H_0, F_1\} = \{F_0, H_1\} \text{ etc.}

Remark. Regularity of \(H\) \(\Rightarrow\) commutativity of the centralizer

Example. Hopf equation

\[
  u_t = u u_x = \partial_x \frac{\delta H_0}{\delta u(x)}, \quad H_0 = \int \frac{1}{6} u^3(x) \, dx
\]

is integrable: commuting first integrals are

\[
  F_0 = \int f(u(x)) \, dx
\]

for an arbitrary function \(f(u)\)
Proof.

\[
\{ F_0, H_0 \} = \int \frac{\delta F_0}{\delta u(x)} \partial_x \frac{\delta H_0}{\delta u(x)} \, dx = \int f'(u) u \, u_x \, dx = \int dF(u) = 0
\]

where

\[
\frac{dF}{du} = f'(u) u
\]

An example of integrable deformation: KdV

\[
H = H_0 + \epsilon^2 H_2 = \int \left[ \frac{1}{6} u^3 - \frac{\epsilon^2}{24} u_x^2 \right] \, dx
\]
Claim. For any function $f(u)$ there exists a first integral

$$H_f = \int \left[ f(u) - \frac{\epsilon^2}{24} f'''(u) u_x^2 + \epsilon^4 \left( \frac{1}{480} f^{(4)} u_{xx}^2 - \frac{1}{3456} f^{(6)} u_x^4 \right) + \ldots \right] dx$$

of KdV equation $\{H, H_f\} = 0 \quad \forall \ f$
An explicit formula in terms of Lax operator

\[ L = \frac{\epsilon^2}{2} \frac{d^2}{dx^2} + u(x) \]

Then

\[ H_f = \int h_f \, dx \]

where

\[ h_f = \text{res} \, f^{(1/2)}(L) \]
More general second order perturbations

\[ H = H_0 + \epsilon^2 H_2 = \int \left[ \frac{1}{6} u^3 + \frac{\epsilon^2}{12} c(u) u_x^2 \right] \, dx \]

Thm. (B.D., J.Ekstrand) Integrability iff

\[ c(u) = (u - u_0)^{-3} \]

Lax operator

\[ \epsilon \Psi' = U(\lambda) \Psi, \quad U(\lambda) = \begin{pmatrix} 0 & (u-u_0)^2 \\ \frac{1}{2} u_0 (u-u_0)^2 & 0 + \lambda \end{pmatrix} \]

It gives only half of the first integrals \( H_f \), for odd \( f \)
Back to general case

The main goal: to compare the properties of solutions to the perturbed system

\[ u_t = A(u)u_x + \epsilon A_2(u; u_x, u_{xx}) + \epsilon^2 A_3(u; u_x, u_{xx}, u_{xxx}) + \ldots \]

with solutions to the “dispersionless limit” \( \epsilon \to 0 \)

\[ u_t = A(u)u_x \]
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- Hamiltonian

\[ u_t = A(u)u_x \]
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- Hamiltonian
- completely integrable
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with solutions to the “dispersionless limit \( \epsilon \to 0 \)

\[ u_t = A(u)u_x \]

- Hamiltonian
- completely integrable
- finite life span (nonlinearity!)
For the dispersionless system \( u_t = A(u)u_x \)

a gradient catastrophe takes place: the solution exists for \( t < t_0 \), there exists the limit \( \lim_{t \to t_0} u(x, t) \)

but, for some \( x_0 \)

\( u_x(x, t), u_t(x, t) \to \infty \) for \( (x, t) \to (x_0, t_0) \)

The problem: to describe the asymptotic behaviour of the generic solution \( u(x, t; \epsilon), u(x, 0; \epsilon) = u_0(x) \)

to the perturbed system for \( \epsilon \to 0 \)

in a neighborhood of the point of catastrophe \( (x_0, t_0) \)
Gradient catastrophe for Hopf equation

\[ u_t + u u_x = 0 \]
Perturbation: Burgers equation \[ u_t + u u_x = \epsilon u_{xx} \]
(dissipative case)
Perturbation: KdV equation \[ u_t + u u_x + \epsilon^2 u_{xxx} = 0 \]

(Hamiltonian case)
The smaller is $\varepsilon$, the faster are the oscillations.
Nonlinear Schrödinger equation (the focusing case)

\[ i\epsilon \psi_t + \epsilon^2 \psi_{xx} + |\psi|^2\psi = 0 \]

NB: the dispersionless limit is a PDE of elliptic type

\[
\begin{pmatrix}
  u_t \\
  v_t
\end{pmatrix} + \begin{pmatrix}
  v & u \\
  -1 & v
\end{pmatrix} \begin{pmatrix}
  u_x \\
  v_x
\end{pmatrix} = 0 , \text{ eigenvalues } \lambda = v \pm i\sqrt{u} \]
Main Conjecture (B.D., 2005): a finite list of types of the critical behaviour
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(Universality)
Universality for the generalized Burgers equation
(A.II’ in 1985)

\[ u_t + a(u)u_x = \epsilon u_{xx} \]

\[ u(x, t; \epsilon) = u_0 + \gamma \epsilon^{1/4} \Gamma \left( \frac{x - x_0 - a_0(t - t_0)}{\alpha \epsilon^{3/4}}, \frac{t - t_0}{\beta \epsilon^{1/2}} \right) + \mathcal{O} \left( \epsilon^{1/2} \right) \]

Here \( \Gamma(\xi, \tau) \) is the logarithmic derivative of the Pearcey function

\[ \Gamma(\xi, \tau) = -2 \frac{\partial}{\partial \xi} \log \int_{-\infty}^{\infty} e^{-\frac{1}{8} \left( z^4 - 2z^2 \tau + 4z\xi \right)} dz \]
What kind of special functions is needed for the Hamiltonian case?

Painlevé-1 equation \( U'' = 6U^2 - X \) \((P_1)\)

and its 4th order analogue

\[
X = TU - \left[ \frac{1}{6}U^3 + \frac{1}{24}(U'^2 + 2UU'') + \frac{1}{240}U^{IV} \right] \quad (P_{I^2})
\]

Painlevé property: any solution is a meromorphic function in \( X \in \mathbb{C} \)
Isomonodromy representation for $P_{I}^{2}$

$$
\frac{d\Psi}{d\lambda} = W\Psi, \quad \frac{d\Psi}{dX} = U\Psi, \quad \frac{d\Psi}{dT} = V\Psi
$$

$$
W = \begin{pmatrix}
12UU' + 8\lambda U' + U''' & 2(16\lambda^2 + 8\lambda U + 6U^2 + U'' - 60T') \\
2w_{21} & -12UU' - 8\lambda U' - U'''
\end{pmatrix}
$$

$$
w_{21} = 32\lambda^3 - 16\lambda^2 U - 2\lambda(2U^2 + U'' + 60T) + 8U^3 + 2U''U - U'^2 + 120X
$$

$$
U = \begin{pmatrix}
0 & -1 \\
2U - 2\lambda & 0
\end{pmatrix} \quad V = \frac{1}{6} \begin{pmatrix}
U' & 2U + 4\lambda \\
8\lambda^2 - 4\lambda U - 4U^2 - U'' & -U'
\end{pmatrix}
$$