Short notice on works

Laurent STOLOVITCH, 2020

My inital works study **local analytic foliations in a neighborhood of a singular point in the eucliean complex space**. In particular, we are interested in the the behavior of the flow of a vector field X in a neighborhood of a singular point (i.e. a fixed point of the dynamic) in $\mathbb{C}^n$, $n \geq 2$. We study those, for instance, the associated system of differential equations of which, can be written as

$$\frac{dx_i}{dt} = \lambda_i x_i + f_i(x) \quad i = 1, \dots, n$$

where λ_i (the eigenvalues of the linear part) are complex numbers, not all zero and where the f_i's are germs of "non-linear" holomorphic functions ($f_i(0) = 0$ et $Df_i(0) = 0$).

In order to understand the geometry of the foliation, one tries to transform, by a change of coordinates preserving the singularity, the differential system into another one, supposed to be "simpler"(there is a formal definition), called **normal form**. In general, such a transformation exists only at a formal power series level: there exists a formal change of coordinates $\hat{\Phi}$ which conjuguates X to $\hat{\Phi}_* X = S + N$ where S denotes the linear part of X at the origin and where N denotes a formal nonlinear vector field (understood as a "formal nilpotent"), commuting with S. This has to be thought as an infinite dimensional version "Jordan normal form decomposition of matrices".

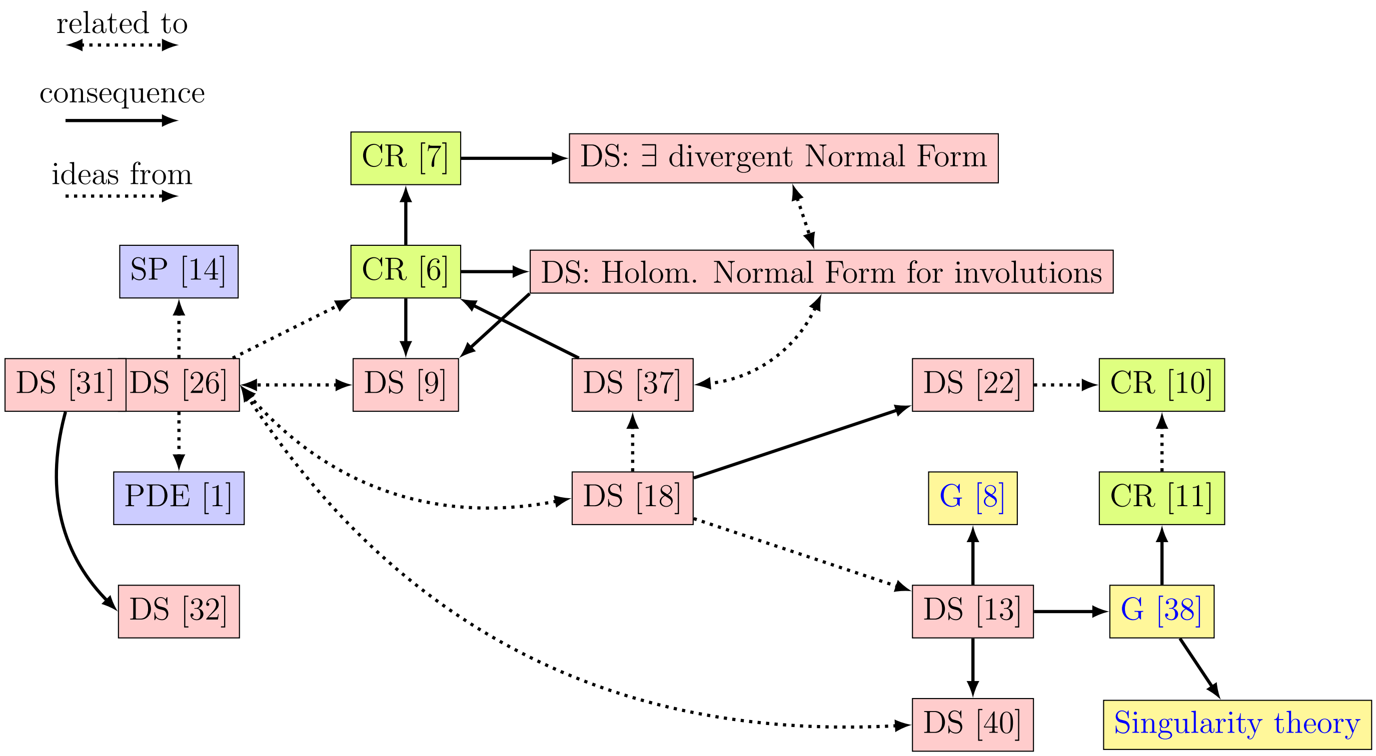
In order to obtain dynamical or geometrical information on the initial dynamical system from the normal form, one need the transformation to be regular. The more regular it is, the more faithful the information transferred from the normal form are. We mainly concerned with analytic (real or complex) regularity, the finest one. So the problem lead to know under which circumstances there exists an analytic normalizing diffeomorphism (i.e transformation from the original to a normal form). This problem is related to *resonances* (i.e formal obstruction to linearization) and to the so-called *small divisors problems* (which measure how close the system is to resonances).

Our works shed a new light on these problems on both analytic and geometrical aspects.

For some years, I've been trying to transfer this point of view, these methods and technics to understanding general rigidity problems in Geometry as for instance, Cauchy-Riemann geometry, Poisson structures and also in some PDE's problems.

In particular, I've been studying Cauchy-Riemann structures and their singularities. The introduction of "new technologies" in these areas led to deep new results.

Just to mention two examples : Together with Xianghong Gong (U. Wisconsin-Madison), we have opened up a new era in the studies of higher dimensional singularities of Cauchy-Riemann structures. For more than 40 years following the seminal article by Moser and Webster, works only considered 2 dimensional problems. Indeed, the geometry of these submanifolds is induced from the properties of an associated holomorphic dynamical systems, the dynamic of which is automatically "rather simple" in dimension 2 with straightforward consequences. On the other hand, together with I. Kossovskiy (Masaryk U., Brno) and B. Lamel (U. Wien, Vienna), we have shown that two real analytic hypersurfaces in $\mathbb{C}^2$ (these are CR maniifolds) which are formally CR-equivalent are also smoothly equivalent (i.e there exists a smooth CR-map conjugating both submanifolds). The proof relies on many subtilities of the *multisummability theory* of formal solutions of holomorphic differential systems at an irregular singularity as devised by Ramis, Malgrange, Braaksma. !...



The graph above gives an illustration of some synergies between our different works (DS=Dynamical Systems; CR= Cauchy-Riemann geometry; G=Geometry; SP= Spectral Theory). Numbers in brackets refer to the "bibliography section" below.

1 Noveltiess 2020-2015

- [8] **Equivalence of neighborhoods of embedded compact complex manifolds and higher codimension foliations (submitted, 71p. 2020).** Together with X. Gong (U. Wisconsin-Madison), we go back to the problem left by H. Grauert in the 60's and named by him *Formale Prinzip*. The problem is the following : Let's consider 2 neighborhoods of a complex compact manifolds C embedded into 2 complex manifolds. Assume that these neighborhoods are formally equivalent. Are they automatically holomorphically equivalent ? This is indeed the case when the normal bundle of C into a manifold is *negative* (Grauert/Hironaka-Rossi). As shown by V.I. Arnold the the 70's, the situation is quite different when considering a torus embedded into a surface with a zero self-intersection number. In our work we give a **sufficient condition ensuring that a germ of neighborhood of C of a manifold which is formally equivalent to a neighborhood of the zeroth section of its normal bundle is actually holomorphically equivalent to it.** This allows for instance, to extends holomorphically a section or a cohomology class only defined on C by doing so just in a neighborhood of a zeroth section of the normal bundle. Furthermore, we give conditions that ensure that **there exist a holomorphic foliation in a neighborhood of C having C as a leaf.** This extends Ueda's result obtain in the cas of a complex curve embedded into a surface as **we do not impose any restriction neither on the dimension nor on the codimension of the C.**

These problems can both be formulated as a kind of "linearization problem" for which we need to define appropriate *resonances* and *small divisors*. This article used technics developed in our article [13] "Normal forms of analytic perturbations of quasihomogeneous vector fields..."

- **[9] Complete integrability of diffeomorphisms and their local normal forms (J. Dynamics and Differential Equations 25p., 2020)** In this joint work with K. Jiang (Postdoc Nice/Pekin U.), we study germs of smooth (or holomorphic) diffeomorphisms at a fixed point and more precisely finite family of pariwise commuting of them. We define the notion of *complete integrability* in the same spirit as Nguyen Tien Zung (U. Toulouse) did for vector fields. We show that under the assumption of *weakly hyperbolicity*, it is possible to smoothly transform such a family to a normal form. In the analytic case, we show implies the one we devise with X. Gong for the study of analytic CR singularities[6]

- **[1] Convergence to normal form of integrable PDEs (Comm. Math. Phys. 30p., 2020).** Together with D. Bambusi (Università degli studi di Milano), we consider the problem of transformation to a normal form of an infinite abelian family of (germs of) analytic vector fields at a common fixed point in some (infinite dimensional) Hilbert space. We give a sufficient condition that ensures that such a family can be analytically transformed into a normal form. We apply our result to the normal form problem of completely integrable PDE's such as KdV, NLS (NonLinear Schrödinger) or Toda. **This allows to obtain** *long term behavior* **of solutions that are not otherwise accessible throught classical PDEs analysis.** Our approach is radically different than those of Kappeler-Pöschel and of Kuksin-Pereleman as it is not based on symplectic geometry but rather on a "Newton scheme" applied to the family of "Hierarchy" associated to the initial PDE. This work can be seen, to some extend, as an infinite dimensional version of our work on "Singular Complete integrability"[26].

- **[10] Equivalence of Cauchy-Riemann manifolds and multisummability theory (submitted, 33p. 2020).** Together with I. Kossovskiy(U. Masaryk, Brno) and B. Lamel (U. Wien, Vienna), we prove that 2 real analytic hypersurfaces of $\mathbb{C}^2$ that are CR-formally equivalent are also CR-smoothly equivalent. It has recently been found that one may not expect CR-analytic equivalence (Kossovskiy-Shafikov). **Our proof is really unexpected.** Indeed, it rests on *multisummability theory* of formal solution of holomorphic system of ODEs at an irregular singular point (Ramis, Malgrange,Braaksama....) ! **To the best of our knowledge, it is the first time that such method of complex analysis is used in order to solve a problem of geometry.**

- **[11] Convergence of the Chern-Moser-Beloshapka normal forms (J. Reine Angew. Math. 43p., 2019).**

  In this article in collaboration with B. Lamel (U. Wien, Vienna), we define the notion a normal form for *Levi-nondegenerate* real analytic submanifolds of $\mathbb{C}^n$ of **codimension** $d \geq 1$ under the action of the group of germs of formal biholomorphisms. We give a sufficient condition ensuring that there exists a holomorphic transformation to such a normal form. This gives, in the case of hypersurfaces ($d = 1$), a new proof of the celebrated Chern-Moser theorem. **It took almost 50 years to overcome the issue of codimension > 1. To overcome this difficulty, we introduced ideas coming from Dynamial Systems.** Our proof uses in an essential way our former work "Big denominators.... "[11](see below).

- **[6] Real submanifolds of maximum complex tangent space at a CR singular point I, (Invent. Math., 85p., 2016).** In this article in collaboration with X. Gong (U. Wisconsin-Madison), we study real analytic submanifolds of $\mathbb{C}^n$ that have a singularity of their Cauchy-Riemann (CR) structure. Submanifolds that are totally real everywhere but at a point where it has a complex tangent space are peculiar examples. The study of these objects started with the seminal work of Bishop ('60s) as they carry very interesting properties for analysts : all (germs of) holomorphic functions defined on such manifold (at point) are likely to be holomorphically extended to a common (strictly) larger set called the *local hull of holomorphy.* It also became clear that it is interesting to classify these through the action of biholomorphisms that preserve the singular point (say 0). It can be shown that these submanifolds are higher order (≥ 3) perturbations $M : Q_2 + M_3 + M_4 + \cdots$ of some quadric Q_2, where M_i is homogeneous of degree i in local coordinates. In dimension $n = 2$, pioneering works by Bishop and Moser-Webster considered the case where the complex dimension at the singularity is *minimal*.

In the present work, we are interested the *maximal* dimension case. We aim at understanding in which case the submanifold is biholomorphic to a the quadric it is a pertubation of. If not, what are the obstructions to do so and what are sufficient conditions ensuring that it is holomorphically conjugated to a *normal form* ? What can be read directly off these normal forms ? These geometric problems are in fact very related to a Dynamical System ones. Indeed, to such a submanifold, one can associate a pair of (germs of) holomorphic involutions at a fixed point, which in turn, serve to built up such a submanifold (in a non-trivial way). The goal is then to *linearize* or to *normalize* such a pair of involutions through a suitable biholomorphism in order to conjugate the submanifold either to the quadric or to a normal form. In the first case, this a consequence of our former Dynamical System work "Family of intersecting totally real manifolds..."[37] and is related to *small divisors condition* built up upon the quadric. Our "higher degeneracy concern" led us to **discoveries of situations that does not occur in dimension** 2 (Moser-Webster case). In particular, there exists a 4-dimensional quadric which is not equivalent to product of Bishop quadrics. Furthermore, we have defined the notion of *abelian CR singularity* which are shown to be biholomorphically equivalent to a normal form near the origin (CR singularity). To do so, we had to devise and **prove a new theorem on holomorphic normalization of abelian families of biholomorphisms at a common fixed point**. For those submanifolds which are not of abelian type, we can still give some insights of the geometry near the singularity : under some assumptions, there exists a complex manifold that intersects the submanifold along totally real transversal analytic submanifolds. This is also a consequence (absolutely not trivial) of the existence of invariant analytic sets of germs of biholomorphisms. **This is the very first work in almost 40 years that overcome the issue** of higher dimension of the degeneracy (or the dimension $n > 2$). One of the reason, is that, not only the geometric realization is conceptually more difficult but also because of the complexity of the underlying dynamics. That is certainly a starting point for further results.

- **[7] Real submanifolds of maximum complex tangent space at a CR singular point II, (J. Diff. Geometry, 78p., 2019)** In this joint work with X. Gong, we continue our study of CR singularities. We give a holomorphic classification "à la Bishop" of quadrics having a CR singularity of maximal dimension at the origin. We give also the formal invariants of their **non-degenerate** higher order perturbations and show that they**have a unique formal normal form**. We also show the existence of an analytic submanifold the normal forms of which are all divergent power sries at the the origin. This situation cannot appear in dimension 2 (Moser-Webster). **Doing so, we solve a century old**

Dynamical Systems problem : the existence of germs of biholomorphism all the Poincaré-Dulac normal forms of which are divergent at the origin. Although, the existence of *divergent transformation* to a normal form is well known and related to *small divisors*, the problem of *existence of divergent normal forms* was open since Poincaré !

- **[38] Big demoninators and analytic normal form, with an appendix of M. Zhitomirskii (J. Reine Angew. Math. 44p; 2016)** This article is our first attempt to transfer concepts and methods from (the local study of) Dynamical Systems to a genral geometric framework.

  In this article, we study the general problem of transformation to *normal form* of an "analytic object" acted on by a group of some (analytic) transformations. Considering the **conjugacy equation** between two such objects, we define the appropriate notion of normal form. Considering its *linearization*, we build up a sequence of positive numbers that plays the rôle of *small divisors* in Dynamical systems at a fixed point as well as in Celestian Mecanics. If instead to accumulating the origin, this sequence of numbers tends to infinity with a "sufficient high speed", then we prove the existence of an analytic conjugacy to a normal form. We say that the problem has the *Big denominators property* (BD). Applying this theorem to the classical case of conjugating (germs of) analytic vector fields at fixed point, the DB property amount to saying that the linear part of the vector field at the origin is in the *Poincaré domain*. We apply this result to the **normal from problem of** *holomorphic functions with non-isolated singularity*. **This is the very first systematic result of the kind**. In the case of isolated singularity, we give a new kind of proof of the conjugacy to a polynomial, this was known from Arnold, Tougeron (late 60's-early 70's).... This work has been used in a crucial manner for the generalization fo Chern-Moser theorem for higher codimensional problems [11].

- **[40] Holomorphic normal form of nilpotent vector fields (Regular and Chaotic Dynamics, 27p., 2016)** In this joint work with F. Verstringe (Obs. Royale de Belgique, Belgium), we study germs of holomorphic vector fields at a fixed point in $\mathbb{C}^n$, $n \geq 2$, the linear part of which is nilpotent. After having defined, the appropriate notion of normal form (this is not trivial; this notion was devised by Takens('70) in dimension 2), we give a sufficient condition on the formal normal form ensuring that the vector field is actually holomorphically conjugate to a normal form. The only previous work in that direction was due to Zoladek-Strozyna ('00), considering only dimensional $n = 2$ case. They proved that, in this situation, no condition is required in order to have an holomorphic conjugacy to a Takens normal form. This is in sharp contrast to higher dimensional problem we considered as we know that conditions on the normal form are required to obtain convergence of the formal normalizing transformation. **This is the very first result in any dimension for analytic nilpotent singularties**. This can be seen as the **nilpotent counterpart of**

Brjuno's theorem for vector fields with semi-simple linear part. Our work in not only base on $\mathfrak{sl}_2(\mathbb{C})$ representation theory but also on our joint work with E. Lombardi [13] and those of Verstinge.

- **[2, 3] Existence of Quasipattern Solutions of the Swift-Hohenberg Equation (Arch. Rational Mech. Anal., 30p.,2013+ erratum 2014) ; Proof of quasipatterns for the Swift-Hohenberg equation (Comm. Math. Phys., 32p., 2017)** In this joint work with B. Braaksma(U. Groningen) and G. Iooss(U. Nice), we show the existence of quasiperiodic stationary solutions in all direction of the plane and invariant by a fixed angle rotation of the Swift-Hohenberg Equation $(1 + \Delta)^2 u - \mu u + u^3 = 0$. To the best of our knowledge, **it is mathematical proof of the exitence such a phenomena that has been observed both numerically and in physical world**. The main difficulties are due to *small divisors problems* in infinite dimension. The solution of the problem passes through the use of delicate recent versions of *Nash-Moser theorem* due to Bourgain, Berti, Bolle; Procesi.....

- **[14] Quantum singular complete integrability (J. Funct. Analysis, 57p., 2016)** In this joint work with T. Paul (Polytechnique), we study family of pariwise commuting quantum sytems that are small perturbations of quantum Hamitonians on a torus. The goal is to obtain **result on the spectrum of these perturbations**. Under some assumptions (that we call *complete integrability*) , we prove that these spectra can be expressed as analytic functions of the unperturbed spectra. This is shown through the use of quantum Birkhoff normal form theory that we developed on purpose. **The point of view and the technics we use are completely new as no classical theorem (of spectral theory) can be applied**. Indeed, the spectra fo the unperturbed system are dense.

- **[5] Analytic reducibility of resonant cocycles to a normal form (J Inst. Math. Jussieu 21p., 2016)** In this joint work with C. Chavaudret (Paris 7), we study the *reducibility* of analytic quasi-periodic cocycles in any dimension. This means the existence (or not) of an analytic transformation conjugating a linear system of ODEs with quasiperiodic coefficents to a constant linear system. One of the main feature of this article is that it does not rule out *resonant frequencies* as all other works do. In general, one cannot expect to conjugate to a linear constant system but rather to linear system the coefficient of which have Fourier expansion depending only on the resonant frequencies.

- **[37] Family of intersecting totally real manifolds of** $(\mathbb{C}^n, 0)$ **and germs of holomorphic diffeomorphisms (Bull. S.M.F.,17p., 2015)** In this article, we study some analytic geometry problems through Dynamical Systems methods. The goal is the classify, up to conjugacy by germs of biholomorphisms of $(\mathbb{C}^n, 0)$, families of germs of n-dimensional totally real real analytic submanifolds intersecting at the origin of $\mathbb{C}^n$. To do so, we study the holomorphic linearization as well as the existence of invariant analytic set of an abelian family of germs of biholomorphisms at a common fixed point, having a semi-simple linear part at that point. This works generalizes Pöschel result and is related to our own work [18] on vector fields. It also plays an important rôle in our study of CR singularities [6, 7] (see above).

2 Laurent Stolovitch's bibliography

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