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INVITED ARTICLE

Progress in normal form theory

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Abstract
In this paper we review recent results about normal forms of systems of analytic differential equations near a fixed point.

1. Introduction

Let us start with a very elementary example. In order to study the iterates of a square complex matrix A of $\mathbb{C}^n$, that is the orbits $\{A^k x\}_{k\in\mathbb{N}}$ for $x \in \mathbb{C}^n$ near the fixed point 0, it is very convenient to transform, with the help of a linear change of coordinates P, the matrix A into a Jordan matrix $J = S + N$, with S a diagonal matrix, N an upper triangular nilpotent matrix commuting with S: $PAP^{-1} = S + N$. Using the (block diagonal) structure of $S + N$, it is easy to study its iterates. Since $A^k = P^{-1}J^kP$, we have $A^n x = P^{-1}(J^n y)$ where $x = P^{-1}y$. We thus obtain all information needed for the study of the iterates of A.

If we consider a vector field (or a system of differential equations) near a point where it does not vanish, then the 'flow box' theorem shows that the vector field can be smoothly straightened to a constant vector field. One of the great ideas of Poincaré was to try to proceed in the same way for vector fields near a fixed point. Is it possible to transform a given vector field X, vanishing at the origin of $\mathbb{R}^n$ (respectively $\mathbb{C}^n$), into a 'simpler' one with the help of a local diffeomorphism Φ near the origin which maps the origin to itself? The group of germs of C^k (respectively holomorphic, formal) diffeomorphisms at $0 \in \mathbb{C}^n$ and tangent to $Id_{\mathbb{C}^n}$ at the origin, acts on the space of germs of holomorphic (respectively formal) vector fields at $0 \in \mathbb{C}^n$ by conjugacy: if X is any representative of a germ of vector field $\boldsymbol{X}$, and ϕ is any representative of a germ of diffeomorphism $\boldsymbol{\Phi}$, then $\boldsymbol{\Phi}_*\boldsymbol{X}$ is the germ of vector field defined by

$$\phi_*X(\phi(x)) := D\phi(x)X(x),$$

where $D\phi(x)$ denotes the derivative of ϕ at the point x. One may first attempt to linearize formally X, that is to find a formal change of coordinates $\hat{\Phi}$, such that $\hat{\Phi}_*X(y) = DX(0)y$. Assuming it is so, then one could expect to understand all about the dynamics of X since the flow of the linear vector field $DX(0)y$ is easy to study. Nevertheless, this cannot be the case unless we are able to pull-back the information by $\hat{\Phi}$, and this requires some 'regularity' conditions on $\hat{\Phi}$. Is there a C^k (respectively smooth) linearizing diffeomorphism? When we are working in the analytic category, this regularity condition should be that $\hat{\Phi}$ is holomorphic in a neighbourhood of the origin. What happens in this situation?

These ideas have been widely developed by Arnol'd and his school. Our main reference for this topic is the great book by Arnol'd [Arn88a]. We refer also to [AA88] which contains a lot of references on this topic.

In the second section we will review recent results about holomorphic vector fields having a nonzero linear part at the fixed point. In the last section we will present a brand new normal form theory relative to holomorphic vector fields having a 'quasihomogeneous principal part' at the fixed point.

1.1. Vector fields and differential equations

Let us consider a germ of vector field X at a point p: in a coordinate chart at p, it can be written $X(z) = \sum_{i=1}^n X_i(z)(\partial/\partial z_i)$. It is equivalent to considering the system of autonomous differential equations:

$$\begin{aligned}\dot{z}_1 &= X_1(z)\\ &\vdots\\ \dot{z}_n &= X_n(z).\end{aligned}$$

The *Lie derivative* of a germ of function f along the vector field X is the germ of function

$$\mathcal{L}_X f(z) := \sum_{i=1}^n X_i(z)\frac{\partial f}{\partial z_i}(z).$$

It will also be denoted by $X(f)$. We will denote by $[X, Y]$ the *Lie bracket* of the vector fields $X = \sum_{i=1}^n X_i(\partial/\partial z_i)$ and $Y = \sum_{i=1}^n Y_i(\partial/\partial z_i)$. It is defined to be

$$[X, Y] = \sum_{i=1}^n \left(\sum_{j=1}^n X_j\frac{\partial Y_i}{\partial z_j} - Y_j\frac{\partial X_i}{\partial z_j}\right)\frac{\partial}{\partial z_i}.$$

It is skew-symmetric and satisfies the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Moreover, if X, Y are vector fields and f a function

$$[X, fY] = f[X, Y] + \mathcal{L}_X(f)Y. \tag{1}$$

Two vector fields X, Y are said to be *commuting pairwise* whenever $[X, Y] \equiv 0$.

2. Normal form of systems of differential equations with a nonzero linear part

In this section, we will assume that the linear part of X at the origin is semi-simple:

$$S := DX(0)x = \sum_{i=1}^n \lambda_i x_i\frac{\partial}{\partial x_i}$$

is a nonzero diagonal vector field. If $Q = (q_1, \ldots, q_n) \in \mathbb{N}^n$, we will write $(Q, \lambda) := \sum_{i=1}^n q_i\lambda_i$, $|Q| := q_1 + \cdots + q_n$ and $x^Q := x_1^{q_1}\cdots x_n^{q_n}$.

Proposition 2.1 (Poincaré–Dulac normal form). *Let* $X = S + R_2$ *be a nonlinear perturbation of the linear vector field* S *(*R_2 *is of order* $\geqslant 2$ *at the origin). Then there exists a formal change of coordinates* $\hat{\Phi}$ *tangent to the identity such that*

$$\hat{\Phi}_* X = S + \hat{N},$$

where the nonlinear formal vector field $\hat{N}$ *commutes with* S*:* $[S, \hat{N}] = 0$.

By a formal change of coordinates $\hat{\Phi}$ tangent to the identity, we mean that there exists formal power series $\hat{\phi}_i(x) = \sum_{Q\in\mathbb{N}^n,|Q|\geqslant 2} \phi_{i,Q} x^Q \in \mathbb{C}[[x_1, \ldots x_n]]$ of order $\geqslant 2$, such that $\hat{\Phi}_i(x) = x_i + \hat{\phi}_i(x)$, the ith-component of $\hat{\Phi}$.

Let us describe a normal form in local coordinates. First of all, we note that

$$\left[S, x^Q \frac{\partial}{\partial x_i}\right] = ((Q, \lambda) - \lambda_i)\, x^Q \frac{\partial}{\partial x_i}.$$

Therefore, such an elementary vector field commutes with S if and only if

$$(Q, \lambda) = \lambda_i.$$

This is called a *resonance relation* and $x^Q(\partial/\partial x_i)$ the associated *resonant vector field*.

Therefore, the formal normal form proposition can be rephrased as: there exists a formal diffeomorphism $\hat{\Phi}$ (which is not unique in general) such that

$$\hat{\Phi}_* X = \sum_{i=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \left(\sum_{(Q,\lambda)=\lambda_i} a_{i,Q} x^Q \right) \frac{\partial}{\partial x_i},$$

where the sum is over the multiintegers $Q \in \mathbb{N}^n$, $|Q| \geqslant 2$ and the index i which satisfies to $(Q, \lambda) = \lambda_i$ and where the $a_{i,Q}$'s are complex numbers.

Example 2.2. *Let* ζ *be a positive irrational number. Let us consider the vector field* X

$$\dot{x} = x + f(x, y),$$
$$\dot{y} = -\zeta y + g(x, y),$$

where f, g *are smooth functions vanishing at the origin as well as their first derivatives. It is formally linearizable since the only integer solution* (q_1, q_2) *of* $q_1 - \zeta q_2 = 0$ *is* $(0, 0)$*. Hence, there are no resonance relations satisfied.*

Example 2.3. *Let us consider the analytic vector field* X

$$\begin{aligned} \dot{x} &= x + f(x, y), \\ \dot{y} &= -y + g(x, y), \end{aligned} \tag{2}$$

for some holomorphic functions f, g *vanishing at first order at the origin. It is clear that the only solutions of the resonance relation* $q_1\lambda_1 + q_2\lambda_2 = \lambda_1$ *(respectively* $q_1\lambda_1 + q_2\lambda_2 = \lambda_2$*) are of the form* $q_1 = q_2 + 1$ *(respectively* $q_2 = q_1 + 1$*). Thus, the resonant vector fields are generated by* $(xy)^l x(\partial/\partial x)$ *and* $(xy)^l y(\partial/\partial y)$ *where* l *is a positive integer. Applying the Poincaré–Dulac theorem to equation (2) leads to a formal normal form*

$$\begin{aligned} \dot{x} &= x\hat{F}(xy), \\ \dot{y} &= -y\hat{G}(xy), \end{aligned} \tag{3}$$

where $\hat{F}, \hat{G}$ *are formal power series whose values at* 0 *is* 1.

We will focus on (germs of) holomorphic vector fields in a neighbourhood of the origin, a fixed point. One of the main issues is the existence of a holomorphic transformation in a neighbourhood of the origin to a normal form. The lack of existence is due to two main sources of problems

(a) Infinite number of resonances: equations $(Q, \lambda) = \lambda_i$ have an infinite number of solutions as in example 2.3. This will induce generally a Gevrey character

(b) Small divisors: the nonzero numbers $(Q, \lambda) - \lambda_i$ can accumulate the origin too rapidly. In example 2.2, if ζ is a Liouville number, then the vector field might not be holomorphically linearizable.

Alexander Brjuno was one of the first to give sufficient conditions ensuring the existence of a holomorphic transformation to a normal form. When these conditions are not satisfied what can be said? It might happen that the vector field to study belongs to a 'large enough' family of commuting vector fields sharing some nice properties. The existence of these 'symmetries' may help to transform simultaneously and holomorphically the members of the family into a normal form. This is the singular complete integrability phenomenon. As we shall see, vector fields which are suitable perturbations of these completely integrable systems have a lot of analytic invariant sets (or manifolds) in a neighbourhood of the origin. This is the singular KAM theory. If the vector field is not of this type, then it might happen that it admits a holomorphic transformation not in a full neighbourhood of the origin but rather in some 'sectorial domain' of the form $\alpha < \arg(x^r) < \beta$ in a neighbourhood of the origin for some monomial x^r. This is the sectorial normalization.

2.1. Singular complete integrability

The main progress in the holomorphic conjugacy to a normal formal problem is due to Brjuno who gave sufficient conditions that ensure that there is a convergent normalizing transformation to a normal form. These conditions are of two different types. The first one is a condition (ω) about the rate of accumulation to zero of the small divisors of the linear part (see below).

The second one is linked to the nonlinearities of the perturbation we are considering. It is a condition about a formal normal form of the perturbation.

The aim of this section is to present recent results about holomorphic normalization of vector fields with semi-simple linear parts. We refer to our recent lecture notes [Sto08] devoted to these questions or to the original papers [Sto00, Sto05b] for more details.

We shall say that $S = \sum_{i=1}^{n} \lambda_i x_i (\partial/\partial x_i)$ satisfies the Brjuno diophantine condition:

$$(\omega) \qquad -\sum_{k\geqslant 0} \frac{\ln \omega_k}{2^k} < +\infty,$$

where $\omega_k = \inf\{|(Q, \lambda) - \lambda_i| \neq 0,\ 1 \leqslant i \leqslant n,\ Q \in \mathbb{N}^n, 2 \leqslant |Q| \leqslant 2^k\}$. It is weaker than Siegel condition: there exists $c, \tau > 0$ such that

$$0 \neq |(Q, \lambda) - \lambda_i| \geqslant \frac{c}{|Q|^{\tau}}.$$

Theorem 2.4 (Bruno [Bru72]). *Let $X = S + R$ be a holomorphic vector field as above. We assume that S satisfies the Bruno condition (ω). If X has a formal normal form of the form $\hat{a}.S$ for some formal power series $\hat{a}$ (with $\hat{a}(0) = 1$), then X is holomorphically normalizable.*

In example 2.3, there are no small divisors (i.e. there exists $\epsilon > 0$ such that for all k, $\omega_k > \epsilon$) so that (ω) holds. The Brjuno theorem then says that if $\hat{F}(XY) = \hat{G}(XY)$ in the normal form (3), then there is a holomorphic transformation to a normal form.

What happens if these conditions are not satisfied? It might happen that there are other vector fields of the same type, 'independent' of each other and commuting pairwise and the family of these vector fields might be transformed simultaneously into a normal form by a holomorphic transformation. Jacques Vey has proved a couple of theorems in this direction:

Theorem 2.5 (Vey [Vey79]). *Let $X_1, \dots, X_{n-1}$ be $n-1$ holomorphic vector fields in a neighbourhood of $0 \in \mathbb{C}^n$, vanishing at this point. We assume that:*

- *each X_i is a volume preserving vector field (say $\operatorname{div} X_i := \sum_{j=1}^n (\partial X_i/\partial x_j) = 0$ to be simple),*
- *the linear parts $L_1, \dots, L_{n-1}$ of $X_1, \dots, X_{n-1}$ at the origin are diagonal and independent over $\mathbb{C}$ (this means that if there are complex constants c_i such that $\sum_{i=1}^{n-1} c_i L_i = 0$, then $c_i = 0$ for all i.),*
- *$[X_i, X_j] = 0$ for all indices i, j.*

Then, $X_1, \dots, X_{n-1}$ are holomorphically and simultaneously normalizable.

The second one [Vey78] (generalized by Ito [Ito89] and by Zung [Zun05]) is a similar statement but with n Hamiltonian vector fields in $\mathbb{C}^{2n}$ instead. The main differences between these results and Brjuno theorem are the following.

(a) There is apparently no diophantine condition.
(b) There is no assumption on the formal normal form but rather a geometrical assumption.

We were able to give a general result about normalization of a commutative family of holomorphic vector fields vanishing at the same point that unifies both Vey's and Brjuno's theorems. Here is the framework: let us consider the family $S = \{S_1, \dots, S_l\}$, $l \leqslant n$, of linearly independent linear diagonal vector fields

$$S_i = \sum_{j=1}^n \lambda_{i,j} x_j \frac{\partial}{\partial x_j}.$$

This means that if $\sum_{i=1}^l c_i S_i = 0$ for some complex numbers c_i, then all the c_i's are zero. Let us define the sequence of positive numbers

$$\omega_k(S) = \inf \left\{ \max_{1 \leqslant i \leqslant l} |(Q, \lambda^i) - \lambda_{i,j}| \neq 0,\ 1 \leqslant j \leqslant n,\ Q \in \mathbb{N}^n, 2 \leqslant |Q| \leqslant 2^k, \right\},$$

where $\lambda^i = (\lambda_{i,1}, \dots, \lambda_{i,n})$.

Definition 2.6. *We shall say that S is diophantine if*

$$(\omega(S)) - \sum_{k \geqslant 0} \frac{\ln \omega_k(S)}{2^k} < +\infty.$$

We can show that this condition can be satisfied while none of the S_i's satisfy the Brjuno condition. Let $\left(\widehat{\mathcal{X}}_n^1\right)^S$ (respectively $\left(\widehat{\mathcal{O}}_n\right)^S$) be the formal centralizer of S (respectively the ring of formal first integrals), that is the set of formal vector fields X (respectively formal power series f) such that $[S_i, X] = 0$ (respectively $\mathcal{L}_{S_i}(f) = 0$) for all $1 \leqslant i \leqslant l$.

Let $X = \{X_1, \dots, X_l\}$ be a family of germs of commuting vector fields at the origin such that the linear part of X_i is S_i; that is $[X_i, X_j] = 0$ for all i, j. We shall call X a nonlinear deformation of S.

Definition 2.7. *We shall say that a nonlinear deformation X of S is a normal form (with respect to S) if*

$$[S_i, X_j] = 0, \qquad 1 \leqslant i, j \leqslant l.$$

Definition 2.8. *We shall say that X, a nonlinear deformation of S, is formally completely integrable if there exists a formal diffeomorphism $\hat{\Phi}$ fixing the origin and tangent to the identity at that point which conjugate the family X to normal form of the type*

$$\hat{\Phi}_* X_i = \sum_{j=1}^{l} \hat{a}_{i,j} S_j, \qquad i = 1, \ldots, l, \tag{4}$$

where the $\hat{a}_{i,j}$'s belong to $\widehat{\mathcal{O}}_n^S$.

Theorem 2.9 (Stolovitch [Sto00]). *Under the above assumptions, if S is diophantine, then any formally completely integrable nonlinear deformation of S is holomorphically normalizable.*

This means that there exists a genuine germ of biholomorphism $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ tangent to the identity at 0 which conjugates the family X to a normal form of the type

$$\Phi_* X_i = \sum_{j=1}^{l} a_{i,j} S_j, \qquad i = 1, \ldots, l, \tag{5}$$

where the $a_{i,j}$'s are germs of holomorphic invariant functions, i.e. they belong to $\mathcal{O}_n^S$.

Remark 2.10. The theorem does not say that neither $\hat{\Phi}$ nor the $\hat{a}_{i,j}$'s converge but rather that there is another normalizing diffeomorphism that converges.

Remark 2.11. One way to use this theorem is to have 'a magic word in hand' (such as Hamiltonian, volume preserving and reversible) that will imply that the formal normal form is formally completely integrable using a formal transformation preserving an associated geometric structure P (this comes from the data of the problem that one wants to solve). Then one applies the theorem to obtain one germ ϕ of holomorphic transformation to a normal form. Then, following Vey's argument [Vey79, section 4], one shows that we can slightly perturb it by a germ of resonant holomorphic diffeomorphism ψ so that not only $\psi \circ \phi$ still conjugates the vector field to a normal form but it also preserves the geometric structure P.

Remark 2.12. We can show that Vey theorems are corollaries of theorem 2.9: first of all, the geometric property implies that the nonlinear family is formally completely integrable. On the other hand, we can show that the family of linear parts is automatically diophantine.

Let us give a geometrical interpretation of theorem 2.9. First of all, we can show that if the ring of invariants $\widehat{\mathcal{O}}_n^S$ is not reduced to constants, then there exists a finite number of monomials $x^{R_1}, \ldots, x^{R_p}$ such that $\widehat{\mathcal{O}}_n^S = \mathbb{C}[[x^{R_1}, \ldots, x^{R_p}]]$. For instance, in the volume preserving case, we have $p = 1$ and $x^{R_1} := x_1 \cdots x_n$. For simplicity, we will assume that the monomials x^{R_i} are algebraically independent. Let $\pi : \mathbb{C}^n \to \mathbb{C}^p$ defined by $\pi(x) = (x^{R_1}, \ldots, x^{R_p})$. By definition, the linear vector fields $S_1, \ldots, S_l$ are tangent and independent on each fibre $\pi^{-1}(b)$ of π; the latter is called a *toric variety*. Note that we must have $l \leqslant n - s$. Now, we come to the nonlinear deformation. Let $\{X_i = S_i + R_i\}_{i=1,\ldots,l}$ be a family of pairwise commuting germs of holomorphic vector fields. Let us assume that it is formally completely integrable. Then, according to our result, there exists a neighbourhood U of 0 in $\mathbb{C}^n$ and a holomorphic diffeomorphism Φ on U such that, in the new coordinate system, the vector fields $\Phi_* X_1, \ldots, \Phi_* X_l$ are commuting linear diagonal vector fields on each fibre restricted to U and their eigenvalues depend only on the fibre. Indeed, in these new coordinates, we have $\Phi_* X_i = \sum_{j=1}^{l} a_{i,j} S_j$ where $a_{i,j} \in \mathcal{O}_n^S$. By definition, these S_i's are all tangent to the fibers of π. As a consequence, the $\Phi_* X_i$'s are all tangent to the fibers of π. On each fibre, the functions $a_{i,j}$ are constant so that each $\Phi_* X_i$ reads as a linear diagonal vector field, that is a

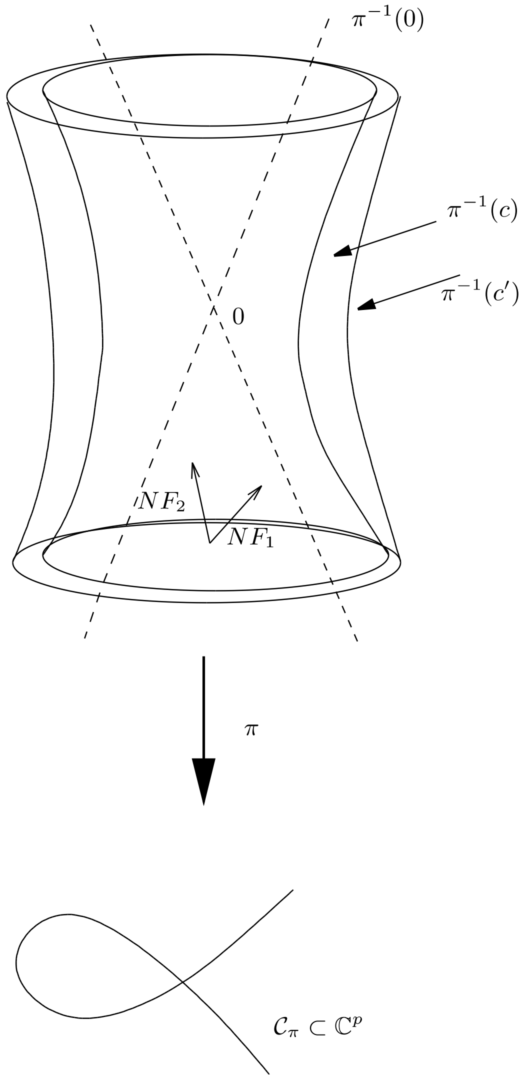

Figure 1. Singular complete integrability: in the new holomorphic coordinate system, all fibers (intersected with a fixed polydisc) are left invariant by the vector fields and their motion on it is a linear one.

linear motion of a toric variety. So, it sounds like the classical complete integrability theorem of Hamiltonians on tori (figure 1).

The 'smoothing' effect of the existence of a large centralizer was known for smooth linearization problems [DR80, Cha86]. In the analytic context, Jürgen Moser used a Siegel type diophantine condition for a family of rotations in the study of commuting diffeomorphisms of the circle [Mos90].

2.2. Singular KAM theory

With respect to what has already been said, the natural question one may ask is the following: starting from a holomorphic singular completely integrable system in a neighbourhood of the origin of $\boldsymbol{C}^n$ (a common fixed point), we consider a holomorphic perturbation (in some sense) of one of its vector fields. Does this perturbation still have invariant varieties in some neighbourhood of the origin? Are these varieties biholomorphic to resonant varieties? To which vector field on a resonant variety does the biholomorphism conjugate the restriction of the perturbation to an invariant variety? Is there a 'big set' of surviving invariant varieties?

The aim of [Sto05a] is to answer these questions. Let us give a taste of what it is all about.

Let $S_1 \dots, S_l$ be a collection of linear diagonal vector fields on $\boldsymbol{C}^n$ as above. Let X_0 be a nondegenerate singular integrable vector field (in the sense of Rüssmann). We mean that X_0 is of the form

$$X_0 = \sum_{j=1}^{l} a_j(x^{R_1}, \dots, x^{R_p}) S_j,$$

where the a_j's are polynomials and the range of the map $(a_1, \dots, a_l)$ from $(\boldsymbol{C}^n, 0)$ to $(\boldsymbol{C}^l, 0)$ is not included in any complex hyperplane. This is our unperturbed motion.

Then, we consider a small holomorphic perturbation X of X_0. Let us set $X = X_0 + R_{m_0}$ where R_{m_0} is a germ of holomorphic vector field at the origin and of order greater than or equal to m_0 at that point. One of the difficulties is that, contrary to the classical KAM theory of Hamiltonians, there are no natural actions-angles coordinates to play with. Nevertheless, we shall construct something similar: we add new variables $u_1, \dots, u_p$ which correspond to the resonant monomials $x^{R_1}, \dots, x^{R_p}$. These are the 'slow variables'. With the holomorphic vector field X in $(\boldsymbol{C}^n, 0)$, we shall associate a holomorphic vector field $\tilde{X}$ in $(\boldsymbol{C}^{n+p}, 0)$ where the coordinate along $\partial/\partial u_j$ is the Lie derivative of the resonant monomial x^{R_j} along X. This vector field is tangent to the variety

$$\Sigma = \{(x, u) \in \boldsymbol{C}^n \times \boldsymbol{C}^p, \mid \; u_j = x^{R_j}, \quad j = 1, \dots, p\}$$

and its restriction to it is nothing but X. We shall say that $\tilde{X}$ is *fibred over* X. We shall conjugate $\tilde{X}$ by germs of diffeomorphisms which preserve the variety Σ. Such a germ will be built in the following way: let $\Phi(x, u) := \{y = x + U(x, u)\}$ be a family of germs of biholomorphisms of $(\boldsymbol{C}^n, 0)$, tangent to the identity at the origin and parametrized over an open set $\mathcal{U}$. Let us set $v := u + \pi(y) - \pi(x)$ and $\tilde{\Phi}(x, u) := (y, v)$. The latter is a germ of diffeomorphism at $(0, b)$ and tangent to the identity at this point, for any b. It leaves Σ invariant. We shall say the $\tilde{\Phi}$ is *fibred over* Φ. We shall define the notion of *Lindstedt–Poincaré normal form* of $\tilde{X}$ of order k as follows: there exists a fibred diffeomorphism $\tilde{\Phi}_k$ such that

$$(\tilde{\Phi}_k)_* \tilde{X} = \begin{cases} \dot{y} = NF^k(y, v) + R_{k+1}(y, v) + r_{\Sigma,k}(y, v), \\ \dot{v} = \pi_*(NF^k(y, v) + R_{k+1}(y, v) + r_{\Sigma,k}(y, v)), \end{cases}$$

where the restriction of the Lie bracket to Σ

$$\left[X_0, \widetilde{NF^k(y, v)}\right]_{|\Sigma} = 0,$$

and where R_{k+1} is of order greater than or equal to $k + 1$ with respect to y and r_Σ vanishes on Σ. Moreover, we shall choose $NF^k(v, y)$ in such a way that any occurrence of y^{R_i} has been replaced by v_i, for all i. The notation $\dot{v} = \pi_*(NF^k(y, v) + R_{k+1}(y, v) + r_{\Sigma,k}(y, v))$ means, for all $1 \leqslant i \leqslant p$,

$$\dot{v}_i = \mathcal{L}_{NF^k(y,v)+R_{k+1}(y,v)+r_{\Sigma,k}(y,v)}(y^{R_i}) = \sum_{j=1}^{n} (NF_j^k + R_{j,k+1} + r_{j,\Sigma,k}) \frac{\partial y^{R_i}}{\partial y_j}.$$

If we were dealing with an integrable symplectic vector field X_0, we would require the perturbation to be also symplectic. The analogue in our general setting is an assumption on the

Lindstedt–Poincaré normal form of X (compare with the previous section where the geometric assumptions in the Vey results were replaced by an algebraic assumption on a formal normal form). Namely, we require that

$$NF^k(y,v) = \sum_{j=1}^{l} a_j^k(v) S_j(y).$$

We shall write

$$NF^k(y,v) = \sum_{i=1}^{n} \lambda_{i,k}(v) y_i \frac{\partial}{\partial y_i}. \tag{6}$$

Hence, the Lindstedt–Poincaré normal form reads

$$(\tilde{\Phi}_k)_*\tilde{X} = \begin{cases} \dot{y} = \sum_{j=1}^{l} a_j^k(v) S_j(y) + R_{k+1}(y,v) + r_{\Sigma,k}(y,v), \\ \dot{v} = \pi_*(R_{k+1}(y,v) + r_{\Sigma,k}(y,v)). \end{cases}$$

A perturbation X of X_0 which has a Lindstedt–Poincaré normal form of this type for any k will be called *good perturbation* of X_0. In this case, by applying an infinite sequence of change of variables, we would find a formal diffeomorphism with coefficient defined in some set $\mathcal{K}_\infty$ in $\mathbb{C}^p$ such that

$$(\tilde{\Phi}_\infty)_*\tilde{X} = \begin{cases} \dot{y} = \sum_{j=1}^{l} a_j^\infty(v) S_j(y) + r_{\Sigma,\infty}(y,v), \\ \dot{v} = \pi_*(r_{\Sigma,\infty}(y,v)). \end{cases}$$

By restricting to Σ, that is by setting $v_j = y^{R_j}$, we would obtain

$$(\tilde{\Phi}_\infty)_*\tilde{X} = \begin{cases} \dot{y} = \sum_{j=1}^{l} a_j^\infty(y^{R_1},\ldots,y^{R_p}) S_j(y), \\ \dot{v} = 0. \end{cases}$$

This means that, for any $b \in \mathcal{K}_\infty$, the *toric variety* $\cap_j\{y \in \mathbb{C}^n \mid v_j = b_j = y^{R_j}\}$ would be invariant and that the motion on it would be a linear motion $\sum_{j=1}^{l} a_j^\infty(b) S_j(y)$.

The main goal of [Sto05a] is to give a real meaning of the previous formal scheme and to show that indeed there are a lot of invariant manifolds. Let us be more precise.

Let $\omega = \{\omega_k\}_{k\in N^*}$ be a sequence of positive numbers such that

- $\omega_k \leqslant 1$,
- $\omega_{k+1} \leqslant \omega_k$,
- the series $\sum_{k>0}(-\ln \omega_k/2^k)$ converges.

Such a sequence will be called a *diophantine sequence*.

Let ρ be a sufficiently small positive number less than $1/2$. Let $\mathcal{K}$ be a nonvoid compact set of $\pi(D_n(0,\rho))$. Let γ be a positive real number and less than some γ'. We define the decreasing sequence $\{\mathcal{K}_k(X,\mathcal{K},\omega,\gamma)\}_{k\in N}$ of compact sets of $\pi(D_n(0,\rho))$ as follows:

$$\mathcal{K}_0 = \mathcal{K},$$
$$\mathcal{K}_k = \left\{ b \in \mathcal{K}_{k-1} \mid \forall Q \in \mathbb{N}^n, |Q| \leqslant 2^{k+1}, \ \left|(Q, \lambda^{2^k}(b)) - \lambda_{2^k,i}\right| \geqslant \gamma\omega_{k+1} \right\}$$

where $\lambda^{2^k}(v) = (\lambda_{2^k,1}(v),\ldots,\lambda_{2^k,n}(v))$ is defined by (6).

Theorem 2.13. *Under the above assumptions, let ω be a diophantine sequence; let $\mathcal{K}$ and γ be defined as above. If m_0 (the order of the perturbation at the origin) is large enough and if the set $\mathcal{K}_\infty(X,\mathcal{K},\omega,\gamma) := \bigcap_{k\in N^n} \mathcal{K}_k$ is nonvoid then, for any $b \in \mathcal{K}_\infty$,*

(a) the sequence $\{NF^m(x,b)\}$ converges to a linear diagonal vector field
$NF(x,b) = \sum_{j=1}^{l} \tilde{a}_j(b) S_j(x)$,

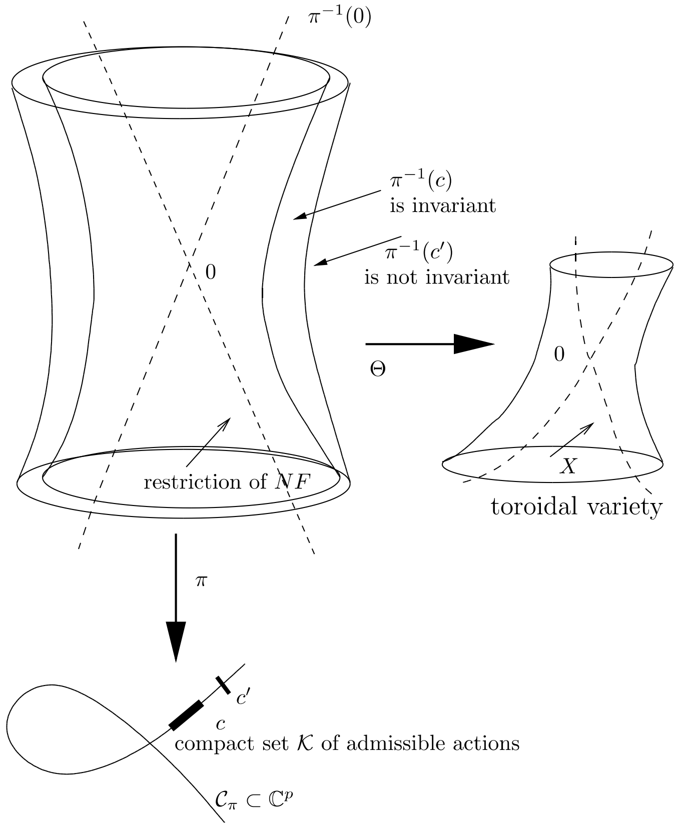

Figure 2. KAM phenomenon: X has an invariant analytic set which is biholomorphic to the restriction to a fixed polydisc of $x_1^R = c_1, \ldots, x^{R_p} = c_p$ for any c in a compact set of positive measure. Its restriction to it is conjugate to a linear motion

(b) *there is a biholomorphism of analytic subsets of open sets in* $\boldsymbol{C}^n$,

$$\Theta_b : \pi^{-1}(b) \cap D_0(\rho) \to V_b \subset \boldsymbol{C}^n,$$

which conjugates the restriction of $NF(x, b)$ *to* $\pi^{-1}(b) \cap D_0(\rho)$ *to the restriction of* X *to* V_b.

As a consequence, the perturbation X *is tangent to the toroidal analytic subset* V_b*; its restriction to it is conjugate to the restriction to the toric analytic subset* $\pi^{-1}(b) \cap D_n(0, \rho)$ *of the linear diagonal vector field* $\sum_{j=1}^{l} \tilde{a}_j(b) S_j$.

This theorem says that X has an invariant manifold 'of the form $x^{R_1} = b_1, \ldots, x^{R_p} = b_p$' with a linear motion on it whenever $b \in \mathcal{K}_\infty$ unless $\mathcal{K}_\infty$ is void. We shall give a sufficient condition which ensures that it is not the case (figure 2).

Definition 2.14. *Let* $\omega = \{\omega_k\}_{k\geqslant 1}$ *be a diophantine sequence and* μ_0 *be a positive integer. We shall say that* S *is strictly diophantine relatively to* (ω, μ_0) *if*

$$\lim_{k\to+\infty} \left(2^k + n + 1\right)^{n+1} \left(\frac{\omega_k}{\omega_k(S}\right)^{2/\mu_0} = 0.$$

Theorem 2.15. *Let $\mathcal{K}$ be a compact set of $\pi(D_n(0,\rho))$ of a positive $2p$-measure. Assume that S is strictly diophantine relative to the sequence $(\omega = \{\omega_i\}_{i\geqslant 1}, \mu_0)$ where μ_0 is the index of nondegeneracy of X_0 with respect to $\mathcal{K}$ (see below). Then, under the assumptions of theorem 2.13, $\mathcal{K}_\infty$ is nonvoid and has a positive $2p$-measure.*

Hence, there are a lot of invariant manifolds.

The definition of strict diophantiness is derived from Rüssmann's work on KAM theory in the symplectic case [Rüs01] from which the previous theorem is an adaptation.

Let $\mathcal{U}$ be a connected open set on $\boldsymbol{C}^n$, $f:\mathcal{U}\to \boldsymbol{C}^l$ a nondegenerate holomorphic map and $\mathcal{K}\subset\mathcal{U}$ a nonvoid compact set. Let $\mathbb{S}^l$ denote the unit sphere. Let us set

$$\beta(\mu, f, \mathcal{K}) = \min_{\substack{y\in\mathcal{K},\\ c\in\mathbb{S}^l}} \max_{0\leqslant k\leqslant\mu} \left| D^k |(c,f)|^2(y)\right|.$$

Clearly, $\{\beta(\mu, f,\mathcal{K})\}_{\mu\geqslant 0}$ is a nondecreasing sequence of nonnegative numbers. We can show that there exists a positive integer N_0 such that $\beta(N_0, f, \mathcal{K})$ is positive. The smallest of these integers N_0 will be called the *index of nondegeneracy* (of f with respect to $\mathcal{K}$) and will be denoted by $\mu_0 = \mu_0(f,\mathcal{K})$.

The previous theorems are to be regarded as a KAM theory for *a priori* nonsymplectic vector fields near a fixed point. The toric varieties $x^{R_1} = b_1, \dots, x^{R_p} = b_p$ play the rôle of tori and the vector field $\sum_{i=1}^l a_i(b)S_j$ plays the rôle of the constant motion on the tori.

This idea that 'the torus may not be the natural generalization of the circle' was also found recently by Chaperon and López de Medrano [CKCDVLDM06, Cha07, CLDM08] in a different context, namely, higher-dimensional smooth Hopf bifurcation.

Example 2.16 (Hamiltonian vector fields). *Let us give a taste of how we can recover the 'classical' KAM theory with genuine real tori. Let us consider a real analytic Hamiltonian H in a neighbourhood of the origin in $\boldsymbol{R}^{2n}$ which is a perturbation of a nondegenerate integrable Hamiltonian h_0. We assume that*

$$H(x,y) = \sum_{l=1}^{N_0}\sum_{i=1}^{n} \mu_{i,l}(x_i^2+y_i^2)^l + h_{M_0+1}(x,y) = h_0 + h_{M_0+1}(x,y),$$

where $h_{M_0+1}(x,y)$ is a real analytic function of order greater than or equal to $M_0+1 \geqslant 2N_0+1$. Here, the $\mu_{i,l}$'s are real numbers and $\omega = \sum_{i=1}^n \mathrm{d}x_i\wedge\mathrm{d}y_i$ denotes the canonical symplectic form of $\boldsymbol{R}^{2n}$. Let us first write the Hamiltonian using complex coordinates $z_j = x_j + \mathrm{i}y_j$, $j = 1,\dots,n$. and then let us complexify it. We obtain a holomorphic Hamiltonian G in a neighbourhood of the origin in $\boldsymbol{C}^{2n}$ with (z,w) as complex symplectic coordinates:

$$G(z,w) = \sum_{l=1}^{N_0}\sum_{i=1}^{n} \mu_{i,l}(z_iw_i)^l + \tilde{h}_{M_0+1}(z,w) =: H_0 + \tilde{h}_{M_0+1}(z,w).$$

We recover $\tilde{H}$ (or H) by restricting G to the set $\bigcap_{i=1}^n\{w_i = \bar{z}_i\}$.

Let us consider the family $S := \{S_1,\dots,S_n\}$, where $S_i = z_i(\partial/\partial z_i) - w_i(\partial/\partial w_i)$. We can show that there exists a positive ϵ such that for all k, $\omega_k(S)\geqslant\epsilon$. The ring of invariants of S is $\widehat{\mathcal{O}}_{2n}^S = \mathbb{C}[[z_1w_1,\dots,z_nw_n]]$ We refer to our previous [Sto00, chapter 10] for more details. Since the vector field X_G associated with G is symplectic, using a symplectic change of coordinates we can show that its Lindstedt–Poincaré normal form of order any order m is of the form

$$\sum_{j=1}^{n} a_j^m(u_1,\dots,u_n)S_j.$$

Therefore, we can apply our result: X_G has invariant analytic subsets which are biholomorphic to the intersection of $\cap_{i=1}^n \{w_i z_i = c_i\}$ with a fixed polydisc, for some well chosen constants c_i. Since the normalization process is compatible with restriction to the set $\cap_{i=1}^n \{w_i = \bar{z}_i\}$ (see as in [Bry88] for Poincaré-Dualce normal form), if one chooses a set of real constants, the Hamiltonian vector field will have invariant (real) analytic subsets analytically isomorphic to the intersection of a fixed polydisc with

$$\left(\bigcap_{i=1}^n \{w_i z_i = c_i\}\right) \bigcap \left(\bigcap_{i=1}^n \{w_i = \bar{z}_i\}\right) = \bigcap_{i=1}^n \{z_i \bar{z}_i = x_i^2 + y_i^2 = c_i\}$$

for some real constants c_i. These are the genuine real tori. It should be noticed that our result concerns vector fields in the neighbourhood of a fixed point.

For classical KAM theory (named after its authors Kolmogorov–Arnold–Moser [Kol54, Kol57, Arn63a, Arn63b, Mos62]), we refer to a very nice introduction by Bost [Bos86], the book by Moser [Mos73] and the book [BHS96] by Broer, Huitema and Sevryuk, which contains an extensive bibliography. We also refer to [Arn88b] for KAM theory and the Lindstedt method. For one of the most ultimate versions of classical KAM theory, we refer to [Rüs01].

Example 2.17 (Volume preserving vector fields). *Let us consider a holomorphic volume preserving vector field X which is a deformation of a nondegenerate volume preserving polynomial vector field X_0 in a neighbourhood of the origin of $\boldsymbol{C}^n$. We consider the family $S := \{S_1, \ldots, S_{n-1}\}$ where $S_i = x_i(\partial/\partial x_i) - x_{i+1}(\partial/\partial x_{i+1})$. We can show that there exists a positive ϵ such that for all k, $\omega_k(S) \geqslant \epsilon$. The ring of invariants of S is defined to be $\widehat{\mathcal{O}}_n^S = \mathbb{C}[[x_1 \cdots x_n]]$. Since the vector field X is volume preserving, then using a volume preserving change of coordinates, we can show that its Lindstedt–Poincaré normal form of order any order m is of the form*

$$\sum_{j=1}^{n-1} a_j^m(u) S_j.$$

Therefore, we can apply our result: X has invariant analytic subsets which are biholomorphic to to the intersection of a fixed polydisc with

$$\left\{x \in \mathbb{C}^n \mid x_1 \cdots x_n = c_i\right\}$$

for some well chosen constants c_i. Hence, the invariant manifolds to be expected are not tori at all!

2.3. Sectorial normalization

Let us start with an example.

Example 2.18. *Let us consider the two-dimensional system*

$$\begin{aligned} \dot{x} &= x^2, \\ \dot{y} &= x + y. \end{aligned} \tag{7}$$

There is a unique formal change of variables $x = X$, $y = Y + \hat{\psi}(X)$ that transforms the previous system into its normal form

$$\begin{aligned} \dot{x} &= x^2, \\ \dot{y} &= y. \end{aligned} \tag{8}$$

This normal form does not satisfy the Brjuno condition since it is not proportional to the linear part $\dot{x} = 0$, $\dot{y} = y$ (and there is no small divisor). We can show that

$$\hat{\psi}(X) = -\sum_{k\geqslant 1}(k-1)!X^k.$$

This power series does not converge in a neighbourhood of the origin! It is a 1*-Gevrey formal power series. Nevertheless, one can show that there exist sectorial normalizations. This means that there exist germs of holomorphic diffeomorphisms defined only in the product of the sector with an edge at the origin (in the x plane) and a disc around* 0 *(in y) which conjugate equation* (7) *into its normal form. This is done by applying the formal Borel transform* $\mathcal{B}\hat{\psi}(t) := -\sum_{k\geqslant 1}((k-1)!/k!)t^{k-1}$ *of* $\hat{\psi}$*. It converges in a neighbourhood of the origin and can be analytically continued in almost all directions with at most an exponential growth at infinity. Then, one applies the Laplace transform in these directions to obtain the desired function. We say that the formal power* $\hat{\phi}$ *is resummable.*

This is the starting point of a long story that has been developed by Martinet and Ramis [MR82, MR83] for two-dimensional vector fields and by Ecalle, Voronin and Malgrange for germs of local diffeomorphisms near a fixed point in the complex plane [Eca, Vor81, Mal82, Il′93].

In higher dimension, the general theory has been developed by Ecalle and Stolovitch [Eca92, Sto96] for 'one-resonant' linear parts which means that all the resonances are generated by a single monomial resonant x^r. Let us consider $S = \sum_{i=1}^{n} \lambda_i x_i(\partial/\partial x_i)$ such that there exists a nonzero $r \in \mathbb{N}^n$ such that if $[S, x^Q(\partial/\partial x_i)] = 0$ then $x^Q = (x^r)^l x_i$ where $l \in \mathbb{N}$. Then, we consider nonlinear perturbations of S for which x^r is not a first integral of a formal normal form. For instance, we can consider higher order perturbations of

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 + \alpha_1 x_1 x^r \\ &\vdots \\ \dot{x}_n &= \lambda_n x_n + \alpha_n x_n x^r, \end{aligned}$$

where $r_1\alpha_1 + \cdots + r_n\alpha_n \neq 0$. In this situation, we show that the transformation to a normal form is a divergent series with respect to the monomial x^r. Nevertheless, we can prove the existence of a holomorphic transformation to a normal form in a sectorial domain of the form $\{\alpha < \arg x^r < \beta\}$. Recently, the interplay between these 'Stokes phenomena' and small divisors phenomena have been investigated by Braaksma and Stolovitch [BS07].

3. Normal form of perturbation of quasihomogeneous vector fields

In this section, we shall focus on vector fields which are degenerate and which may not have a nonzero linear part at the origin.

We shall be given a 'reference' polynomial vector field S to which we would like to compare a suitable perturbation of it. This means that we would like to know if some of the geometric or dynamical properties of the model can survive for the perturbation. For instance, the model $S_1 = y(\partial/\partial x)$ and $S_2 = y(\partial/\partial x) + x^2(\partial/\partial y)$ are quite different although they have the same linear part at the origin of $\mathbb{C}^2$. In fact, for S_1, each point of $\{y = 0\}$ is fixed, whereas the set $\{2x^3 - 3y^2 = 0\}$ is globally invariant by S_2.

Our framework is the following: the *unperturbed vector field S is quasihomogeneous* with respect to some weight $p = (p_1, \ldots, p_n) \in (\mathbb{N}^*)^n$. This means that each variable x_i has the weight p_i while $(\partial/\partial x_i)$ has the weight $-p_i$. Hence, the monomial x^Q is quasihomogeneous of quasidegree $(Q, p) := \sum_{i=1}^{n} q_i p_i$. In particular, the vector field $S = \sum_{i=1}^{n} S_i(x)(\partial/\partial x_i)$ is quasihomogeneous of quasidegree s if and only if each S_i is a quasihomogeneous polynomial of degree $s + p_i$.

We then consider a germ of *holomorphic vector field X which is a good perturbation of a quasihomogeneous vector field S*, which means that the smallest quasidegree of nonzero terms in the Taylor expansion of $X - S$ is greater than s. In the homogeneous case ($p = (1, \ldots, 1)$), a linear vector field S is quasihomogeneous of degree 0 and a good perturbation is a nonlinear perturbation of S (i.e. the order at 0 of the components of $X - S$ is greater then equal to 2).

We develop a *normal form theory* for these objects. As we have seen, the formal normal form theory of vector fields which are nonlinear perturbations of a semi-simple (respectively nilpotent, general) linear vector field is well known (respectively [CS86, Bel79, Mur03]), It is much more difficult to handle the problem when the vector field does not have a nonzero linear part. We define a *'diophantine condition' on the quasihomogeneous initial part S* which ensures that if such a perturbation of S is formally conjugate to S then it is also holomorphically conjugate to it. We give a condition on S that ensures that there always exists a holomorphic transformation to a normal form. If this condition is not satisfied, we also show that under some reasonable assumptions, each perturbation of S admits a Gevrey formal normalizing transformation. Finally, we give an exponentially good approximation by a partial normal form. All these results can be found in [LS09b] and were announced in [LS09a].

3.1. Quasihomogeneous polynomials and vector fields

Let $p = (p_1, \ldots, p_n) \in (\mathbb{N}^*)^n$ be fixed such that the largest common divisor of its components $p_1 \wedge \cdots \wedge p_n$ is equal to 1. Let $n \geqslant 2$ be an integer. The elements of the set

$$\Delta = \{d \in \mathbb{N} /\ d = (\alpha, p),\ \text{with}\ \alpha \in \mathbb{N}^n\}$$

are called the *quasidegree*. For $\delta \in \Delta$, we shall denote by $\mathcal{P}_\delta$ the vector space of p-quasihomogeneous polynomials of degree δ. If $\delta \notin \Delta$, we set $\mathcal{P}_\delta := \{0\}$. Hence, for any $\delta \in \mathbb{N}$,

$$\mathcal{P}_\delta := \left\{ f \in \mathbb{C}[x],\ f(x) = \sum_{(Qp)=\delta} f_Q x^Q \right\} \qquad \text{if } \delta \in \Delta,\ \mathcal{P}_\delta := \{0\} \quad \text{otherwiwse.}$$

A vector field $X = \sum_{i=1}^{n} X_i(\partial/\partial x_i)$ is *quasihomogeneous* of quasidegree $\delta \geqslant 0$ if, for each $1 \leqslant i \leqslant n$, X_i is belongs to $\mathcal{P}_{\delta+p_i}$. For $\delta \in \tilde{\Delta}$, we shall denote by $\mathcal{H}_\delta$ the complex vector space of p-quasihomogeneous polynomials of quasidegree δ. If $\delta \notin \tilde{\Delta}$, we shall set $\mathcal{H}_\delta := \{0\}$. The set of quasidegree of the vector fields is defined to be

$$\tilde{\Delta} = \{\tilde{\delta} \in \mathbb{Z} / \tilde{\delta} = \delta - p_i, \text{with } \delta \in \Delta, 1 \leqslant i \leqslant n\}.$$

First of all, we shall define a special Hermitian product $\langle ., . \rangle_{p,\delta}$ on each space $\mathcal{H}_\delta$ of quasihomogeneous vector fields of quasidegree δ. Its main property is that the associated norm of a product is less than or equal to the product of the norm. Let us set

$$\langle x^R, x^P \rangle_{p,\delta} := \begin{cases} \dfrac{(r_1!)^{p_1} \cdots (r_n!)^{p_n}}{\delta!} & \text{if } R = Q, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

We shall set $(R!)^p := (r_1!)^{p_1} \cdots (r_n!)^{p_n}$ and let $|.|_p$ be the associated norm. If $p = (1, \ldots, 1)$ (i.e. in the homogeneous case), this is the Fischer scalar product [Sha89, Fis17, IL05]

Let $f = \sum_{(Q,p)=\delta} f_Q x^Q \in \mathcal{P}_\delta$, then

$$|f|_{p,\delta}^2 := \langle f, f \rangle_{p,\delta} = \sum_{(Q,p)=\delta} |f_Q|^2 \frac{(Q!)^p}{\delta!}. \tag{10}$$

For a quasihomogeneous vector field of degree delta $\delta \in \tilde{\Delta}$ we define the associated Hermitian product and norm given by

$$\langle UV\rangle_{p,\delta} := \sum_{i=1}^{n} \langle U_i V_i\rangle_{p,\delta+p_i} \qquad \text{and} \qquad \|U\|^2_{p,\delta} := \sum_{i=1}^{n} |U_i|^2_{p,\delta+p_i},$$

where $U = \sum_{i=1}^{n} U_i(\partial/\partial x_i) \in \mathcal{H}_\delta$ and $V = \sum_{i=1}^{n} V_i(\partial/\partial x_i) \in \mathcal{H}_\delta$.

One of the main features of these Hermitian products is its good behaviour with respect to the product. More precisely, we have the following proposition.

Proposition 3.1 (submultiplicativity of the norms).

(a) Let f, g be p-quasihomogeneous polynomials of δ, δ', respectively. Then,

$$|fg|_{p,\delta+\delta'} \leqslant |f|_{p,\delta}\, |g|_{p,\delta'}.$$

(b) For a formal power series f, the following properties are equivalent:

(i) f is uniformly convergent in a neighbourhood of the origin,

(ii) There exist $M, R > 0$ such that for every $\delta \in \Delta$, $\quad |f_\delta|_{p,\delta} \leqslant (M/R^\delta)$.

In the homogeneous case, the first point is due to Iooss and Lombardi [IL05, lemma A.8], while the second one is due to Shapiro [Sha89, lemma 1].

3.2. Normal forms of their perturbations

Let S be a quasihomogeneous vector field of $\mathbb{C}^n$ of quasidegree s (with respect to p which is fixed once for all). We are interested in suitable holomorphic perturbations of S.

Definition 3.2. *Let X be a germ of holomorphic vector field at the origin of $\mathbb{C}^n$. We shall say that X is a good perturbation of S if the Taylor expansion of $X - S$ at the origin is of quasiorder greater than s.*

Remark 3.3. If one is given a vector field, it might not be obvious how to choose the weight p. It really depends on the dynamics one would like to compare to. Furthermore, it might be helpful to consider parameters as variables with weights. For instance, let us consider the weight $p = (1, 2, 3, 2, 2)$ associated with the variables $(\epsilon, \alpha, \beta, z, \bar{z})$. Consider the following five-dimensional differential system:

$$\begin{cases} \dot{\epsilon} = & 0 \\ \dot{\alpha} = & 0 & +\beta & +\alpha^2 + |z|^2 \\ \dot{\beta} = & 0 & +\epsilon^2\alpha - c\alpha^2 + d|z|^2 & +0 \\ \dot{z} = & \mathrm{i}\omega z & +0 & +\mathrm{i}\omega z(\alpha^2 + |z|^2) \\ \dot{\bar{z}} = & \underbrace{-\mathrm{i}\omega z}_{0} & \underbrace{+0}_{1} & \underbrace{-\mathrm{i}\omega z(\alpha^2 + |z|^2)}_{2}. \end{cases}$$

The number assigned to each column is the quasidegree of the vector field relative to the weight p. It represents a normal form $S + N_1 + N_2$ of quasidegree 2 of a perturbation of a rotation $S = \mathrm{i}\omega z(\partial/\partial z) - \mathrm{i}\omega\bar{z}(\partial/\partial\bar{z})$ which has homocline orbits to circles (this problem was studied in [IL04]). If instead we chose $p = (1, 1, 2, 1, 1)$, then

$$\begin{cases} \dot{\epsilon} = & 0 \\ \dot{\alpha} = & 0 & +\beta + \alpha^2 + |z|^2 \\ \dot{\beta} = & -c\alpha^2 + d|z|^2 & +\epsilon^2\alpha & 0 \\ \dot{z} = & \mathrm{i}\omega z & +0 & +\mathrm{i}\omega z(\alpha^2 + |z|^2) \\ \dot{\bar{z}} = & \underbrace{-\mathrm{i}\omega z}_{0} & \underbrace{+0}_{1} & \underbrace{-\mathrm{i}\omega z(\alpha^2 + |z|^2)}_{2}. \end{cases}$$

The dynamical properties of the part a 0-quasidegree is quite different.

Let us consider the *cohomological operator*:

$$d_0 : \mathcal{H}_\delta \to \mathcal{H}_{s+\delta},$$
$$U \mapsto [S, U],$$

where $[., .]$ denotes the usual Lie bracket of vector fields. We emphasize that, contrary to the case where S is linear ($s = 0$), d_0 does not not leave $\mathcal{H}_\delta$ invariant. Let $d_0^* : \mathcal{H}_{\delta+s} \to \mathcal{H}_\delta$ be the *adjoint of* d_0 with respect to the Hermitian product. An element of the kernel of this operator will be called *resonant or harmonic*.

Proposition 3.4 (Partial normal form). *Let S be a p-quasihomogeneous vector field of $\mathbb{C}^n$. Let $X := S + R$ be a good holomorphic perturbation of S in a neighbourhood of the origin of $\mathbb{C}^n$ (i.e. the order of R at the origin is greater than s). Then, for every $\alpha \in \tilde{\Delta}$, there exists a polynomial diffeomorphism tangent to identity $\Phi_\alpha^{-1} = \mathrm{Id} + \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha = \sum_{0<\delta\leqslant\alpha-s} U_\delta$, with $U_\delta \in \mathcal{H}_\delta \cap (\ker d_0)^\perp$ such that*

$$(\Phi_\alpha)_*(X) = S + \mathcal{N}_\alpha + \mathcal{R}_{>\alpha}, \qquad \text{where} \quad \mathcal{N}_\alpha = \sum_{s<\delta\leqslant\alpha} N_\delta, \quad N_\delta \in \ker d^*_{0|\mathcal{H}_\delta}, \tag{11}$$

and where $\mathcal{H}_{>\alpha}$ is of quasiorder $> \alpha$.

Therefore, there exists a formal diffeomorphism $\hat{\Phi}$ tangent to the identity which conjugate X to a formal normal form; that is $\hat{\Phi}_* X - S$ is resonant (i.e. each quasihomogeneous component is resonant). Moreover, there exists a unique normalizing diffeomorphism $\Phi = \mathrm{Id} + \mathcal{U}$ such that $\mathcal{U}$ has a zero projection on the kernel of $d_0 = [S, .]$.

Proof. Let us give the proof: a basic identification of the quasihomogeneous components for $\delta \in \tilde{\Delta}$ with $s < \delta \leqslant \alpha$ in (11) with $X = S + R$ leads to

$$\Big\{\mathcal{N}_\alpha + [S, \mathcal{U}_\alpha]\Big\}_\delta = \Big\{R(\mathrm{Id} + \mathcal{U}_\alpha) - D\mathcal{U}_\alpha.\mathcal{N}_\alpha + S(\mathrm{Id} + \mathcal{U}_\alpha) - S - DS.\mathcal{U}_\alpha\Big\}_\delta. \tag{12}$$

We get the following hierarchy of cohomological equations in $\mathcal{H}_\delta$ for $\delta \in \tilde{\Delta}$ with $s < \delta \leqslant \alpha$:

$$N_\delta + d_0(U_{\delta-s}) = K_\delta, \tag{13}$$

where K_δ depends only on R, S which are given and on N_β and $U_{\beta-s}$ for $s < \beta < \delta$. So the hierarchy of equations (13) for $s < \delta \leqslant \alpha$ can be solved by induction starting with the smallest $\delta \in \tilde{\Delta}$ greater than s.

If $\delta - s \notin \tilde{\Delta}$, then $\mathcal{H}_{\delta-s} = \{0\}$. Hence, $d_{0|\mathcal{H}_{\delta-s}} \equiv 0$ so that $K_\delta \in \ker d^*_{0|\mathcal{H}_\delta} = \mathcal{H}_\delta$. Hence, if $\delta - s \notin \tilde{\Delta}$, we set $U_{\delta-s} := 0$ and $N_\delta := K_\delta \in \ker d_0^*$.

If $\delta - s \in \tilde{\Delta}$ (and $\delta \in \tilde{\Delta}$), then let us decompose H_δ along the direct sum

$$\mathcal{H}_\delta = \mathrm{Im}\, d_{0|\mathcal{H}_{\delta-s}} \overset{\perp}{\bigoplus} \ker d^*_{0|\mathcal{H}_\delta}.$$

Then, denoting π_δ the orthogonal projection onto $(\ker d_0^*)^\perp$, the cohomological equation (13) is equivalent to

$$N_\delta = (\mathrm{Id} - \pi_\delta)(K_\delta) \in \ker d^*_{0|\mathcal{H}_\delta}, \qquad d_0(U_{\delta-s}) = \pi_\delta(K_\delta) \in \mathrm{Im}\, d_{0|\mathcal{H}_{\delta-s}}.$$

Then since d_0 induces an isomorphism from $\ker(d_{0|\mathcal{H}_{\delta-s}})^\perp$ on to $\mathrm{Im}\, d_{0|\mathcal{H}_{\delta-s}}$, there exists a unique $U_{\delta-s} \in (\ker(d_{0|\mathcal{H}_{\delta-s}}))^\perp$ such that $d_0(U_{\delta-s}) = \pi_\delta(K_\delta) \in \mathrm{Im}\, d_{0|\mathcal{H}_{\delta-s}}$. □

A similar definition of the normal form of perturbation of homogeneous vector fields was given by Belitskii [Bel79, Bel82] using a different scalar product. Another definition of the normal form of perturbation of quasihomogeneous vector fields was given by Kokubu *et al* [KOW96]. It is a general scheme that provides a unique normal form. This scheme can be combined with our techniques to provide a unique normal form as well.

The perturbation of a nilpotent linear vector field has been treated by Cushman and Sanders [CS86] using $\mathfrak{sl}_2$-triple representation. Computational aspects with another definition of the normal forms in any dimension was done by Stolovitch [Sto92]. Two-dimensional aspects were initiated by Bogdanov [Bog79] and Takens [Tak74].

For very particular examples of S in dimension 2, normal forms have been obtained by Basov (see [Bas06] and references therein). When the perturbation of $S = y(\partial/\partial x)+x^2(\partial/\partial y)$ (with $p = (2, 3)$) is tangent to the germ of $x^2 = y^3$ at the origin, then a formal normal form of vector fields tangent to the 'cusp' has been devised by Loray [Lor99]. It is described in terms of a basis of the local algebra of the function $x^2 - y^3$.

Example 3.5. *Let* $S = \sum_{i=1}^{n} \lambda_i x_i(\partial/\partial x_i)$ *be a linear diagonal vector field. It is* $(1, \ldots, 1)$*-quasihomogeneous of degree* 0*. An easy computation shows that* $(ad_S)^* = ad_{\bar{S}}$*, where* $\bar{S} = \sum_{i=1}^{n} \bar{\lambda}_i x_i(\partial/\partial x_i)$*. Hence, we have*

$$\text{Ker}\,(ad_S)^* = \text{Ker}\,ad_{\bar{S}} = \text{Vect}\left\{x^Q\frac{\partial}{\partial x_i}|(Q,\lambda) = \lambda_i\right\}.$$

Example 3.6. *Let* $S = y(\partial/\partial x)$ *in* $\mathbb{C}^2$*. It is* $(1, 1)$*-quasihomogeneous of degree* 0*. The adjoint of the Lie derivative is* $\mathcal{L}^* = x(\partial/\partial y)$*; the adjoint of the Lie bracket with* S *is*

$$(ad_S)^*v = -x\frac{\partial v_1}{\partial y}\frac{\partial}{\partial x} + \left(v_1 - x\frac{\partial v_2}{\partial y}\right)\frac{\partial}{\partial y}.$$

We can show that any holomorphic (or formal) perturbation $X = S + R$ *of* S *of positive quasiorder (i.e the components of* R *are of order* > 1*) admits a formal normal form of the type:*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y + \hat{f}(x)x, \qquad (14)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \hat{f}(x)y + x\hat{g}(x)$$

for some power series $\hat{f}, \hat{g}$ *of positive order at* 0*.*

Its formal kernel is the $\mathbb{C}[[x]]$*-module generated by the radial vector field* $R = x(\partial/\partial x) + y(\partial/\partial y)$ *and* $x(\partial/\partial y)$*.*

Example 3.7. *Let us consider the case where* $S = x^2(\partial/\partial x) + xy(\partial/\partial y)$*,* $p = (1, 1)$ *and* $s = 1$*. We have* $S^* = x(\partial^2/\partial x^2) + y(\partial^2/\partial x\partial y)$*. We have*

$$(ad_S)^*\left(v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}\right) = \begin{pmatrix} S^* - 2\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} \\ 0 & S^* - \frac{\partial}{\partial x}\end{pmatrix}\begin{pmatrix} v_1 \\ v_2\end{pmatrix}.$$

We can show that any holomorphic (or formal) perturbation $X = S + R$ *of* S *of quasiorder* > 1 *(i.e the components of* R *are of order* > 2*) admits a formal normal form of the type:*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 + P_3(x, y) + x\sum_{k\geqslant 3}\frac{k+1}{k-2}f_{k+1}y^k + \hat{h}_4(y), \qquad (15)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = xy + \sum_{k\geqslant 3} f_{k+1}y^{k+1}$$

for some power series $\hat{h}_4$ of order $\geqslant 4$, some numbers f_k and some homogeneous polynomial P_3 of degree 3.

We should emphasize that the computations of the adjoint operator and so of the normal form are easiest in the homogeneous case ($p = (1, \dots, 1)$). Anyhow, the algebraic structure of the space of resonances is not clear as in the linear diagonal case.

3.3. Small divisors associated with a quasihomogeneous vector field

Let us set some notation which will be useful. Let $\delta \in \tilde{\Delta}$ such that $\delta > s$. Let us set

$$\delta_* := \frac{\min\{\delta + p_i \mid \delta + p_i \in \Delta\}}{\bar{p}} \qquad \text{and} \qquad \delta^* := \frac{\max\{\delta + p_i \mid \delta + p_i \in \Delta\}}{\underline{p}},$$

where

$$\bar{p} := \max_i p_i, \qquad \underline{p} := \min_i p_i.$$

Let us set

$$\tilde{\Delta}^- := \tilde{\Delta} \cap (\tilde{\Delta} - s), \qquad \tilde{\Delta}^+ := \tilde{\Delta} \cap (\tilde{\Delta} + s), \qquad \delta_0 := \max(\min_{\delta \in \tilde{\Delta}^-} \delta, 1).$$

The integer δ_0 is the smallest positive integer of $\tilde{\Delta}^-$.

Let us consider the *box operator*:

$$\begin{aligned} \Box : \mathcal{H}_\delta &\to \mathcal{H}_\delta \\ U &\mapsto \Box(U) = d_0(d_0^*(U)). \end{aligned}$$

It is self-adjoint and has a nonnegative real spectrum. *The 'small divisors' of the problem of conjugacy are the square roots of nonzero eigenvalues of* $\Box$.

Let us set

$$a_\delta := \min_{\lambda \in \mathrm{Spec}(\Box_\delta) \setminus \{0\}} \sqrt{\lambda}.$$

Let us define the sequence of positive real numbers $\{\eta_\delta\}_{\delta \in \tilde{\Delta}^- \cap \mathbb{N}^* \cup \{0\}}$ as follows: $\eta_0 = 1$; for any positive $\delta \in \tilde{\Delta}^-$ (i.e. $\delta \geqslant \delta_0$),

$$a_{s+\delta}\eta_\delta = \max_{s \leqslant \mu \leqslant s+\delta,\, \mu \in \tilde{\Delta}} \quad \overset{*}{\max_{\substack{\delta_1 + \cdots + \delta_r + \mu = s + \delta \\ \mu_* \leqslant r \leqslant \mu^*}}} \eta_{\delta_1} \cdots \eta_{\delta_r}, \tag{16}$$

where if $\mu = s$ then the maximum is taken over the r-tuples $(\delta_1, \dots, \delta_r)$ of nonnegative integers such that at least two of the δ_i's are positive. Moreover, the maximum is taken over the indices δ_i (respectively μ) which belongs to $\tilde{\Delta}^- \cap \mathbb{N}^* \cup \{0\}$ (respectively $\tilde{\Delta}$). It can happen that $\delta_0 = 1$.

Remark 3.8. The sequence η_δ is well defined by induction since the maximum only involves terms η_d's with $d < \delta$.

Definition 3.9. *The quasihomogeneous vector field S will called diophantine if the formal power series* $\sum_{\delta > 0,\, \delta \in \tilde{\Delta}} \eta_\delta z^\delta$ *converges in a neighbourhood of the origin in* $\mathbb{C}$*; that is to say that there exists* $c, M > 0$ *such that* $\eta_\delta \leqslant Mc^\delta$.

Remark 3.10. The diophantiness condition means that the sequence $\{a_\delta\}$ is not accumulating 0 too rapidly.

Example 3.11. *Let us return to example 3.5 where S is linear and diagonal. It is known [Sto94, lemma 2.3] that S is diophantine in the above sense if and only if it satisfies the Brjuno condition:*

$$(\omega) - \sum_{k\geqslant 0} \frac{\ln(\omega_{k+1})}{2^k} < +\infty,$$

where

$$\omega_k = \inf \left\{ |(Q,\lambda) - \lambda_i| \neq 0,\ i = 1,\ldots,n,\ Q \in \mathbb{N}^n,\ 2 \leqslant |Q| \leqslant 2^k \right\}.$$

One of our main results in this direction is the following theorem.

Theorem 3.12 (Lombardi and Stolovitch [LS09b]). *If the quasihomogeneous vector field S is diophantine and if the holomorphic good perturbation X is formally conjugate to S, then X is holomorphically conjugate to S.*

In the case where S is linear and diagonal (example 3.5), this result reduces to the Brjuno–Siegel linearization theorem.

What about holomorphic conjugacy to a normal form if the small divisors happen to be large? By this, we mean that numbers $\min_{\lambda\in\mathrm{Spec}(\Box_\delta)\setminus\{0\}} \sqrt{\lambda}$ may be pushed away to infinity instead of accumulating the origin. In this case, the convergence is helped and we have the following theorem:

Theorem 3.13 (Lombardi and Stolovitch [LS09b]). *Let us set $\nu := \max(1, \bar{p}/2)$. Assume that there exists constant $c > 0$ such that for all $\delta \in \tilde{\Delta}^+$,*

$$\min_{\lambda\in\mathrm{spec}\Box_\delta\setminus\{0\}} \sqrt{\lambda} > c^{-1}(\delta - s)^{\nu}.$$

Then, any good holomorphic perturbation X of S is holomorphically conjugate to a normal form.

Example 3.14 (Example 3.5 continued). *Assume that the linear diagonal vector field S is in the Poincaré domain. This means that the convex hull of the eigenvalues λ_i in the complex plane does not contain the origin [Arn80]. This implies that there exists a positive ε such that for all $Q \in \mathbb{N}^n$, $|Q| \geqslant 2$, we have*

$$\frac{q_1}{|Q|}\lambda_1 + \cdots + \frac{q_n}{|Q|}\lambda_n > \varepsilon.$$

Hence, if $|Q|$ is large enough, there exists a positive M such that for all $Q \in \mathbb{N}^n$, $|Q| \geqslant 2$, $|q_1\lambda_1 + \cdots + q_n\lambda_n - \lambda_i| \geqslant M|Q|$.

Example 3.15 (Example 3.6 continued). *The following result was obtained by Stróżyna–Żołądek about holomorphic conjugacy to Bogdanov–Takens normal form.*

Theorem 3.16 (Stróżyna and Żoładek [SZ02]). *Any nonlinear holomorphic perturbation $(y + f(x,y))(\partial/\partial x) + g(x,y)(\partial/\partial y)$ of $y(\partial/\partial x)$ is holomorphically conjugate to a normal form $(y + a(x))(\partial/\partial x) + b(x)(\partial/\partial y)$. Here the functions f, g, a, b vanish at 0 as well as their first derivative.*

Example 3.17 (Example 3.7 continued). *We can show, that for each $n \geqslant 3$, the one-dimensional vector space generated by $x^n(\partial/\partial y)$ is left invariant by $d_0 d_0^*$:*

$$n d_0 d_0^* \left(x^n \frac{\partial}{\partial y} \right) = n(n-2)(n-3)x^n \frac{\partial}{\partial y}.$$

For each $Q = (p, q) \in \mathbb{N}^2$ with $p \geqslant 1$, the vector subspace E_Q generated by $e_{1,Q} = x^p y^q(\partial/\partial x)$ and $e_{2,Q} = x^{p-1}y^{q+1}(\partial/\partial y)$ is invariant by $d_0 d_0^*$. Its smallest eigenvalue of its restriction to E_Q for $Q = (p, n-p)$ and $1 \leqslant p \leqslant n$ is

$$n\lambda_-(n, p) := \left(p - \frac{1}{2}\right)n^2 + \frac{5}{2}n(1-2p) + \left(6p - \frac{3}{2}\right) - (1/2)\sqrt{9 + 72p + (31 + 12p)n^2 - 30n(1-2p) - 10n^3 + n^4}.$$

We can show that, if n is large enough, there exists a constant $M > 0$ such that $\lambda_-(n, p) > Mn$.

We should emphasize that it is not an easy task either to compute or to estimate the behaviour of the spectrum.

In general, it is known that a transformation of a perturbation to a normal form usually diverges in a neighbourhood of the origin. The next theorem formalizes that: *there is a normalizing transformation which is at worst a Gevrey power series*. We recall that a formal power series $\sum_{Q\in\mathbb{N}^n} f_Q x^Q$ is said to be Gevrey of order α if there exist positive constants C, M such that $|f_Q| \leqslant MC^{|Q|}(|Q|!)^\alpha$. We have the following theorem.

Theorem 3.18 (Lombardi and Stolovitch [LS09b]). *Assume that there exists constant $c > 0$; $\tau > 0$ such that for all $\delta \in \tilde{\Delta}^+$,*

$$\min_{\lambda\in\operatorname{spec}\square_\delta\setminus\{0\}} \sqrt{\lambda} > c^{-1}(\delta - s)^{-\tau}.$$

Any good holomorphic perturbation of S admits a formal transformation to a formal normal form both of which are $\bar{p}((a/\delta_0) + \tau)$-Gevrey power series where $\delta_0 := \max(\min_{\delta\in\tilde{\Delta}^-} \delta, 1)$ and $a := \max(1, [(\bar{p} + 1)/2])$.

So, in the homogeneous case, we obtain a $(1 + \tau)$-Gevrey formal normalizing transformation and a $(1 + \tau)$-Gevrey formal normal form as well.

In the case of the 'cusp', $S = 2y(\partial/\partial x) + 3x^2(\partial/\partial y)$ a very precise study of this case with sharp estimates of the Gevrey order was done by Canalis-Durand and Schäfke [CDS04]. They even obtained more: they proved the summability of a formal transformation with respect to the first integral $y^2 - x^3$ of S.

Gramchev and Yoshino studied the cohomological equation (i.e. the linearized equation of the conjugacy equation) of a pair of commuting four-dimensional vector fields having linear part with a Jordan block in [YG08].

We have shown that this Gevrey property can be used in order to obtain an *exponentially small approximation of the flow by partial normal form*. First of all, we need to define the quasinorm, for $x \in \mathbb{C}^n$, $d_p(x) := \left(\sum_{i=1}^n p_i|x_i|^{2/p_i}\right)^{1/2}$. Then, for a complex-valued function f defined in a neighbourhood of the twisted ball $d_p(x) < \epsilon$ we shall set

$$|f|_{qh,\epsilon} := \sup_{d_p(x)<\epsilon} |f(x)|.$$

If X is a vector field defined in a neighbourhood of the 'twisted ball' $d_p(x) < \epsilon$, we shall set

$$\|X\|^2_{qh,\epsilon} := \sum_{i=1}^n \frac{1}{\epsilon^{2p_i}}|X_i|^2_{qh,\epsilon}.$$

The subscript *qh* stands for *quasihomogeneous* as these norms are adapted to quasihomogeneous objects.

Theorem 3.19. *Let S be a p-quasihomogeneous vector field of $\mathbb{C}^n$. Let $X := S + R$ be a good holomorphic perturbation of S in a neighbourhood of the origin of $\mathbb{C}^n$. Proposition 3.4 ensures that for every $\alpha \in \tilde{\Delta}$, there exists a polynomial diffeomorphism tangent to identity* $\Phi_\alpha^{-1} = \mathrm{Id} + \mathcal{U}_\alpha$, *where* $\mathcal{U}_\alpha = \sum_{0<\delta\leqslant\alpha-s} U_\delta$, with $U_\delta \in \mathcal{H}_\delta$ *such that*

$$(\Phi_\alpha)_*(X) = S + \mathcal{N}_\alpha + \mathcal{R}_{>\alpha},$$

where $\mathcal{N}_\alpha = \sum_{s<\delta\leqslant\alpha} N_\delta$, $N_\delta \in \ker \square_\delta$, *and where* $\mathcal{R}_{>\alpha}$ *is of quasiorder* $> \alpha$.

Assume that there exist $c \geqslant 1$ and $\tau > 0$ such that for every $\delta \in \tilde{\Delta}$ with $\delta \geqslant s$, we have

$$\frac{1}{(\delta - s)^\tau} \leqslant c \min_{\lambda \in \mathrm{spec}\square_\delta \setminus \{0\}} \sqrt{\lambda}. \tag{17}$$

Then, there exists $\theta \geqslant 4$, $M_{\mathrm{opt}} > 0$, $w_{\mathrm{opt}} > 0$ and $\varepsilon_0 > 0$ such that for every $\epsilon \in]0, \epsilon_0[$, $\alpha_{\mathrm{opt}} = \left[1/(\theta C \epsilon)^b\right] + s - 2$ *satisfies* $\alpha_{\mathrm{opt}} > s$ *and* $\alpha_{\mathrm{opt}} \geqslant \delta_*$, *and*

$$\|\mathcal{R}_{>\alpha_{\mathrm{opt}}}\|_{qh,\epsilon} \leqslant M_{\mathrm{opt}} \mathrm{e}^{-\frac{w_{\mathrm{opt}}}{\epsilon^b}}, \tag{18}$$

where $(1/b) = \tau + (a/\delta_0)$.

In the homogeneous case, $(1/b) = \tau + 1$. Hence, in a twisted ball of radius ε, the partial normal form gives an exponentially good approximation of the vector field. This result can be regarded as a Nekhoroshev type theorem.

References

[AA88] Anosov D V and Arnol′d V I (ed) 1988 Ordinary differential equations and smooth dynamical systems *Dynamical Systems: I* (*Encyclopaedia of Mathematical Sciences* vol 1) (Berlin: Springer) (Translated from the Russian)

[Arn63a] Arnold V I 1963 Proof of a theorem by A.N. Kolmogorov on the persistence of quasi-periodic motions under small perturbations of the hamiltonian. *Russ. Math. Surv.* **18** 9–36

[Arn63b] Arnold V I 1963 Small denominators and the problem of stability of motion in the classical and celestian mechanics *Russ. Math. Surv.* **18** 85–191

[Arn80] Arnold V I 1980 *Chapitres Supplémentaires de la Théorie des Équations Différentielles Ordinaires* (Paris: Mir)

[Arn88a] Arnol′d V I 1988 *Geometrical Methods in the Theory of Ordinary Differential Equations* (*Grundlehren der Mathematischen Wissenschaften* vol 250) 2nd edn (Berlin: Springer)

[Arn88b] Arnold V I (ed) 1988 *Dynamical Systems III* (*Encyclopaedia of Mathematical Sciences* vol 28) (Berlin: Springer)

[Bas06] Basov V V 2006 The generalized normal form and the formal equivalence of two-dimensional systems with zero quadratic approximation: III *Diff. Uravn.* **42** 308–19 429

[Bel79] Belitskii G R 1979 Invariant normal forms of formal series *Funct. Anal. Appl.* **13** 46–7

[Bel82] Belitskii G R 1982 *Normal Forms Relative to a Filtering Action of a Group* (*Transactions of the Moscow Mathematical Society* 1981) (Providence, RI: American Mathematical Society) pp 1–39
Belitskii G R 1979 *Trudy Moskov. Mat. Obšč* **40** 3–46 (Transl.)
Belitskii G R 1982 *Trans. Moscow Math. Soc.* **2**

[BHS96] Broer H W, Huitema G W and Sevryuk M B 1996 Quasi-periodic motions in famillies of dynamical systems (*Lecture Notes in Mathematics* vol 1645) (Berlin: Springer)

[Bog79] Bogdanov R I 1979 Local orbital normal forms of vector fields on the plane *Trudy Sem. Petrovsk.* **5** 51–84

[Bos86] Bost J-B 1986 Tores invariants des systèmes dynamiques hamiltoniens (d'après Kolomogorov, Arnol'd, Moser, Rüssmann, Zehnder, Herman, Pöschel,...) *Séminaire Bourbaki* (*Astérisque* vol 133–34) (*Société Mathématiques de France* pp 113–57 exposé 639

[Bru72] Bruno A D 1971 Analytical form of differential equations *Trans. Mosc. Math. Soc* **25** 131-288
Bruno A D 1972 *Trans. Mosc. Math. Soc* **26** 199-239

[Bry88] Bryuno A D 1988 The normal form of a Hamiltonian system *Usp. Mat. Nauk* **43** 23–56, 247

[BS07] Braaksma B and Stolovitch L 2007 Small divisors and large multipliers *Ann. Inst. Fourier (Grenoble)* **57** 603–28

[CDS04] Canalis-Durand M and Schäfke R 2004 Divergence and summability of normal forms of systems of differential equations with nilpotent linear part *Ann. Fac. Sci. Toulouse Math. (6)* **13** 493–513

[Cha86] Chaperon M 1986 Géométrie différentielle et singularités de systèmes dynamiques *Astérisque* **138–139**

[Cha07] Chaperon M 2007 Birth control in generalized Hopf bifurcations *C. R. Math. Acad. Sci. Paris* **345** 453–8

[CKCDVLDM06] Chaperon M, Kammerer-Colin De Verdière M and López De Medrano S 2006 More compact invariant manifolds appearing in the non-linear coupling of oscillators *C. R. Math. Acad. Sci. Paris* **342** 301–05

[CLDM08] Chaperon M and López De Medrano S 2008 Birth of attracting compact invariant submanifolds diffeomorphic to moment-angle manifolds in generic families of dynamics *C. R. Math. Acad. Sci. Paris* **346** 1099–102

[CS86] Cushman R and Sanders J A 1986 Nilpotent normal forms and representation theory of $\mathrm{sl}(2, \boldsymbol{R})$ *Multiparameter Bifurcation Theory (Arcata, CA, 1985)*, (*Contemporary Mathematics* vol 56) (Providence, RI: American Mathematical Society) pp 31–51

[DR80] Dumortier F and Roussarie R 1980 Smooth linearization of germs of $\mathbb{R}^2$-actions and holomorphic vector fields *Ann. Inst. Fourier, Grenoble* **30** 31–64

[Eca] Ecalle J 1981 Sur les fonctions résurgentes *Publ. Math. d'Orsay* **I, II**; 1985 **III**

[Eca92] Ecalle J 1992 Singularités non abordables par la géométrie *Ann. Inst. Fourier (Grenoble)* **42** 1–2
Ecalle J 1992 *Ann. Inst. Fourier (Grenoble)* **42** 73–164

[Fis17] Fischer E 1917 Über die Differentiationsprozesse der Algebra *J. für Math.* **148** 1–78

[Il′93] Il′yashenko Yu S (ed) 1993 *Nonlinear Stokes Phenomena* (*Advances in Soviet Mathematics* vol 14) (Providence, RI: American Mathematical Society)

[IL04] Iooss G and Lombardi E 2004 Normal forms with exponentially small remainder: application to homoclinic connections for the reversible $0^{2+}\mathrm{i}\omega$ resonance *C. R. Math. Acad. Sci. Paris* **339** 831–8

[IL05] Iooss G and Lombardi E 2005 Polynomial normal forms with exponentially small remainder for analytic vector fields *J. Diff. Eqns* **212** 1–61

[Ito89] Ito H 1989 Convergence of birkhoff normal forms for integrable systems *Comment. Math. Helv.* **64** 412–61

[Kol54] Kolmogorov A N 1954 On the preservation of conditionally periodic motions under small variations of the hamilton function *Dokl. Akad. Nauk SSSR* **98** 527–30 (English translation in 'Selected Works')

[Kol57] Kolmogorov A N 1954 The general theory of dynamical systems and classical mechanics *Proc. Int. Congress of Mathematicians (Amsterdam, Netherlands, 1954)* vol 1, pp 315–33 (Amsterdam: North-Holland) (English translation in 'Collected Works') (Dordrecht: Kluwer)

[KOW96] Kokubu H, Oka H and Wang D 1996 Linear grading function and further reduction of normal forms *J. Diff. Eqns* **132** 293–318

[Lor99] Loray F 1999 Réduction formelle des singularités cuspidales de champs de vecteurs analytiques *J. Diff. Eqns* **158** 152–73

[LS09a] Lombardi E and Stolovitch L 2009 Forme normale de perturbation de champs de vecteurs quasi-homogènes *C.R. Acad. Sci, Paris, Série I* **347** 143–6

[LS09b] Lombardi E and Stolovitch L 2009 Normal forms of analytic perturbations of quasihomogeneous vector fields: Rigidity, analytic invariants sets and exponentially small approximation, unpublished

[Mal82] Malgrange B 1982 Travaux d'Ecalle et de Martinet-Ramis sur les systèmes dynamiques *Séminaire Bourbaki 1981-1982*, exposé 582

[Mos62] Moser J 1962 On invariant curves of aera-preserving mappings of an annulus *Nachr. Akad. Wiss. Göttett. Math.-Phys. Kl. II* **1** 1–20

[Mos73] Moser J 1973 Stable and random motions in dynamical systems, with special emphasis on celestian mechanics (*Annals of Mathematics Studies* vol 77) (Princeton, NJ: Princeton University Press)

[Mos90] Moser J 1990 On commuting circle mappings and simultaneous diophantine approximations *Math. Z.* **205** 105–21

[MR82] Martinet J and Ramis J-P 1982 Problèmes de modules pour des équations différentielles non linéaires du premier ordre *Publ. Math. Inst. Hautes Études Sci.* **55** 63–164

[MR83] Martinet J and Ramis J-P 1983 Classification analytique des équations différentielles non linéaires résonantes du premier ordre *Ann. Sci. Ecole Norm. sup.*, 4ème série **16** 571-621

[Mur03] Murdock J 2003 *Normal Forms and Unfoldings for Local Dynamical Systems (Springer Monographs in Mathematics)* (New York: Springer)

[Rüs01] Rüssmann H 2001 Invariant tori in non-degenerate nearly integrable Hamiltonian systems *Regular Chaotic Dyn.* **6** 119–204

[Sha89] Shapiro H S 1989 An algebraic theorem of E. Fischer, and the holomorphic Goursat problem *Bull. London Math. Soc.* **21** 513–37

[Sto92] Stolovitch L 1992 Sur les formes normales de systèmes nilpotents *C.R. Acad. Sci.* **314** 355–8

[Sto94] Stolovitch L 1994 Sur un théorème de Dulac *Ann. Inst. Fourier* **44** 1397–433

[Sto96] Stolovitch L 1996 Classification analytique de champs de vecteurs 1-résonnants de $(\mathbb{C}^n, 0)$ *Asymptotic Anal.* **12** 91–143

[Sto00] Stolovitch L 2000 Singular complete integrabilty *Publ. Math. Inst. Hautes Études Sci.* **91** 133–210

[Sto05a] Stolovitch L 2005 A KAM phenomenon for singular holomorphic vector fields *Publ. Math. Inst. Hautes Études Sci.* **102** 99–165

[Sto05b] Stolovitch L 2005 Normalisation holomorphe d'algèbres de type Cartan de champs de vecteurs holomorphes singuliers *Ann. Math.* **161** 589–612

[Sto08] Stolovitch L 2008 Normal Forms of holomorphic dynamical systems W Craig (ed) *Hamiltonian Dynamical Systems and Applications* (Berlin: Springer) pp 249–84

[SZ02] Stróżyna E and Żoładek H 2002 The analytic and formal normal form for the nilpotent singularity *J. Diff. Eqns* **179** 479–537

[Tak74] Takens F 1974 Singularities of vector fields *Inst. Hautes Études Sci. Publ. Math.* **43** 47–100

[Vey78] Vey J 1978 Sur certains systèmes dynamiques séparables *Am. J. Math.* **100** 591–614

[Vey79] Vey J 1979 Algèbres commutatives de champs de vecteurs isochores *Bull. Soc. Math. France* **107** 423–32

[Vor81] Voronin S M 1981 Analytic classification of germs of conformal mappings $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ with identity linear part *Funct. Anal. Appl.* **15** 1–17

[YG08] Yoshino M and Gramchev T 2008 Simultaneous reduction to normal forms of commuting singular vector fields with linear parts having Jordan blocks *Ann. Inst. Fourier (Grenoble)* **58** 263–97

[Zun05] Zung N T 2005 Convergence versus integrability in Birkhoff normal form *Ann. of Math. (2)* **161** 141–56