Correspondence, Canonicity, and Model Theory for Monotonic Modal Logics

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TACL 2019

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Background: Fine's Theorem

In the early 1970s, every normal modal logic L known was either

- determined by an elementary class of Kripke frames and canonical, i.e., L is valid in the ultrafilter frame of the Lindanbaum-Tarski algebra of L, or
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Fine gave an explanation to this empirical fact:

Theorem (Fine)
Suppose a normal modal logic L is determined by an elementary class of Kripke frames. Then L is canonical.

Background: Monotonic Modal Logics

Our goal is to extend Fine's theorem for monotonic modal logics, which generalize normal modal logics by dropping the K axiom

$$\Box(p \to q) \to (\Box p \to \Box q)$$

and instead requiring only that

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Monotonic modal logics arise in the following:

Game theory $\Box\phi$ means "agents can force ϕ to be true".

Probability theory $\Box\phi$ means "the probability of ϕ holding is greater than p" for some $p \in [0, 1]$.

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In our result, a logic called coalgebraic predicate logic proposed by Chang (1973) plays the rôle of first-order logic.

Frames for Monotonic Modal Logics

Definition

A monotonic neighborhood frame is a pair (F, N^F) of a set F and a neighborhood function $N^F : F \to \mathscr{P}(\mathscr{P}(F))$ s.t. for every $w \in F$ the family $N^F(w)$ is closed under supersets. A member of $N^F(w)$ is a neighborhood of w.

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For a monotonic neighborhood model $M = (F, N^F, P_0, P_1, \dots)$ and $w \in F$:

$$M, w \Vdash_{\mathsf{nbhd}} \Box\phi \iff \{w \in M \mid w \Vdash_{\mathsf{nbhd}} \phi\} \in N^F(w).$$

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- There is an embedding ST from modal logic into CPL s.t. for a nbhd model M and $w \in M$:

$$M, w \Vdash_{\mathsf{nbhd}} \phi \iff (M, w) \models \mathrm{ST}(\phi).$$

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- The image of ST can be characterized à la van Benthem (Litak, Pattinson, and Schröder).

Syntax of CPL

Definition (Chang; Litak et al.)

The language L of coalgebraic predicate logic based on a language L_0 of f.-o. logic has atomic formulas of L_0 and is closed under

- Boolean combinations,
- existential quantification, and
- formation of formulas of the form

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$$F \models w \Box_y \phi(y) \iff \{v \in F \mid F \models \phi(v)\} \in N^F(w).$$

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The language $L_=$ of nbhd frames is based on $\varnothing$. The language of nbhd models is based on $\{P_0, P_1, \ldots\}$.

A Topological Example: Topological Neighborhood Frames

For a topological space $X = (X, \tau)$, we associate a monotonic neighborhood frame $X^* = (X, N)$ defined by

$$U \in N(w) \iff w \in U^\circ.$$

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Note the relationship between this setting and topological semantics:

$$(X, P_0, P_1, \dots), w \Vdash_{\mathsf{top}} \phi \iff (X^*, P_0, P_1, \dots), w \Vdash_{\mathsf{nbhd}} \phi.$$

A Topological Example: What Can One Say in CPL?

The specialization preorder of X is the preorder $\lesssim$ on X defined by $x \lesssim y \iff x \in \overline{\{y\}}$. It is "definable" in $L_=$:

$$\begin{aligned} x \lesssim y &\iff x \notin (-\{y\})^\circ \\ &\iff X^* \models \neg(x \,\Box_z\, z \neq y). \end{aligned}$$

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Hence, there is an $L_=$-sentence ϕ s.t. for topological spaces X

$$X^* \models \phi \iff X \text{ is } \mathrm{T}_0$$

i.e., the *-image of the class of T_0 spaces is CPL-elementary relative to the class of topological neighborhood frames.

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i.e., the *-image of the class of T_0 spaces is CPL-elementary relative to the class of topological neighborhood frames. The same goes for T_1 spaces.

Modal-Logical Examples

For a modal formula α of the form

$$\langle \text{purely propositional positive formula} \rangle \to \langle \text{positive formula} \rangle \quad (1)$$

there exists an $L_=$-sentence ϕ (a correspondent of α) s.t. for monotonic neighborhood frames F

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It follows from Hansen's result that such an α axiomatizes a complete monotonic modal logic.

BAMs

Definition

A monotonic Boolean algebra expansion (BAM) $(B, \Box)$ is a Boolean algebra B expanded with an operation $\Box : B \to B$ that is monotonic, i.e., for $x, y \in B$, $\Box x \leq \Box y$ whenever $x \leq y$.

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Definition

The underlying BAM F^+ of a monotonic neighborhood frame F is the BAM $(\mathscr{P}(F), \Box^F)$, where

$$\Box^F(X) = \{w \in F \mid X \in N^F(w)\}$$

Canonicity

Definition

The ultrafilter frame $\mathrm{Uf}(A)$ of A is a frame $(\mathrm{Uf}(A), N^\sigma)$ where

$$U \in N^\sigma(u) \iff \exists X \subseteq U \, \forall a \in A([a] \supseteq X \Rightarrow \Box(a) \in u),$$

where $\mathrm{Uf}(A)$ is the set of ultrafilters of the Boolean reduct $A|_{\mathsf{Bool}}$ of A, $u \in \mathrm{Uf}(A)$, and X ranges over closed subsets of the Stone space of $A|_{\mathsf{Bool}}$.

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A monotonic modal logic is canonical if it is valid in the ultrafilter frame of the Lindenbaum-Tarski algebra of L.

Result

Theorem

Suppose a monotonic modal logic L is determined by a class of CPL-elementary relative to the classes of monotonic neighborhood frames. Then L is canonical.

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There are several other classes relative to which $\mathcal{K}$ can be CPL-elementary for the result to still obtain (e.g., the class of topological neighborhood frames).

A Consequence

Reconsider an arbitrary modal formula α of the form

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and a correspondent ϕ of α:

$$\mathcal{K} := \{F \mid F \Vdash_{\mathsf{nbhd}} \alpha\} = \{F \mid F \models \phi\}.$$

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By the Theorem, such a logic is canonical.

Proof of the Theorem

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For a monotonic neighborhood frame F, there exists another G s.t.

- *F and G satisfies the same $L_=$ sentences and*
- *there is a surjective bounded morphism $G \twoheadrightarrow \mathrm{Uf}(F^+)$, i.e., for each $w \in G$:*

$$f^{-1}(U') \in N^G(w) \implies U' \in N^\sigma(f(w)) \qquad (\text{“forth”})$$

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One can impose more closure conditions on F and G; e.g., if F is a topological neighborhood frame, so is G.

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5. Show that the surjective function that assigns to each $w \in G$ the "type" realized by w is a bounded morphism. □

Open Questions

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- Are there other classes of monotonic neighborhood frames relative to which similar results hold? Can we characterize such classes?