Intuitionistic and classical non-normal modal logics: An embedding

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Outline

- Motivations
- Intuitionistic non-normal modal logics and sequent calculi
- Neighbourhood semantics
- Embedding into classical non-normal multimodal logics
- Conclusions and future work

Why non-normal modalities

Possible interpretations of $\Box$ and $\Diamond$ incompatible with some theorems of K.

- Deontic logic
- Epistemic logic
- Logic of high probability
- Logic of ability
- Logics of knowledge and belief
- Logic of classical deduction
- Logic of group decision making
- ...

Why non-normal modalities

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- Logical omniscience: If A is valid, then the agent knows that A.

Classical non-normal modal logics (equivalent definitions)

$$A := p \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \mid \Diamond A.$$

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$\Box$ primitive operator:

- Basic system E: CPL $+$ $\Diamond A \supset\subset \neg\Box\neg A$ $+$ $\dfrac{A \supset B \qquad B \supset A}{\Box A \supset \Box B}$
- Extensions: by adding any combination of

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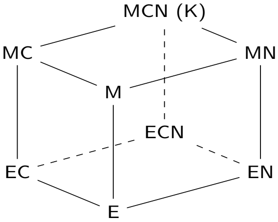
$$\Diamond A \supset \Diamond(A \vee B) \qquad \Diamond(A \vee B) \supset \Diamond A \vee \Diamond B \qquad \neg\Diamond\bot$$

Eight distinct systems



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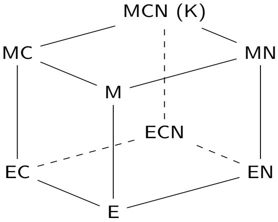

Some properties:

- Sound and complete w.r.t. neighbourhood semantics.
- Decidable.
- NP complexity without C, PSPACE with C.

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- Motivated by applications to computer science.
- Type-theoretic interpretations.
- Verification.
- Knowledge representation.

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Practical distinction: Constructive modal logics do not satisfy $\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$.

$\Rightarrow$ $\Diamond$ is non-normal.

Some intuitionistic non-normal modal logics

Wijesekera's Constructive Concurrent Dynamic Logic (Wijesekera 1990).

- First order logic for the analysis of concurrent programs.
- $\Box A$: "after any execution of the program, A holds".
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Constructive K (Bellin et al. 2001, Mendler & de Paiva 2005).

- Intuitionistic version of logic K for contextual reasoning.
- $\Box A$: "A holds in all contexts"; $\Diamond A$: "A holds in some context".
- $\Box$ is normal and $\Diamond$ is non-normal.
- $\neg\Diamond\bot$ is not valid.

Our aim

State of the art

- Study of specific systems for special motivations.
- No general study of non-normal modalities in intuitionistic/constructive context

  vs. intuitionistic normal modal logics (Simpson 1994) and
  classical non-normal modal logics (Chellas 1980).

Our aim

- General theory of intuitionistic non-normal modal logics.
- Define intuitionistic counterparts of classical non-normal modal logics.

Our approach

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General requirements:

- Intuitionistic modal logics should be conservative over IPL.
- They should satisfy the disjunction property: If $A \vee B$ is derivable, then A is derivable or B is derivable.
- $\Box$ and $\Diamond$ should not be interdefinable.
- They should not contain $\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$.

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$$\neg(\Box\top \wedge \Diamond\bot) \qquad \neg(\Diamond\top \wedge \Box\bot)$$

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(ii) A is equivalent to $\neg B$, or B is equivalent to $\neg A$:

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(iii) A and B are jointly inconsistent:

$$\frac{\neg(A \wedge B)}{\neg(\Box A \wedge \Diamond B)}$$

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$$\text{Cut}\ \frac{\Gamma \Rightarrow A \qquad A, \Gamma \Rightarrow B}{\Gamma \Rightarrow B}$$

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$\Rightarrow$ Not all combinations of the considered rules are accepted.

Sequent calculi

- Sequents: $\Gamma \Rightarrow C$ or $\Gamma \Rightarrow$ C formula, Γ finite multiset of formulas.
- Base calculus for IPL: G3ip.
- Modal rules:

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Rules for $\Box$: (generalised to n principal formulas for $\mathsf{C}_\Box$)

$$\mathsf{E}^{\mathsf{s}}_{\Box}\ \frac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B} \qquad \mathsf{M}^{\mathsf{s}}_{\Box}\ \frac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B} \qquad \mathsf{N}^{\mathsf{s}}_{\Box}\ \frac{\Rightarrow A}{\Gamma \Rightarrow \Box A}$$

Rules for $\Diamond$:

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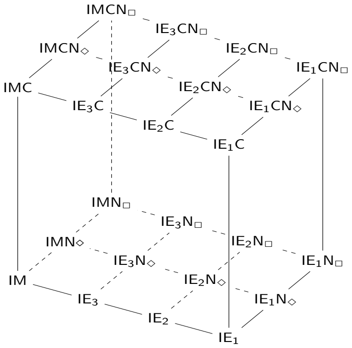
Interaction rules:

$$\mathrm{weak}^{\mathrm{s}}_{\mathrm{a}}\ \frac{\Rightarrow A \qquad B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow C} \qquad \mathrm{weak}^{\mathrm{s}}_{\mathrm{b}}\ \frac{A \Rightarrow \qquad \Rightarrow B}{\Gamma, \Box A, \Diamond B \Rightarrow C}$$

$$\mathrm{neg}^{\mathrm{s}}_{\mathrm{a}}\ \frac{A, B \Rightarrow \qquad \neg A \Rightarrow B}{\Gamma, \Box A, \Diamond B \Rightarrow C} \qquad \mathrm{neg}^{\mathrm{s}}_{\mathrm{b}}\ \frac{A, B \Rightarrow \qquad \neg B \Rightarrow A}{\Gamma, \Box A, \Diamond B \Rightarrow C}$$

$$\mathrm{str}^{\mathrm{s}}\ \frac{A, B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow C}$$

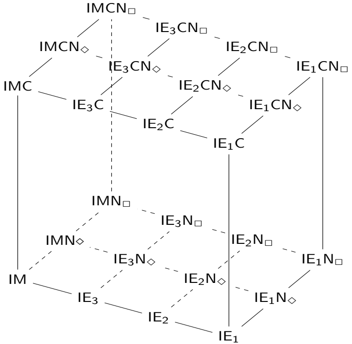
Cut-free calculi



24 cut-free calculi

- $\Box$ and $\Diamond$ either both monotonic or both non-monotonic.
- Monotonicity compatible only with the strongest interaction.
- If a calculus contains $N^s_\Box$, then it also contains $N^s_\Diamond$.
- Counterexamples to cut-elimination for the other cases.

Cut-free calculi



Some properties (proved by means of analytic sequent calculi)

- Definition of 24 different non-normal modal logics (no equivalences).
- All logics satisfy the initial requirements.
- Decidability.
- Craig interpolation (for most of them).

Hilbert systems

Hilbert systems defined as extensions of IPL.

Three basic non-monotonic systems:

$$\mathsf{IE}_1 \quad := \quad \mathsf{E}_\Box, \mathsf{E}_\Diamond \quad + \quad \neg(\Box\top \wedge \Diamond\bot) \qquad \neg(\Diamond\top \wedge \Box\bot)$$

$$\mathsf{IE}_2 \quad := \quad \mathsf{E}_\Box, \mathsf{E}_\Diamond \quad + \quad \neg(\Box A \wedge \Diamond\neg A) \qquad \neg(\Box\neg A \wedge \Diamond A)$$

$$\mathsf{IE}_3 \quad := \quad \mathsf{E}_\Box, \mathsf{E}_\Diamond \quad + \quad \frac{\neg(A \wedge B)}{\neg(\Box A \wedge \Diamond B)}$$

One basic monotonic system:

$$\mathsf{IE}_3 \quad := \quad \mathsf{E}_\Box, \mathsf{E}_\Diamond \quad + \quad \mathsf{M}_\Box, \mathsf{M}_\Diamond \quad + \quad \frac{\neg(A \wedge B)}{\neg(\Box A \wedge \Diamond B)}$$

Extensions:

To each basic system we can add $\mathsf{N}_\Diamond$, $\mathsf{N}_\Box$ or $\mathsf{C}_\Box$.

Theorem

The system L is equivalent to the sequent calculus G.L.

Philosophical remarks

Different intuitionistic logics corresponding to the same classical logics

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Some intuitionistic logics are not counterpart of any classical logic.

- IE_3 does not correspond to classical E.
  - $\dfrac{\neg(A \land B)}{\neg(\Box A \land \Diamond B)}$ is not derivable in E (classically equivalent to $\dfrac{A \supset B}{\Box A \supset \Box B}$).
- IE_3 does not correspond to classical M.
  - Axioms $M_\Box$ and $M_\Diamond$ are not derivable in IE_3.

Semantics

Our semantics for intuitionistic non-normal modal logics

Neighbourhood models for non-normal modal logics
$\langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$

Kripke models for intuitionistic logic
$\langle \mathcal{W}, \preceq, \mathcal{V} \rangle$

Models for intuitionistic non-normal modal logics
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- Preorder $\preceq$ dealing with $\supset$.
- Two neighbourhood functions $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ handling the modalities separately.
- Different connections between $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ corresponding to the interaction axioms.

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- Different connections between $\mathcal{N}_{\Box}$ and $\mathcal{N}_{\Diamond}$ corresponding to the interaction axioms.
- The combination must preserve the hereditary property.

Definition (Coupled Intuitionistic Neighbourhood Models) (CINMs)

$\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$, where

- $\mathcal{W}$ is a non-empty set;
- $\preceq$ is a preorder over $\mathcal{W}$;
- $\mathcal{V}$ is a valuation function $\mathcal{W} \longrightarrow Atm$ s.t. $w \preceq v$ implies $\mathcal{V}(w) \subseteq \mathcal{V}(v)$;
- $\mathcal{N}_\Box$, $\mathcal{N}_\Diamond$ are two neighbourhood functions $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$ s.t.

$$w \preceq v \text{ implies } \mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Box(v);$$
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$$\begin{array}{lll} w \preceq v & \text{implies} & \mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Box(v); \\ w \preceq v & \text{implies} & \mathcal{N}_\Diamond(w) \supseteq \mathcal{N}_\Diamond(v). \end{array}$$

The forcing relation $w \Vdash A$. Standard for p, $\bot$, $B \wedge C$, $B \vee C$.

$$\begin{array}{lll} w \Vdash B \supset C & \text{iff} & \text{for all } v \succeq w,\ v \Vdash B \text{ implies } v \Vdash C; \\ w \Vdash \Box B & \text{iff} & [B] \in \mathcal{N}_\Box(w); \\ w \Vdash \Diamond B & \text{iff} & \mathcal{W} \setminus [B] \notin \mathcal{N}_\Diamond(w). \end{array}$$

Semantic conditions associated to the axioms

Conditions connecting $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$: $-\alpha = \{w \in \mathcal{W} \mid \text{for all } v \succeq w, v \notin \alpha\}$.

weak$_a$ + weak$_b$	$\mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Diamond(w)$
neg$_a$	If $\alpha \in \mathcal{N}_\Box(w)$, then $\mathcal{W} \setminus -\alpha \in \mathcal{N}_\Diamond(w)$
neg$_b$	If $-\alpha \in \mathcal{N}_\Box(w)$, then $\mathcal{W} \setminus \alpha \in \mathcal{N}_\Diamond(w)$
str	If $\alpha \in \mathcal{N}_\Box(w)$ and $\alpha \subseteq \beta$, then $\beta \in \mathcal{N}_\Diamond(w)$

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Conditions for $\mathcal{N}_\Box$ or $\mathcal{N}_\Diamond$:

M$_\Box$	If $\alpha \in \mathcal{N}_\Box(w)$ and $\alpha \subseteq \beta$, then $\beta \in \mathcal{N}_\Box(w)$
M$_\Diamond$	If $\alpha \in \mathcal{N}_\Diamond(w)$ and $\alpha \subseteq \beta$, then $\beta \in \mathcal{N}_\Diamond(w)$
N$_\Box$	$\mathcal{W} \in \mathcal{N}_\Box(w)$
N$_\Diamond$	$\mathcal{W} \in \mathcal{N}_\Diamond(w)$
C$_\Box$	If $\alpha, \beta \in \mathcal{N}_\Box(w)$, then $\alpha \cap \beta \in \mathcal{N}_\Box(w)$

Cf. neighbourhood semantics for the classical cube (Chellas 1980).

Results

Related works

- The reducts without $\mathcal{N}_\diamond$ coincide essentially with Goldblatt's neighbourhood spaces (Goldblatt 1981).
- (Kojima 2012). A different neighbourhood semantics for Wijesekera's CCDL.
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- Decidability (from FMP).

CINMs extended to CCDL and CK by considering the property

$$\text{If } \alpha \in \mathcal{N}_\Box(w) \text{ and } \beta \in \mathcal{N}_\diamond(w)\text{, then } \alpha \cap \beta \in \mathcal{N}_\diamond(w).$$

- Soundness, completeness and FNP.
- Model transformations between neighbourhood and relational semantics.
- Advantage: for CK a natural semantics without fallible worlds ($w \Vdash \bot$).

Embedding into classical non-normal multimodal logics

Embedding

Previous work

- E. of intuitionistic normal modal logics into classical normal multimodal logics (Fischer Servi 1980, Wolter & Zakharyaschev 1999).
- E. of classical non-normal modal logics into classical normal multimodal logics (Kracht & Wolter 1999, Gasquet & Herzig 1996)

We show

- Embedding of intuitionistic non-normal modal logics into classical non-normal multimodal logics.

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We show

- Embedding of intuitionistic non-normal modal logics into classical non-normal multimodal logics.

Classical multimodal logics of the form (S4, cL_2, cL_3)

S4	$\boxed{\leq}$	$\Diamond_{\leq}$
cL_2	$\boxed{\Box}$	$\Diamond_{\Box}$
cL_3	$\boxed{\Diamond}$	$\Diamond_{\Diamond}$

cL_2 and cL_3 are classical non-normal modal logics.

$$\mathcal{L}_3: \quad A ::= p \mid \bot \mid A \land A \mid A \lor A \mid A \supset A \mid \boxed{\leq} A \mid \boxed{\Box} A \mid \boxed{\Diamond} A.$$

Let L be an intuitionistic non-normal modal logic.
e(L) is the classical multimodal logic (S4, cL_2, cL_3) in $\mathcal{L}_3$ where:

Characteristic axioms

- cL_2 contains the $\Box$-axioms of L.
  - e.g. if $\Box A \land \Box B \supset \Box(A \land B) \in$ L, then $\boxed{\Box} A \land \boxed{\Box} B \supset \boxed{\Box}(A \land B) \in \mathrm{cL}_2$.
- cL_3 contains the $\Diamond$-axioms of L.
  - e.g. if $\neg\Diamond\bot \in$ L, then $\neg\boxed{\Diamond}\bot \in \mathrm{cL}_3$.

Embedding

Connecting axioms

- For all logics

  $(\boxed{\leq}\,;\boxed{\Box})$:

  $(\boxed{\leq}\,;\boxed{\Diamond})$:

- For non-monotonic logics

  $(\boxed{\Box}\,;\boxed{\Diamond})$:

- For monotonic logics

  $(\boxed{\Box}\,;\boxed{\Diamond})$:

Embedding

Connecting axioms

- For all logics

  $(\boxed{\leq}\,;\boxed{\Box})$: $\quad \boxed{\Box} A \rightarrow \boxed{\leq}\boxed{\Box} A$

  $(\boxed{\leq}\,;\boxed{\Diamond})$: $\quad \boxed{\Diamond} A \rightarrow \boxed{\leq}\boxed{\Diamond} A$

- For non-monotonic logics

  $(\boxed{\Box}\,;\boxed{\Diamond})$:

- For monotonic logics

  $(\boxed{\Box}\,;\boxed{\Diamond})$:

Embedding

Connecting axioms

- For all logics

  $(\boxed{\leq}\,;\boxed{\Box})$: $\quad \boxed{\Box}A \to \boxed{\leq}\boxed{\Box}A$

  $(\boxed{\leq}\,;\boxed{\Diamond})$: $\quad \Diamond\!\!\!\!\diamond\, A \to \boxed{\leq}\,\Diamond\!\!\!\!\diamond\, A$

- For non-monotonic logics

  $(\boxed{\Box}\,;\boxed{\Diamond})$: depend on the interactions of L:

  weak$_a$ and weak$_b$: $\boxed{\Box}A \to \boxed{\Diamond}A$

  neg$_a$ and neg$_b$: $\boxed{\Box}A \to \boxed{\Diamond}\,\Diamond\!\!\!\!\leq\, A$ and $\boxed{\Box}\boxed{\leq}A \to \boxed{\Diamond}A$

  str: $\dfrac{A \to B}{\boxed{\Box}A \to \boxed{\Diamond}B}$

- For monotonic logics

  $(\boxed{\Box}\,;\boxed{\Diamond})$:

Embedding

Connecting axioms

- For all logics

  $(\boxed{\leq}\,;\boxed{\Box})$: $\quad \boxed{\Box}A \to \boxed{\leq}\boxed{\Box}A$

  $(\boxed{\leq}\,;\boxed{\Diamond})$: $\quad \langle\Diamond\rangle A \to \boxed{\leq}\langle\Diamond\rangle A$

- For non-monotonic logics

  $(\boxed{\Box}\,;\boxed{\Diamond})$: depend on the interactions of L:

weak_a and weak_b:	$\boxed{\Box}A \to \boxed{\Diamond}A$
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- For monotonic logics

  $(\boxed{\Box}\,;\boxed{\Diamond})$: $\quad \boxed{\Box}A \to \boxed{\Diamond}A$

Embedding

Translation $\dagger$: $\mathcal{L} \longrightarrow \mathcal{L}_3$:

$$p^\dagger = \boxed{\leq} p \qquad \bot^\dagger = \bot \qquad (A \wedge B)^\dagger = A^\dagger \wedge B^\dagger \qquad (A \vee B)^\dagger = A^\dagger \vee B^\dagger$$

$$(A \supset B)^\dagger = \boxed{\leq}(A^\dagger \to B^\dagger) \qquad (\Box A)^\dagger = \boxed{\leq}\boxed{\Box} A^\dagger \qquad (\Diamond A)^\dagger = \boxed{\leq}\boxed{\Diamond} A^\dagger$$

Theorem

$$\vdash_{\mathsf{L}} A \quad \textit{iff} \quad \vdash_{e(\mathsf{L})} A^\dagger .$$

Embedding

Translation $\dagger$: $\mathcal{L} \longrightarrow \mathcal{L}_3$:

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Theorem

$$\vdash_{\mathsf{L}} A \quad \textit{iff} \quad \vdash_{e(\mathsf{L})} A^\dagger.$$

Similar embeddings for CK and CCDL by considering

$(\boxed{\Box}\,;\boxed{\Diamond})$: $\qquad \boxed{\Box} A \wedge \boxed{\Diamond} B \to \boxed{\Diamond}(A \wedge B)$

Theorem

$$\vdash_{\mathsf{CK}} A \quad \textit{iff} \quad \vdash_{\mathsf{e(CK)}} A^\dagger.$$

$$\vdash_{\mathsf{CCDL}} A \quad \textit{iff} \quad \vdash_{\mathsf{e(CCDL)}} A^\dagger.$$

Conclusions and future work

Conclusions

Sequent calculi and Hilbert systems

- Beginning of a theory of intuitionistic non-normal modal logics.
- 24 systems satisfying the initial requirements.
- Analytic cut-free sequent calculi.
- Decidability and (for most systems) Craig interpolation.

Conclusions

Sequent calculi and Hilbert systems

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Semantics

- Semantics by combining intuitionistic Kripke models and neighbourhood models.
- Modular treatment for all systems.
- Finite model property.
- New semantics for already existing systems: CK, CCDL.

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- Semantics by combining intuitionistic Kripke models and neighbourhood models.
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Open problems

- Embedding of ntuitionistic non-normal modal logics into intuitionistic normal multimodal logics.
- Properties of classical non-normal multimodal logics.

Thank you!

... Questions?

Dalmonte, T., C. Grellois, and N. Olivetti, Intuitionistic non-normal modal logics: A general framework (2019), ArXiv.

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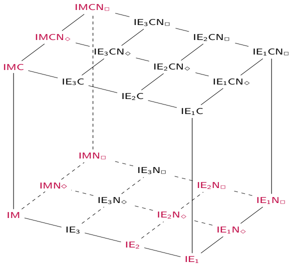
Craig interpolation

Craig interpolation

If $A \Rightarrow B$ is derivable, then there is a formula C with only the common propositional variables of A and B, such that $A \Rightarrow C$ and $C \Rightarrow B$ are derivable.

Theorem

The red systems have Craig interpolation.



- Counterexamples for IE$_3$, IE$_3$N$_\Box$, IE$_3$N$_\Diamond$.
- I'm not aware of proofs of Craig interpolation for classical non-monotonic logics containing C.

Relational models for CK

Relational models for CK (Mendler & de Paiva 2005)

$\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{R}, \mathcal{V} \rangle$,

- $\mathcal{W}$ is a non-empty set;
- $\preceq$ is a preorder over $\mathcal{W}$;
- $\mathcal{R}$ is any binary relation on $\mathcal{W}$.
- $\mathcal{V}$ is a valuation function $\mathcal{W} \longrightarrow Atm$ s.t. $w \preceq v$ implies $\mathcal{V}(w) \subseteq \mathcal{V}(v)$.

If $w \Vdash_r \bot$	then	$w \preceq v$ or $w\mathcal{R}v$ implies $v \Vdash_r \bot$;
If $w \Vdash_r \bot$	then	$w \Vdash_r p$ for every propositional variables $p \in \mathcal{L}$;
$w \Vdash_r B \supset C$	iff	for all $v \succeq w$, $v \Vdash B$ implies $v \Vdash C$;
$w \Vdash_r \Box B$	iff	for all $v \succeq w$, for all $u \in \mathcal{W}$, $v\mathcal{R}u$ implies $u \Vdash_r B$;
$w \Vdash_r \Diamond B$	iff	for all $v \succeq w$, there is $u \in \mathcal{W}$ s.t. $v\mathcal{R}u$ and $u \Vdash_r B$.