On global algebraic completeness of the Gödel-Löb provability logic

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Gödel-Löb provability logic GL

Axiom schemes

- Boolean tautologies
- $\Box(A \to B) \to (\Box A \to \Box B)$
- $\Box(\Box A \to A) \to \Box A$ (Löb's axiom)

Inference rules

$$\text{mp}\ \frac{A \qquad A \to B}{B}\ , \qquad \text{nec}\ \frac{A}{\Box A}\ .$$

The logic GL is sound and complete w.r.t. the arithmetical semantics, where $\Box A$ corresponds to "A is provable in Peano arithmetic".

Non-well-founded derivations

Definition

An *∞-derivation* in GL is a (possibly infinite) tree whose nodes are marked by modal formulas and that is constructed according to the rules (mp) and (nec). In addition, any infinite branch in an ∞-derivation must contain infinitely many applications of the rule (nec).

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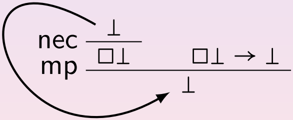
Example



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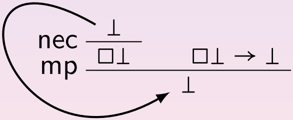

Theorem

GL = K4 + cyclic proofs = K4 + ∞-proofs

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Non-well-founded derivations

Definition

An *assumption leaf* of an ∞-derivation is a leaf that is not marked by an axiom of GL. An assumption leaf is *boxed* if there is an application of the rule (nec) on the path from this leaf to the root of the tree.

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Local and global derivability relations

Definition

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The relation $\vdash$ is a generalization of $\vdash_l$ and $\vdash_g$:

$$\Gamma \vdash_l A \Longleftrightarrow \varnothing; \Gamma \vdash A \quad \text{and} \quad \Gamma \vdash_g A \Longleftrightarrow \Gamma; \Gamma \vdash A.$$

Global neighbourhood completeness of GL

Definition

We set $\Gamma \vDash_g A$ if for any neighbourhood GL-model $\mathcal{M}$

$$(\forall B \in \Gamma\ \mathcal{M} \vDash B) \Rightarrow \mathcal{M} \vDash A\,.$$

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$\Gamma \vdash_g A \Longleftrightarrow \Gamma \vDash_g A$.

Algebraic semantics

Magari algebras

Definition

A *Magari algebra* $\mathcal{A} = (X, \wedge, \vee, \to, 0, 1, \Box)$ is a Boolean algebra $(X, \wedge, \vee, \to, 0, 1)$ together with a unary map $\Box \colon X \to X$ satisfying the identities:

$$\Box 1 = 1, \quad \Box(x \wedge y) = \Box x \wedge \Box y, \quad \Box(\Box x \to x) = \Box x.$$

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Definition

A Magari algebra $\mathcal{A}$ is *$\Box$-founded* if, for every sequence of elements $(b_i)_{i \in \mathbb{N}}$ such that $\Box b_{i+1} \leqslant b_i$, we have $b_0 = 1$.

Definition

A Magari algebra is σ-complete if its underlying Boolean algebra is σ-complete that is every its countable subset S has the least upper bound $\bigvee S$ (equivalently, the greatest lower bound $\bigwedge S$).

Proposition

Any σ-complete Magari algebra is $\Box$-founded.

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Proof

Assume we have a σ-complete Magari algebra $\mathcal{A}$ and a sequence of its elements $(a_i)_{i\in\mathbb{N}}$ such that $\Box a_{i+1} \leqslant a_i$. We shall prove that $a_0 = 1$. Put $b = \bigwedge_{i\in\mathbb{N}} a_i$. For any $n \in \mathbb{N}$, we have $b \leqslant a_{n+1}$ and $\Box b \leqslant \Box a_{n+1} \leqslant a_n$. Hence,

$$\Box b \leqslant b, \qquad \Box b \to b = 1, \qquad \Box b = \Box(\Box b \to b) = 1, \qquad b = 1.$$

We obtain that $a_0 = 1$.

Algebraic consequence relations

A *valuation on $\mathcal{A}$* is a function $\theta\colon Fm \to X$ such that $\theta(\bot) = 0$, $\theta(A \to B) = \theta(A) \to \theta(B)$ and $\theta(\Box A) = \Box\theta(A)$. For a subset S of a Magari algebra $\mathcal{A}$, the filter of (the boolean part of) $\mathcal{A}$ generated by S is denoted by $\langle S \rangle$.

Definition

Given a set of modal formulas Γ and a formula A, we set $\Gamma \Vdash_l A$ if for any $\Box$-founded Magari algebra $\mathcal{A}$ and any valuation θ on $\mathcal{A}$

$$\theta(A) \in \langle \{\theta(B) \mid B \in \Gamma\} \rangle.$$

Definition

We also set $\Gamma \Vdash_g A$ if for any $\Box$-founded Magari algebra $\mathcal{A}$ and any valuation θ on $\mathcal{A}$

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Theorem

$\Sigma; \Gamma \Vdash A \Leftrightarrow \Sigma; \Gamma \vdash A.$

Global neighbourhood completeness of GL is essentially global completeness over atomic complete Magari algebras.

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Plan of the proof

- Correctness
- Compactness
- Completeness via a sequent calculus

Completeness via a sequent calculus

Definition

A *sequent* is an expression of the form $\Sigma; \Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas, and Σ is an arbitrary set of formulas.

Definition

A sequent $\Sigma; \Gamma \Rightarrow \Delta$ is called *valid* if $\Sigma; \{\bigwedge \Gamma\} \vDash \bigvee \Delta$.

Non-well-founded sequent calculus

For a finite multiset of formulas $\Gamma = A_1, \ldots, A_n$, we set $\Box\Gamma := \Box A_1, \ldots, \Box A_n$.

Initial sequents and inference rules of the sequent calculus S have the following form:

$$\Sigma; \Gamma, p \Rightarrow p, \Delta, \qquad \Sigma; \Gamma, \bot \Rightarrow \Delta,$$

$$\to_{\mathsf{L}} \frac{\Sigma; \Gamma, B \Rightarrow \Delta \qquad \Sigma; \Gamma \Rightarrow A, \Delta}{\Sigma; \Gamma, A \to B \Rightarrow \Delta}, \qquad \to_{\mathsf{R}} \frac{\Sigma; \Gamma, A \Rightarrow B, \Delta}{\Sigma; \Gamma \Rightarrow A \to B, \Delta},$$

$$\Box \frac{\Sigma; \Sigma_0, \Gamma, \Box\Gamma \Rightarrow A}{\Sigma; \Pi, \Box\Gamma \Rightarrow \Box A, \Delta} \text{ (Σ_0 is a finite subset of Σ).}$$

Lemma

If a sequent $\Sigma; \Gamma, A \to B \Rightarrow \Delta$ is valid, then sequents $\Sigma; \Gamma, B \Rightarrow \Delta$ and $\Sigma; \Gamma \Rightarrow A, \Delta$ are valid. If $\Sigma; \Gamma \Rightarrow A \to B, \Delta$ is valid, then $\Sigma; \Gamma, A \Rightarrow B, \Delta$ is also valid.

Definition

A sequent $\Sigma; \Gamma \Rightarrow \Delta$ is called *saturated* if Γ and Δ do not contain formulas of the form $A \to B$.

Lemma

If $\Sigma; \Pi, \Box\Gamma \Rightarrow \Delta$ is a valid non-initial saturated sequent, where Π consists only of propositional variables, then there are a finite subset Σ_0 of Σ and a formula $\Box A$ from Δ such that $\Sigma; \Sigma_0, \Gamma, \Box\Gamma \Rightarrow A$ is a valid sequent.

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Corollary (Algebraic completeness)

$\Sigma; \Gamma \models A \Rightarrow \Sigma; \Gamma \vdash A$.

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Theorem

A Magari algebra is $\Box$-founded if and only if it is embeddable into an atomic complete Magari algebra.

Global completeness of GL over σ-complete Magari algebras can be obtained along the same lines.

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Proposition

A Magari algebra is $\Box$-founded if and only if it is embeddable into a σ-complete Magari algebra.

Consequence relation over σ-complete Magari algebras

Definition

We also set $\Gamma \vDash_g A$ if for any σ-complete Magari algebra $\mathcal{A}$ and any valuation θ on $\mathcal{A}$

$$(\forall B \in \Gamma \ \theta(B) = 1) \Rightarrow \theta(A) = 1.$$

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Lemma

$\Sigma; \Gamma \Vdash A \Rightarrow \Sigma; \Gamma \vDash A.$

Lemma

$\Gamma \vDash_g A \Rightarrow \Gamma; \Gamma \vDash A$.

Lemma (Algebraic compactness)

If $\Sigma; \Gamma \vDash A$, then there is a finite subset Γ_0 of Γ such that $\Sigma; \Gamma_0 \vDash A$.

Corollary

A Magari algebra is $\Box$-founded if and only if it is embeddable into a σ-complete Magari algebra.

Proof sketch

Any $\Box$-founded Magari algebra can be embedded into a reduced product of its countable (and finite) subalgebras over a countably complete filter.

Any countable (and finite) $\Box$-founded Magari algebra is can be embedded into a σ-complete Magari algebra.

Any reduced product of σ-complete Magari algebras over a countably complete filter is σ-complete.

It seems that, relying only on DC without LEM, global completeness of KM over δ-complete KM-algebras can be obtained.

Proposition

A KM-algebra is $\Box$-founded if and only if it is embeddable into a σ-δ-complete KM-algebra.