Completion of pseudo-orthomodular posets

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Outline

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Introduction - pseudo-orthomodular posets

In this lecture we introduce the notion of a pseudo-orthomodular poset $\mathbf{P}$.

Our goal is to to determine when its Dedekind-MacNeille completion $\mathbf{DM}(\mathbf{P})$ is an orthomodular lattice.

We get some classes of pseudo-orthomodular posets for which their Dedekind-MacNeille completion is an orthomodular lattice.

Notice that pseudo-orthomodular posets were introduced by Ivan Chajda and Helmut Länger.

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LU-identities

Consider a bounded poset $\mathbf{P} = (P, \leq, 0, 1)$. For $M \subseteq P$ denote by

$$U(M) := \{x \in P \mid y \leq x \text{ for all } y \in M\},$$

the so-called *upper cone of* M, and by

$$L(M) := \{x \in P \mid x \leq y \text{ for all } y \in M\},$$

the so-called *lower cone of* M. If $M = \{a, b\}$ or $M = \{a\}$, we will write simply $U(a,b)$, $L(a,b)$ or $U(a)$, $L(a)$, respectively.

Posets with with antitone involution and complementation

A *poset with antitone involution* is an ordered quintuple $\mathbf{P} = (P, \leq, ', 0, 1)$ such that $(P, \leq, 0, 1)$ is a bounded poset and $'$ is a unary operation on P satisfying the following conditions for all $x, y \in P$:

(i) $x \leq y$ implies $y' \leq x'$,

(ii) $(x')' \approx x$.

A *poset with complementation* is a poset with antitone involution $\mathbf{P} = (P, \leq, ', 0, 1)$ satisfying the following LU-identities:

(iii) $L(x, x') \approx \{0\}$ and $U(x, x') \approx \{1\}$.

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Orthomodular lattices

Recall that a lattice with complementation $(L, \wedge, \vee, ', 0, 1)$ is *orthomodular* if and only if it satisfies the following identity:

$$x \vee y \approx ((x \vee y) \wedge y') \vee y.$$

which in turn is equivalent to the following condition:

if $x, y \in L$, $x \leq y$ and $x' \wedge y = 0$ then $x = y$.

Orthomodular posets

A *poset* with complementation $\mathbf{P} = (P, \leq, ', 0, 1)$ is called *orthomodular* if for all $x, y \in P$ with $x \leq y'$ there exists $x \vee y$ and then $\mathbf{P}$ satisfies the following identity:

$$((x \vee y) \wedge y') \vee y \approx x \vee y$$

where $x \wedge y$ stands for $(x' \vee y')'$ (De Morgan laws).

Pseudo-orthomodular posets - Ivan Chajda and Helmut Länger 2018

The poset $\mathbf{P}$ with complementation is called a *pseudo-orthomodular poset* if it satisfies one of the following equivalent conditions:

$$L(U(L(x,y),y'),y) \approx L(x,y),$$
$$U(L(U(x,y),y'),y) \approx U(x,y).$$

Dedekind-MacNeille completion

It is well-known that every poset $(P, \leq)$ can be embedded into a complete lattice $\mathbf{L}$. We frequently take the so-called Dedekind-MacNeille completion $\mathbf{DM}(P, \leq)$ for this $\mathbf{L}$.

We put $\mathrm{DM}(\mathbf{P}) := \{B \subseteq P \mid LU(B) = B\}$. (We simply write $LU(B)$ instead of $L(U(B))$. Analogous simplifications are used in the sequel.) Then for $\mathrm{DM}(\mathbf{P}) = \{L(B) \mid B \subseteq P\}$, $\mathbf{DM}(\mathbf{P}) := (\mathrm{DM}(\mathbf{P}), \subseteq)$ is a complete lattice and $x \mapsto L(x)$ is an embedding from $\mathbf{P}$ to $\mathbf{DM}(\mathbf{P})$ preserving all existing joins and meets, and an order isomorphism between posets $\mathbf{P}$ and $(\{L(x) \mid x \in P\}, \subseteq)$. We usually identify P with $\{L(x) \mid x \in P\}$.

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Orthogonal sets and orthocomplete posets

A subset $S \subseteq P$ of a poset $\mathbf{P}$ with complementation such that $s \leq t'$ for any pair $s,t \in S, s \neq t$ is called *orthogonal*.

A poset $\mathbf{P}$ with complementation is called an *orthocomplete poset* if every orthogonal subset of $\mathbf{P}$ has a supremum.

A poset $\mathbf{P}$ is said to have a *finite rank* if every orthogonal subset of $\mathbf{P}$ is finite.

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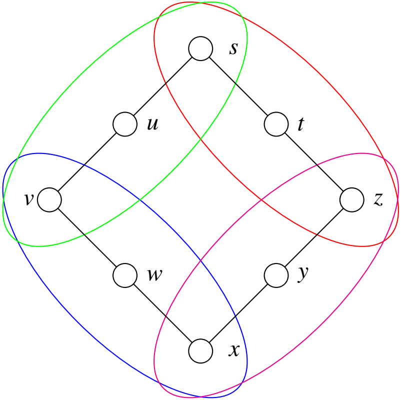
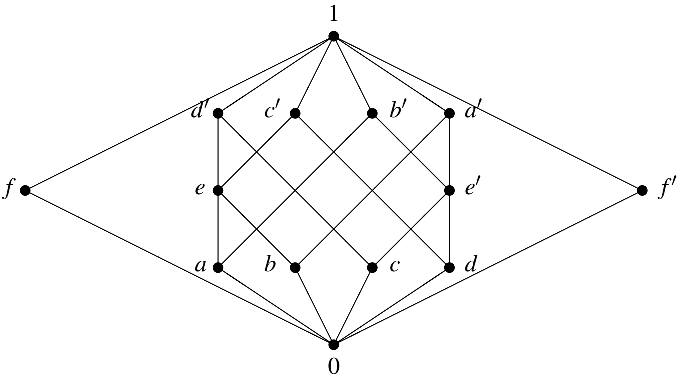
Orthomodular posets and their Dedekind-MacNeille completion

It is known that there is a finite orthomodular poset $\mathbf{P} = (P, \leq, ', 0, 1)$ which is not pseudo-orthomodular such that its Dedekind-MacNeille completion $\mathbf{DM}(\mathbf{P})$ need not be an orthomodular lattice.



Pseudo-orthomodular posets and their Dedekind-MacNeille completion

It is easy to find an example of a finite pseudo-orthomodular poset $\mathbf{P}$ such that $\mathbf{DM}(\mathbf{P})$ is a nonmodular orthomodular lattice.



Outline

Orthomodular Dedekind-MacNeille completion implies pseudo-orthomodularity

Theorem 1

Let $\mathbf{P} = (P, \leq, ', 0, 1)$ be a complemented poset such that $\mathbf{DM}(\mathbf{P})$ is an orthomodular lattice. Then $\mathbf{P}$ is pseudo-orthomodular.

Proof.

Let $\mathbf{DM}(\mathbf{P})$ be an orthomodular lattice and let $x, y \in P$. We compute:

$$L(U(x,y)) = x \vee_{\mathbf{DM}(\mathbf{P})} y = ((x \vee_{\mathbf{DM}(\mathbf{P})} y) \wedge_{\mathbf{DM}(\mathbf{P})} y') \vee_{\mathbf{DM}(\mathbf{P})} y = LU(L(U(x,y),y'),y).$$

Hence $U(L(U(x,y),y'),y) = U(x,y)$.

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Orthocomplete atomic orthomodular posets and their Dedekind-MacNeille completion

Theorem 2

Let $\mathbf{P} = (P, \leq, ', 0, 1)$ be an orthocomplete atomic orthomodular poset. The following conditions are equivalent:

(i) $\mathbf{P}$ is pseudo-orthomodular.

(ii) $\mathbf{P}$ is a complete orthomodular lattice.

(iii) $\mathbf{DM}(\mathbf{P})$ is orthomodular.

Proof.

(ii) $\implies$ (iii) is evident and (iii) $\implies$ (i) follows by Theorem 1.
(i) $\implies$ (ii): Assume that $v, z \in P, v, z \notin \{0, 1\}$ (the case when $v \in \{0, 1\}$ or $z \in \{0, 1\}$ is trivial) and let us prove that $v \vee z$ exists.

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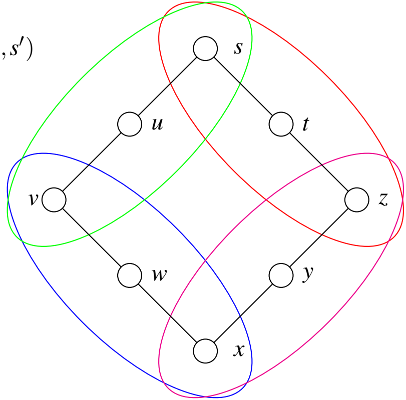
Proof of the implication (i) $\Longrightarrow$ (ii):

$$u \in L(s') = L(\{1\}, s')$$



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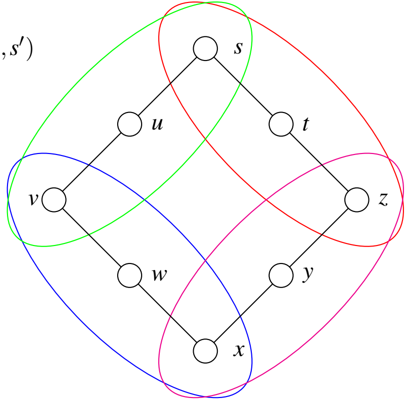

$u \wedge t = 0$

$u' \vee t' = 1$

$(s \vee v) \vee (s \vee z) = 1$

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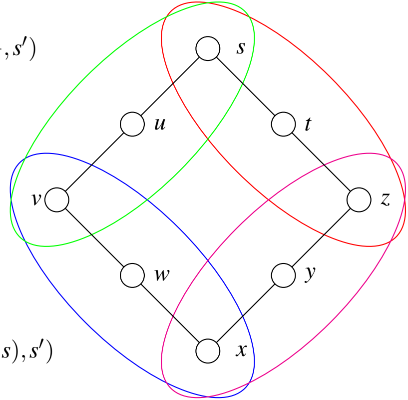

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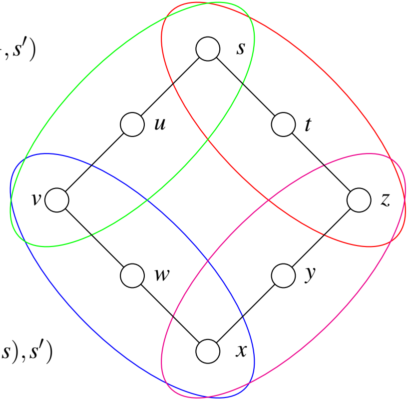

$u' \vee t' = 1$

$(s \vee v) \vee (s \vee z) = 1$

$U(L(x', s'), s) = \{1\}$

$u \in L(U(L(x', s'), s), s')$

$u \not\leq x'$

$L(x', s') \neq L(U(L(x', s'), s), s')$

$u \notin L(x', s')$

Orthocomplete atomic orthomodular posets and their Dedekind-MacNeille completion

Corollary 3

Let $\mathbf{P} = (P, \leq, ', 0, 1)$ be a finite orthomodular poset which is not a lattice. Then its Dedekind-MacNeille completion $\mathbf{DM}(\mathbf{P})$ is not orthomodular.

Dedekind-MacNeille completion of finite pseudo-orthomodular posets

Theorem 4

Let $\mathbf{P} = (P, \leq, ', 0, 1)$ be an atomic pseudo-orthomodular poset with finite rank. Then $\mathbf{DM}(\mathbf{P})$ is orthomodular.

Corollary 5

Let $\mathbf{P} = (P, \leq, ', 0, 1)$ be a finite pseudo-orthomodular poset. Then $\mathbf{DM}(\mathbf{P})$ is a complete orthomodular lattice.

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Thank you for your attention.