Amalgamating poset extensions

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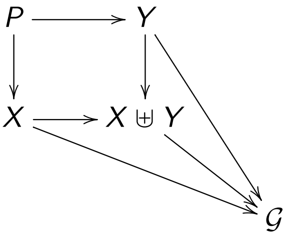
Polarities

- A **polarity** is a triple (X, Y, R) where X and Y are sets and $R \subseteq X \times Y$.
- Define an antitone Galois connection $\wp(X) \leftrightarrow \wp(Y)$ using R.
- Let $\mathcal{G}$ be the complete lattice of stable subsets of X.
  - E.g. for canonical extension use $X = \mathcal{F}$, $Y = \mathcal{I}$, and define R to be 'non-empty intersection'. Then there's a natural embedding of P into $\mathcal{G}$ (Dunn et al. 2005 [2]).
- Given a poset P, there is a 1-1 correspondence between Δ_1-completions of P and a particular class of polarities defined from P (Gehrke et al. 2013 [5]).

The intermediate structure

- For polarities producing Δ_1-completions, X and Y are essentially meet- and join-extensions of P.
- An ‘intermediate structure’ emerges.



- Order restricted to $X \times Y$ is R.

Shifting perspective

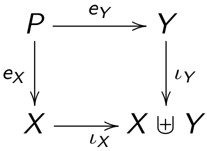
- An **extension polarity** is a triple (e_X, e_Y, R) such that:
  1. $e_X : P \to X$ and $e_Y : P \to Y$ are order extensions.
  2. $R \subseteq X \times Y$.

- Question: Can we always get something like the 'intermediate structure' using X, Y, R?

$$\begin{array}{ccc} P & \xrightarrow{e_Y} & Y \\ {\scriptstyle e_X}\downarrow & & \downarrow{\scriptstyle \iota_Y} \\ X & \xrightarrow[\iota_X]{} & X \uplus Y \end{array}$$

- Answer: No. A little thought reveals that given e_X and e_Y there are constraints on the choice of R.
  - E.g. if $x_1 \leq x_2$ and $x_2 R y$ then we should have $x_1 R y$.

Coherence conditions



- Define (e_X, e_Y, R) to be **Galois** if e_X is a meet-extension, e_Y is a join-extension, and we can order $X \cup Y$ so that:
  1. ι_X and ι_Y are order embeddings.
  2. ι_X preserves meets of subsets of $e_X[P]$, and ι_Y preserves joins of subsets of $e_Y[P]$.
  3. The order on $X \times Y$ is R.
- For Galois polarities the order on $X \cup Y$ meeting the conditions above is unique.
- See [4] for more information.

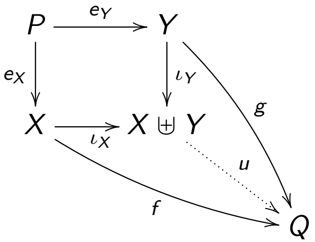
Morphisms

- We can define a concept of a morphism between Galois polarities $(e_X, e_Y, R) \to (e_{X'}, e_{Y'}, R')$.
- This turns the class of Galois polarities into a category.
- There is an adjunction between the category of Galois polarities and the category of Δ_1-completions with complete lattice homomorphisms.
- The **complete** Galois polarities are the fixed points of this adjunction (up to isomorphism).

The universality of R_I

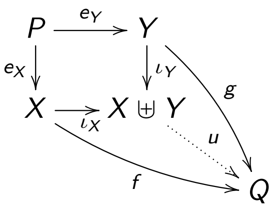
- Given meet-extension e_X and join-extension e_Y, the smallest R making (e_X, e_Y, R) Galois is R_I (non-empty intersection).

- $(X \uplus Y, \iota_X, \iota_Y)$ induced by (e_X, e_Y, R_I) has a universal property.

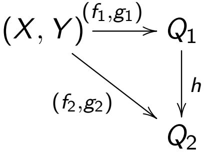


$$\iota_Y(y) \leq \iota_X(x) \implies g(y) \leq f(x).$$

As an initial object

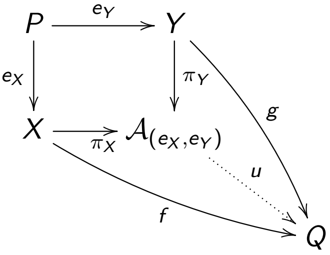


- (ι_X, ι_Y) is initial in the category whose objects are pairs of monotone maps (f, g) and whose maps are commuting triangles.



Defining canonical amalgamations

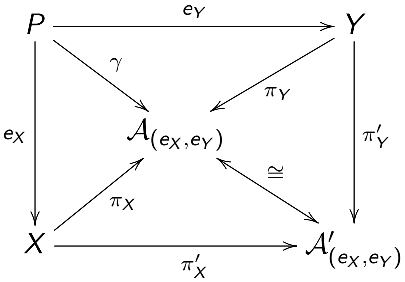
$e_X : P \to X$ is a meet-extension, $e_Y : P \to Y$ is a join-extension. The **canonical amalgamation of e_X and e_Y** is a triple $(\mathcal{A}_{(e_X,e_Y)}, \pi_X, \pi_Y)$:



π_X completely meet-preserving, π_Y completely join-preserving, both order embeddings.

Basic properties of canonical amalgamations

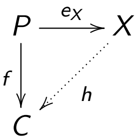
- Canonical Amalgamations always exist.
- They are unique up to isomorphism:



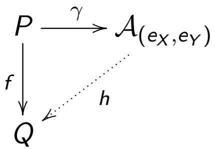
- γ preserves meets iff e_X does, and joins iff e_Y does.

Lifting maps to canonical amalgamations

- Every meet-completion $e_X : P \to X$ is defined by a universal property (J. Schmidt 1974 [7]).



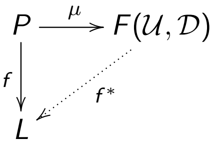
- If (e_X, e_Y, f) are '(α, β)-nice':



  h has some preservation properties parametrized by α and β.
- See [3] for more information.

The free lattice generated by P preserving selected meets and joins

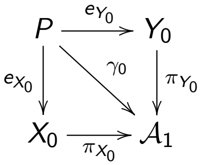
- $F(\mathcal{U}, \mathcal{D})$ is a lattice.
- μ is a $(\mathcal{U}, \mathcal{D})$-embedding.
- If f is a $(\mathcal{U}, \mathcal{D})$-morphism then:



f^* is a lattice homomorphism (see Dean 1964 [1], Lakser 2012 [6]).

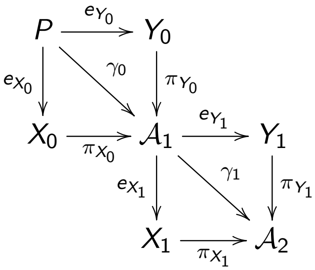
Constructing the free lattice - preserving $\mathcal{U}$ and $\mathcal{D}$

Choose e_{X_0} and e_{Y_0} carefully to preserve meets from $\mathcal{D}$ and joins from $\mathcal{U}$ respectively, and so that X_0 and Y_0 are 'finitely generated' by P.



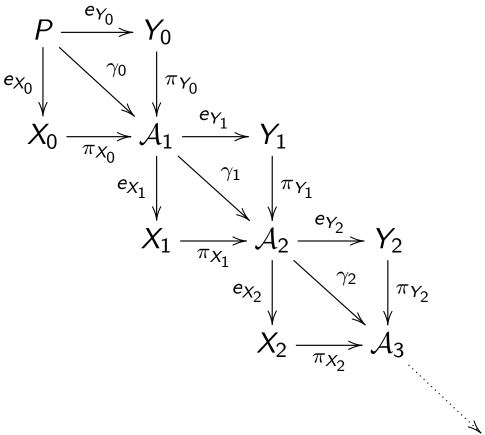
Constructing the free lattice - filling out the lattice

Choose e_{X_1} and e_{Y_1} carefully to preserve all finite meets and, respectively, joins of subsets of $\gamma_0[P]$, and so that X_1 and Y_1 are 'finitely generated'.



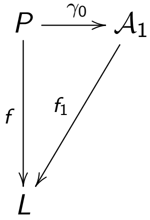
Constructing the free lattice - etc.

Choose e_{X_2} and e_{Y_2} carefully to preserve all meets and, respectively, joins of finite subsets of $\gamma_1[\mathcal{A}_1]$, and so that X_2 and Y_2 are 'finitely generated', and so on.



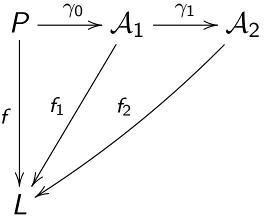
Constructing the free lattice - a universal property

Let f be a $(\mathcal{U}, \mathcal{D})$-morphism. Then (e_{X_0}, e_{Y_0}, f) is (ω, ω)-nice, so we get unique f_1.



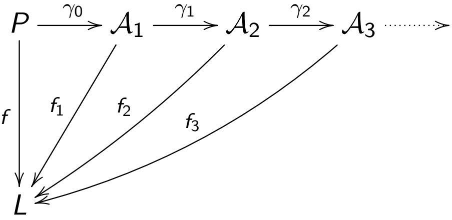
Constructing the free lattice - extending universality

(e_{X_1}, e_{Y_1}, f_1) is (ω, ω)-nice so we get unique f_2...



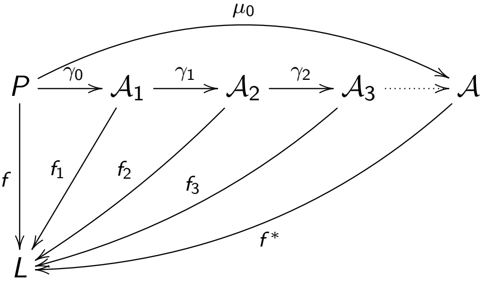
Constructing the free lattice - universality step by step

...and so on...



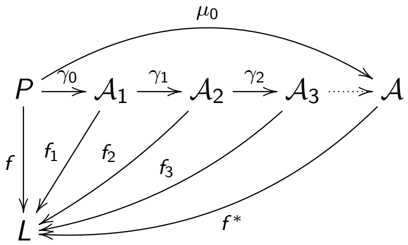
Constructing the free lattice - using the colimit

Directed colimits exist in Pos.



Properties of $\mathcal{A}$.

- $\mathcal{A}$ is a lattice.
- μ_0 is a $(\mathcal{U}, \mathcal{D})$-embedding.



  - f^* is the unique lattice homomorphism making the outer triangle commute.

  - I.e. $\mathcal{A} = F(\mathcal{U}, \mathcal{D})$.

Some observations

- Each $\mathcal{A}_n$ is a kind of approximation of $F(\mathcal{U}, \mathcal{D})$.
- If P is finite each $\mathcal{A}_n$ will be finite, though $F(\mathcal{U}, \mathcal{D})$ will usually be infinite.
- Finite joins and meets defined from $\gamma_n[\mathcal{A}_n]$ in $\mathcal{A}_{n+1}$ are preserved in $\mathcal{A}$.
- The order on $\mathcal{A}_n$ is unchanged in $\mathcal{A}$.
- The chain $P \to \mathcal{A}_1 \to \mathcal{A}_2 \to \ldots$ builds $F(\mathcal{U}, \mathcal{D})$, and guarantees the 'correctness' of the order structure, and most of the existing lattice structure, at each stage.

[1] R. A. Dean.
Free lattices generated by partially ordered sets and preserving bounds.
Canad. J. Math., 16:136–148, 1964.

[2] J. Dunn, M. Gehrke, and A. Palmigiano.
Canonical extensions and relational completeness of some substructural logics.
J. Symb. Logic, 70:713–740, 2005.

[3] Egrot.
Something like “amalgamating poset extensions”.
In preparation.

[4] R. Egrot.
Order polarities.
preprint.

[5] M. Gehrke, R. Jansana, and A. Palmigiano.
Δ_1-completions of a poset.
Order, 30(1):39–64, 2013.

[6] H. Lakser.
Lattices freely generated by an order and preserving certain bounds.
Algebra Universalis, 67(2):113–120, 2012.

[7] J. Schmidt.
Each join-completion of a partially ordered set is the solution of a universal problem.
J. Austral. Math. Soc., 17:406–413, 1974.