From MV-semirings to Involutive semirings

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Semirings

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Definition (Semirings)

An *semiring* S is an algebra $(S, +, \cdot, 0, 1)$ of type (2,2,0,0) such that:

1. $(S, +, 0)$ is a commutative monoid;
2. $(S, \cdot, 1)$ is a monoid;
3. $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$, for every $x, y, z \in S$.
4. $0 \cdot s = 0 = s \cdot 0$, for each $s \in S$.

Definition (Semifield)

Let $(S, +, \cdot, 0, 1)$ be a semiring. S is a *semifield* if $(S \setminus \{0\}, \cdot, 1)$ is a commutative group.

Examples (Semirings and semifields)

1. *boolean semifield* $\mathbb{B} := (\{0, 1\}, +, \cdot, 0, 1)$;
2. *tropical semifield* $\mathbb{T} := (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$.
3. bounded distributive lattices.

Definition (Additively idempotent semirings)

A semiring S is called *additively idempotent* if $s + s = s$, for any $s \in S$.

Remark

On every $(S, +, \cdot, 0, 1)$ additively idempotent semiring it is possible to define a natural order by $s \leq s' \iff s + s' = s'$, for any $s, s' \in S$. In this case $(S, +)$ is a join semilattice and the sum is usually denoted with $\vee$.

Definition (Zero-free semirings)

A semiring $(S, +, \cdot, 1)$ is called *0-free* if it doesn't have the element 0. In particular we have that $(S, +)$ is a commutative semigroup and the axiom 4 isn't assumed to hold.

Definition (Commutative semirings)

A semirings S is called *commutative* if $x \cdot y = y \cdot x$ for every $x, y \in S$.

MV-algebras

Definition (MV-algebra)

An *MV-algebra* is an algebra $(A, \oplus, ^*, 0)$ of type (2,1,0) such that, for every $x, y, z \in A$ we have:

1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $x \oplus y = y \oplus x$;
3. $x \oplus 0 = x$;
4. $(x^*)^* = x$;
5. $x \oplus 0^* = 0^*$;
6. $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

On every MV-algebra A, it is possible to define another constant $1 := 0^*$ and the operation $x \odot y := (x^* \oplus y^*)^*$.

The Natural order on an MV-Algebra

For any MV-algebra A, there exists a natural order given by:

$$x \leq y \iff x^* \oplus y = 1,$$

for every $x, y \in A$.
The natural order determines a structure of bounded distributive lattice on A, with 0 and 1 respectively bottom and top element and the operations of *sup* and *inf* defined by:

$$x \vee y := (x \odot y^*) \oplus y$$

and

$$x \wedge y := (x^* \vee y^*)^*.$$

MV-semirings

Definition (MV-semiring)

Let $A = (A, \vee, \cdot, 0, 1)$ be a commutative, additively idempotent semiring. A is a *MV-semiring* iff exists a map $^* : A \to A$, called *negation*, such that:

1. $a \cdot b = 0$ iff $b \leq a^*$;
2. $a \vee b = (a^* \cdot (a^* \cdot b)^*)^*$.

From now on we shall include the negation symbol in the signature of the MV-semiring, denoting A with $(A, \vee, \cdot, 0, 1, ^*)$.

MV-Algebras and MV-semirings are isomorphic categories

Theorem

Let A be an MV-algebra. Then $A^{\vee\odot} = (A, \vee, \odot, 0, 1)$ and $A^{\wedge\oplus} = (A, \wedge, \oplus, 1, 0)$ are semirings and the involution $^* : A \to A$ is an isomorphism between them. In particular $A^{\vee\odot}$ and $A^{\wedge\oplus}$ are MV-semirings with negation *.

Theorem

If $(A, +, \cdot, 0, 1, ^*)$ is an MV-semiring, the structure $(A, \oplus, \cdot, ^*, 0, 1)$ with, for all $x, y \in A$

$$x \oplus y = (x^* \cdot y^*)^*$$

is an MV-algebra.

Involutive residuated lattices

Definition (Involutive residuated lattice)

An *involutive residuated lattice* is an algebra $(A, \vee, \wedge, \cdot, 1, \backslash, /, 0)$ of type $(2, 2, 2, 0, 2, 2, 0)$ such that:

1. $(A, \vee, \wedge)$ is a lattice,
2. $(A, \cdot, 1)$ is a monoid,
3. (res) $\quad xy \leq z \iff x \leq z/y \iff y \leq x\backslash z$ and
4. $\sim{-x} = -{\sim x} = x$, where $\sim x = x\backslash 0$ and $-x = 0/x$.

Involutive residuated lattices form a variety, denoted by InRL.

Involutive 0-free semirings

Definition (Involutive 0-free semiring)

An *In-semiring* is an algebra $(A, \vee, \cdot, 1, \sim, -)$ of type $(2, 2, 0, 1, 1)$ such that:

1. $(A, \vee, \cdot, 1)$ is a 0-free idempotent semiring;
2. $\sim - x = - \sim x = x$, for every $x \in A$;
3. $x \leq -y \iff x \cdot y \leq -1$.

In-semirings are term equivalent to InRL

Let $(A, \vee, \wedge, \cdot, 1, \backslash, /, 0)$ be an involuitve residuated lattice, then it is well known (and easy to see) that $\backslash, /, 0$ in the signature of InRL can be replaced by $\sim, -$: $x \backslash y = {\sim}((-y)x)$,
$x/y = -(y({\sim}x)$ and $0 = {\sim}1 = -1$.
So, $(A, \vee, \cdot, 1, \sim, -)$ is an involutive semiring.
Vicerversa, given $(A, \vee, \cdot, 1, \sim, -)$ any involutive semiring, we can define a lattice structure on it with $x \wedge y = {\sim}(-x \vee -y)$.

The semiring perspective helps us find a necessary and sufficient condition for $[0,1]$ to be a subalgebra of an involutive residuated lattice.

Definition (Multiplicative Idempotent Elements)

Let A be an In-semiring, an element $a \in A$ is *multiplicative idempotent* iff $a \cdot a = a$.

Proposition

Let A be an involutive residuated lattice and let $[0,1] = \{a \in A | 0 \leq a \leq 1\}$. Then $[0,1]$ is a subalgebra of A iff 0 is a multiplicative idempotent element.

We also import some results and techniques of semimodule theory in the study of this class of semirings. Following what was already done with MV-semirings, we generalize some results about injective and projective semimodules.

Proposition

Let A be an involutive residuated lattice such that 0 is the bottom element, then 1 is the top element. From now on we will denote the class of 0-bounded residuated lattices with $InRL_0$.

Definition (In_0-semirings)

An *In_0-semiring* is an algebra $(A, \vee, \cdot, 0, 1, \sim, -)$ of type $(2, 2, 0, 0, 1, 1)$ such that:

1. $(A, \vee, \cdot, 0, 1)$ is an idempotent semiring;
2. $\sim - x = - \sim x = x$, for every $x \in A$;
3. $x \leq -y \iff x \cdot y = 0$.

Remark

In_0-semirings are term equivalent to $InRL_0$. From now we will call In_0 semirings *involutive semirings*.

Semimodules

Definition (Semimodule)

Let S be a semiring. A (left) $S-$ *semimodule* is a commutative monoid $(M, +, 0)$ with a scalar multiplication $\cdot : (a, x) \in S \times M \to a \cdot x \in M$, such that the following conditions hold for all $a, b \in S$ and $x, y \in M$:

1. $(ab) \cdot x = a \cdot (b \cdot x)$;
2. $a \cdot (x + y) = (a \cdot x) + (a \cdot y)$;
3. $(a + b) \cdot x = (a \cdot x) + (b \cdot x)$;
4. $0_S \cdot x = 0_M = a \cdot 0_M$;
5. $1 \cdot x = x$.

   Right semimodules are defined similarly. If the semiring S is commutative, then left and right S-semimodules are the same and we will simply call them S-semimodules.

Semimodule-morphisms

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Definition

Let S be a semiring and $(M, +, 0_M)$, $(N, +, 0_N)$ two left S-semimodules. A morphism from M to N is a map $f : M \to N$ such that

1. $f(m + m') = f(m) + f(m')$;
2. $f(0_M) = 0_N$;
3. $f(s \cdot m) = s \cdot f(m)$, for any $s \in S$.

Given two S-semimodules M and N, we will denote with $Hom_S(M, N)$ the set of semimodule-morphisms from M to N.

Examples (Semimodules)

1. Any semiring S is a left and right semimodule over itself;
2. any additively idempotent semiring S is a left and right semimodule over the booleand semifield $\mathbb{B}$;
3. for any additively idempotent semiring S, $Hom_{\mathbb{B}}(S, \mathbb{B})$ is a left S-semimodule with the monoidal operation defined pointwise;
4. for any semiring S and for any $s \in S$, $S \cdot s = \{x \cdot s \mid x \in S\}$ is a left S-semimodule and it is called the cylic semimodule generated by s.

Remark

If $(M, +, 0)$ is a (left) semimodule over an additively idempotent semiring S, then $(M, +, 0)$ is idempotent and the sum is usually denoted with $\vee$. In particular, $(M, \vee, 0)$ is a join-semilattice where 0 is the bottom element.

Injective semimodules- Preliminary results

Theorem

Let S be an additively idempotent semiring and M a left S-semimodule. Then, M is injective if and only if there exists a set X such that M is a retract of the left S-semimodule $Hom_{\mathbb{B}}(S, \mathbb{B})^X$.

Ideals of join-semilattices

Definition

Let $(L, \vee)$ be a join-semilattice. An *ideal* I of L is a subset of L such that:

1. $a \vee b \in I$, for any $a, b \in I$;
2. I is downward closed.

Lemma

Let S be an additively idempotent semiring and consider the monoid $(Id(S), \cap, S)$. If S is also commutative it is possible to define a scalar multiplication $/ : S \times Id(S) \to Id(S)$ which makes $Id(S)$ an S-semimodule. In particular $s/I = \{x \in S \mid sx \in I\}$ for any $s \in S$ and $I \in Id(S)$.

Injective semimodules - Preliminary results

Let S be an additively idempotent and commutative semiring, then the map $\phi : Hom_{\mathbb{B}}(S, \mathbb{B}) \to Id(S)$, defined by $\phi(f) := Ker(f)$, determines an isomorphism of S-semimodules.

Proposition

Let S be an additively idempotent commutative semiring and M an S-semimodule. Then, M is injective iff there exists a set X such that M is a retract of the S-semimodule $Id(S)^X$.

Corollary

Let S be an additively idempotent and commutative semiring and M an injective S-semimodule. Then M is a complete semimodule. If in addition the join-semilattice S is a distributive lattice, then M is a complete and infinitely distributive semimodule.

Projective semimodules - Preliminary results

As regards projective semimodules, we have that projective objects are the retracts of the free ones. Given a semiring S and a set X we know that the free S-semimodule generated by X is the set of functions from X to S with finite support that we denote with $S^{(X)}$. So, we have the following

Proposition

Let S be a semiring and M a S-semimodule. Then M is projective iff it is a retract of $S^{(X)}$ for some set X.

Proposition

Let $(A, \vee, \cdot, 0, 1, \sim, -)$ be an involutive semiring. If A is commutative, then it is symmetric (i.e. $-x = \sim x$, for any $x \in A$).

Let A be a finite, commutative involutive semiring, then the map $\phi : (A, \vee, 0) \to (Id(A), \cap, A)$, defined by $\phi(a) := \downarrow_{\sim a}$, determines an isomorphism of A-semimodules.

Theorem

Let A be a finite commutative involutive semiring and M a finitely generated A-semimodule. Then, M is injective if and only if it is projective.

Definition (von Neumann regular semirings)

A semiring A is said *von Neumann regular* if for every $a \in A$, there exists $b \in A$ such that $a = a \cdot b \cdot a$.

Lemma

A is a multiplicatively idempotent involutive semiring iff A is a boolean algebra.

Theorem

For an involutive semiring A, the following statements are equivalent:
(1) Every cyclic semimodule $A \cdot a$ of A is injective;
(2) A is a self-injective and von Neumann regular semiring;
(3) A is a complete Boolean algebra.

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Thank you for your attention!