Polyhedral Completeness in Intermediate and Modal Logics

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From Topological to Polyhedral Semantics

Any topological space X yields a *topological semantics* for intermediate and modal logics.

Theorem (Tarski-McKinsey-Rasiowa-Sikorski Theorem)

Any metrisable space without isolated points provides a complete semantics for **IPC** *and* **S4**.

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Any metrisable space without isolated points provides a complete semantics for **IPC** *and* **S4**.

- This means: topological semantics can't capture much of the geometric content of a space.
- Motivating idea: to express geometric properties like dimension, restrict to subsets which are 'polyhedral'.
- This leads to *polyhedral semantics*.

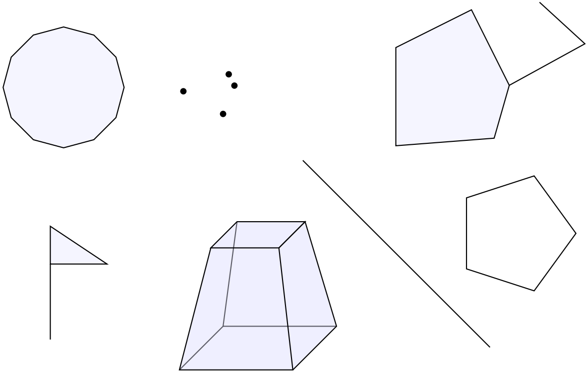
A Summary of the Talk

- Polyhedral semantics is sound and complete for **IPC** and **S4.Grz**.
- We investigate *polyhedral completeness* (poly-completeness): logics sound and complete for a class of polyhedra.
- The *Nerve Criterion* provides a purely combinatorial equivalent of poly-completeness.
- Using this, we show that there are continuum-many poly-incomplete logics with the fmp, and demarcate an infinite class of poly-complete logics of each finite height.
- We give an axiomatisation for the logic of convex polyhedra of each dimension.

Intuitionistic Logic

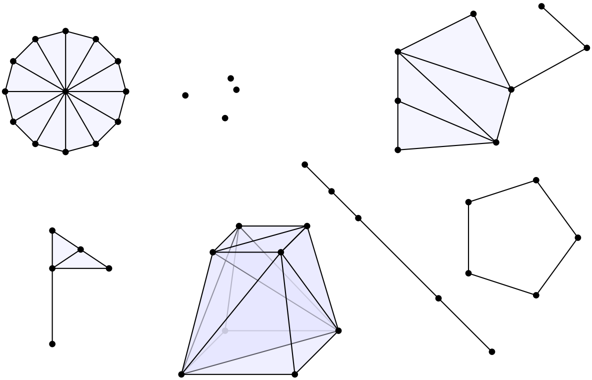
- In this talk, will focus on the intuitionistic side of polyhedral semantics.
- But everything transfers freely to the modal case (we are above **S4**.**Grz**).

Polyhedra



- A polyhedron can have arbitrary dimension, and need not be convex nor connected.
- Our polyhedra are always compact.

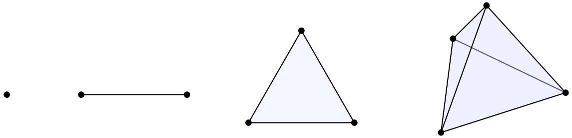
Triangulations I



Intuition: triangulations break polyhedra up into simple shapes.

Triangulations II

- Simplices are the most basic polyhedra of each dimension.
- Points, line segments, triangles, tetrahedra, pentachora, etc.



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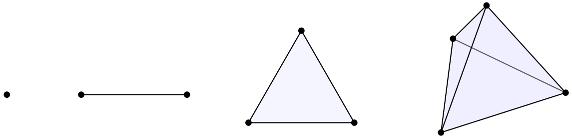

- A *triangulation* is a splitting up of a polyhedron into simplices.
- Represented as a poset $(\Sigma, \preccurlyeq)$ of simplices, where $\sigma \preccurlyeq \tau$ means that σ is a face of τ.
- Its *underlying polyhedron* is $|\Sigma| := \bigcup \Sigma$.
- Every polyhedron admits a triangulation.

The co-Heyting algebra of Subpolyhedra

Theorem (N. Bezhanishvili, Marra, Mcneill, & Pedrini, 2018)

The set of subpolyhedra of a polyhedron forms a co-Heyting algebra.

The Heyting algebra $\mathrm{Sub}_{\mathrm{o}}P$

Since we're interested in logic, let's switch to the dual.

Definition

Let P be a polyhedron. An *open subpolyhedron* of P is the complement in P of a subpolyhedron. $\mathrm{Sub}_{\mathrm{o}}P$ is the set all of open subpolyhedra.

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Corollary

$\mathrm{Sub_o} P$ *is a Heyting algebra.*

So we arrive at a polyhedral semantics for intuitionistic logic.

Some Properties of Polyhedral Semantics

Theorem (N. Bezhanishvili et al., 2018)

The logic of a polyhedron is the logic of its triangulations.

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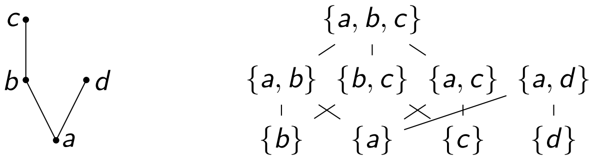
Theorem (N. Bezhanishvili et al., 2018)

IPC *is complete with respect to the class of all polyhedra.*

The Nerve

Definition (Alexandrov's nerve)

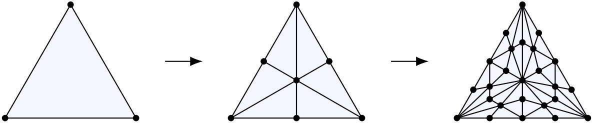
The *nerve*, $\mathcal{N}(F)$, of finite poset F is the set of all non-empty chains in F, ordered by inclusion.

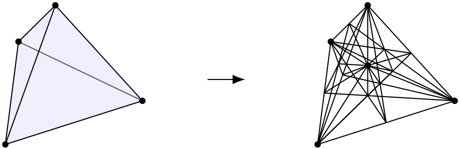


There is always a p-morphism $\mathcal{N}(F) \to F$.

Barycentric Subdivision

Given a triangulation Σ, construct its *barycentric subdivision* Σ' by putting a new point in the middle of each simplex, and forming a new triangulation around it.





$\Sigma' \cong \mathcal{N}(\Sigma)$ as posets.

Barycentric Subdivision and the Nerve Criterion

Theorem (Nerve Criterion)

A logic $\mathcal{L}$ is poly-complete if and only if it is the logic of a class $\mathbf{C}$ *of finite frames closed under* $\mathcal{N}$*.*

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- This is about barycentric subdivision.
- Let $\Sigma^{(n)}$ be the nth iterated barycentric subdivision of Σ.
- Intuition: $(\Sigma^{(n)})_{n\in\mathbb{N}}$ captures everything (logical) about $P = |\Sigma|$.

The Proof of the Theorem

Proof Sketch.

- The algebraic version of the theorem: For any triangulations Σ and Δ of a polyhedron P, there is $n \in \mathbb{N}$ such that the subalgebra generated by Δ is isomorphically contained in subalgebra generated by $\Sigma^{(n)}$.

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- The algebraic version of the theorem: For any triangulations Σ and Δ of a polyhedron P, there is $n \in \mathbb{N}$ such that the subalgebra generated by Δ is isomorphically contained in subalgebra generated by $\Sigma^{(n)}$.
- Show that P, Σ and Δ can be assumed to be rational (i.e. that their vertices lie in $\mathbb{Q}^n$).
- Then use the De Concini-Procesi Lemma from rational polyhedral geometry to find our $n \in \mathbb{N}$. □

Stable Logics

Definition

A logic is *stable* if its rooted frames is closed under monotone images.

Theorem (G. Bezhanishvili & Bezhanishvili, 2017)

The logics **KC**, $\mathbf{LC}_n$, $\mathbf{BW}_n$, $\mathbf{BTW}_n$ *and* $\mathbf{BC}_n$ *are all stable. Moreover, there are continuum-many stable logics.*

Theorem (G. Bezhanishvili & Bezhanishvili, 2017)

Every stable logic has the finite model property.

Stable Logics are Poly-Incomplete

Theorem

If $\mathcal{L}$ is poly-complete, stable and of height at least 2 then $\mathcal{L} = \mathbf{IPC}$.

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If $\mathcal{L}$ is poly-complete, stable and of height at least 2 then $\mathcal{L} = \mathbf{IPC}$.

Proof Idea.

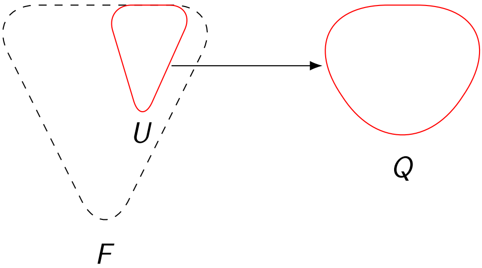
- Purely combinatorial: exploits Nerve Criterion.
- Repeatedly applying $\mathcal{N}$ produces wider and wider frames, which eventually monotone-map to every finite rooted frame.

$\square$

Jankov-Fine Formulas for Forbidden Configurations

Theorem

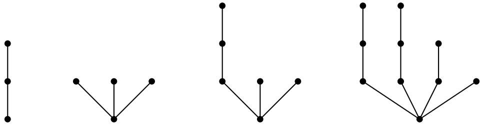
For every finite rooted frame Q, there is a formula $\chi(Q)$, the Jankov-Fine formula of Q, such that for any frame F, we have $F \not\models \chi(Q)$ if and only if F up-reduces to Q.



Starlike Logics

Definition (starlike tree)

A tree T is *starlike* if it has a single branching node at the root.



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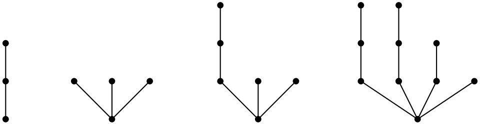

Definition

A logic L is *starlike* if it is of the form $\mathbf{IPC} + \chi(T_1) + \chi(T_2) + \cdots$, where $\{T_1, T_2, \ldots\}$ is a (possibly infinite) set of starlike trees other than .

Starlike Poly-completeness

Theorem

A starlike logic $\mathcal{L}$ is poly-complete if and only if it has the finite model property.

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A starlike logic $\mathcal{L}$ is poly-complete if and only if it has the finite model property.

Corollary

$\mathbf{BD}_n + \chi(T_1) + \chi(T_2) + \cdots$ *is poly-complete. Hence there are infinitely many poly-complete logics of each finite height.*

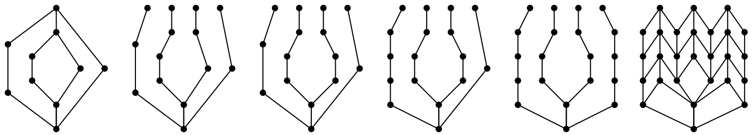
Corollary

$\mathbf{SL} = \mathbf{IPC} + ((\neg\neg p \to p) \to (p \vee \neg p)) \to (\neg\neg p \vee \neg p)$ *(Scott's logic) is poly-complete.*

Proof of Starlike Poly-Completeness

Proof Idea.

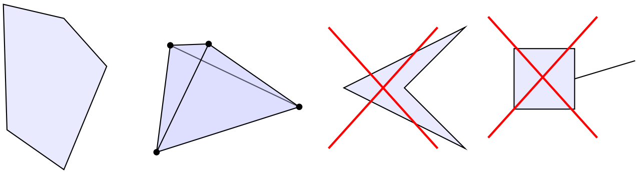
- Exploits the Nerve Criterion.
- A method which, given a finite frame F of $\mathcal{L}$, produces a finite frame F' and a p-morphism $F' \to F$ such that $\mathcal{N}^k(F') \vDash \mathcal{L}$ for every $n \in \mathbb{N}$. □



The Logic $\mathbf{PL}_n$

Definition

A polyhedron P is *convex* if whenever $x, y \in P$, the straight line from x to y is also in P.



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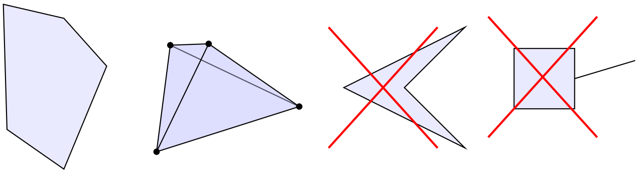
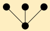


Theorem (An Axiomatisation)

The logic of convex polyhedra of dimension n is axiomatised by $\mathbf{BD}_n + \chi($$) + \chi($$)$.

The Proof of the Axiomatisation

Proof Sketch.

- For **soundness**, we have geometrical arguments exploiting classical dimension theory.
- E.g. for $\chi($$)$ we show that a convex polyhedron can't be partitioned into non-empty sets A, B, C, X such that A, B, C are open subpolyhedra and $X \subseteq \overline{A} \cap \overline{B} \cap \overline{C}$.

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- For **completeness**, we show that every finite frame F of the axiomatisation is realised in a convex polyhedron.
- As an intermediary step we transform F into a more geometrically-amenable form, called a *saw-topped tree*.
- Saw-topped trees are *planar*, which enables the realisation. □

What's Next? Future Directions

- Ultimate goal: a full classification of poly-completeness.
- What is the logic of *all* convex polyhedra? If **IPC** $+ \chi($$) + \chi($$)$ has the fmp, then it is an axiomatisation.
- What is the natural notion of bisimulation for polyhedra?
- Can we use these techniques to prove standard polyhedral geometry results using logic?

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