Simplicial semantics and one-variable fragments of modal predicate logics

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Introduction

- M. Wajsberg (1933) noticed that quantifiers in first order logic can be treated as modalities:

  the 1-variable fragment of classical predicate logic corresponds to propositional **S5**.

$$\forall x \mapsto \Box$$

$$\exists x \mapsto \Diamond$$

$$P_i(x) \mapsto p_i$$

- A. Prior (1957) proposed to study the 1-variable fragment of intuitionistic predicate logic in the same way. This leads to intuitionistic modal logic **MIPC**.

Introduction-2

- G. Fischer-Servi (1977) noticed that the modal transaltion of **MIPC** is the 1-variable fragment of **QS4** (quantified **S4**). This is a bimodal logic (in our notation, **S4 _| S5)**.

MIPC --------------> S4 _| S5

↓ ↓

QH-1 --------------> QS4-1

She proposed to study intuitionistic modal and bimodal logics obtained in such a way, but this work is still only beginning.

Introduction-3

- S.Artemov & G.Dhaparidze (1991) studied the 1-variable fragment of **QGL** (quantified **GL**)= the 1-variable fragment of first-order provability logic of **PA**.
- [D.Gabbay & V.Shehtman, 1998] described the 1-variable fragments of some modal predicate logics with constant domains as *products of modal logics.*
- *Semiproducts* (= *expanding products*) were first studied by F. Wolter, M. Zakharyaschev, A. Kurucz (2003 and later on) and V. Shehtman (2005). They are related to 1-variable fragments of modal predicate logics with expanding domains.

References

[QNL] D.Gabbay, V. Shehtman, D. Skvortsov. Quantification in Nonclassical Logic, Volume 1. Elsevier, 2009.

[MDML] D.Gabbay, A. Kurucz, F. Wolter, M. Zakharyaschev. Many-dimensional modal logics: theory and applications. Elsevier, 2003.

Formulas

Modal predicate formulas are built from:

- the countable set of individual variables $Var=\{v_1,v_2,\ldots\}$
- countable sets of n-ary predicate letters (for every n≥0)
- $\rightarrow, \bot, \Box, \forall$

$\neg, \Diamond, \vee, \wedge, \exists$ are derived.

No equality, constants or function symbols

Variable and formula substitutions [QNL]

$[y_1,\dots,y_n/x_1,\dots,x_n]$ simultaneously replaces all free occurrences of $x_1,\dots,x_n$ with $y_1,\dots,y_n$ (with renaming bound variables if necessary)

To obtain $[C(x_1,\dots,x_n,y_1,\dots,y_m)/P(x_1,\dots,x_n)]A$:

(1) rename all bound variables of A that coincide with the "new" parameters $y_1,\dots,y_m$ of C,

(2) replace every occurrence of every atom $P(z_1,\dots,z_n)$ with $[z_1,\dots,z_n/x_1,\dots,x_n]C$

Strictly speaking, all substitutions are defined up to congruence (α-equivalence): formulas are congruent if they can be obtained by "legal" renaming of bound variables

Modal logics -1

A modal predicate logic (mpl) is a set L of modal formulas containing

- the classical predicate tautologies
- the axiom of **K:** $\Box(p\to q) \to (\Box p \to \Box q)$

and closed under the rules

- Modus Ponens: $A, A \to B \,/\, B$
- Necessitation: $A \,/\, \Box A$
- Generalization: $A \,/\, \forall x A$
- Substitution: A/SA (for any formula substitution S)

Remark Another definition of a modal predicate logic (Kripke, 1963) does not include all predicate tautologies.

Modal logics-2

Modal propositional logics can be regarded as fragments of predicate logics (with only 0-ary predicate letters, without quantifiers).

K := the minimal modal propositional logic

QK:= the minimal modal predicate logic

QΛ :=QK+Λ := the minimal predicate extension of the propositional logic **Λ**

QΛC :=QΛ+ Ba (Barcan axiom)

$$\forall x \Box P(x) \rightarrow \Box \forall x P(x)$$

1-variable fragments-1

A *1-variable predicate formula* is built from monadic predicate letters P_i (i=0,1,...) using a single variable x, free or bound. The *1-variable fragment* of a predicate logic L is $\{A \in L \mid A \text{ is 1-variable}\}$.

Every 1-variable predicate formula A translates into a 2-modal propositional formula A_* :

replace $\forall x \mapsto \blacksquare$ $\qquad P_i(x) \mapsto p_i$

Then put $\quad L\text{-}1 := \{A_* \mid A \in L, A \text{ is 1-variable}\}$.

We also call L-1 the *1-variable fragment* of L.

1-variable fragments-2

Simple remarks Let L be a predicate logic with the propositional fragment **Λ.** Then L-1 is a 2-modal propositional logic, between the *semicommutative join*

Λ_| **S5**:= **Λ ∗ S5** (fusion) + $\Box\blacksquare p \to \blacksquare\Box p$

and **Λ ∗ Triv.** (**Triv**:= **K** + $\blacksquare p \leftrightarrow p$)

Note that $\Box\blacksquare p \to \blacksquare\Box p$ is the translation of the Converse Barcan formula (provable in **QK**):

$$\Box \forall x P(x) \to \forall x \Box P(x)$$

1-variable fragments-3

For the particuar cases **L=QΛ, QΛC** we have

Lemma **Λ** $\_|$**S5** ⊆ **QΛ**-1 ⊆ $\overline{\mathbf{Q\Lambda}}$**-1**

[Λ,S5] ⊆ **QΛC**-1 ⊆ $\overline{\mathbf{Q\Lambda C}}$**-1**

where $\overline{\mathbf{L}}$ denotes the *Kripke-completion* of L,

$$\overline{L} := \mathbf{ML}(\{\Phi \mid \Phi \vDash L\})$$

(the smallest Kripke-complete extension of L),

[Λ,S5] := Λ ∗ S5 + $\Box \blacksquare p \leftrightarrow \blacksquare \Box p$ (the *commutative join*)

The logics $\overline{\mathbf{Q\Lambda}}$**-1**, $\overline{\mathbf{Q\Lambda C}}$**-1** can also be described as *semiproducts* and *products* with **S5**

Kripke semantics-1

A propositional Kripke frame $F=(W, R)$ $(W \neq \varnothing, R \subseteq W^2)$

A predicate Kripke frame: $\Phi = (F,D)$, where

$D=(D_u)_{u \in W}$ is an expanding family of non-empty sets:

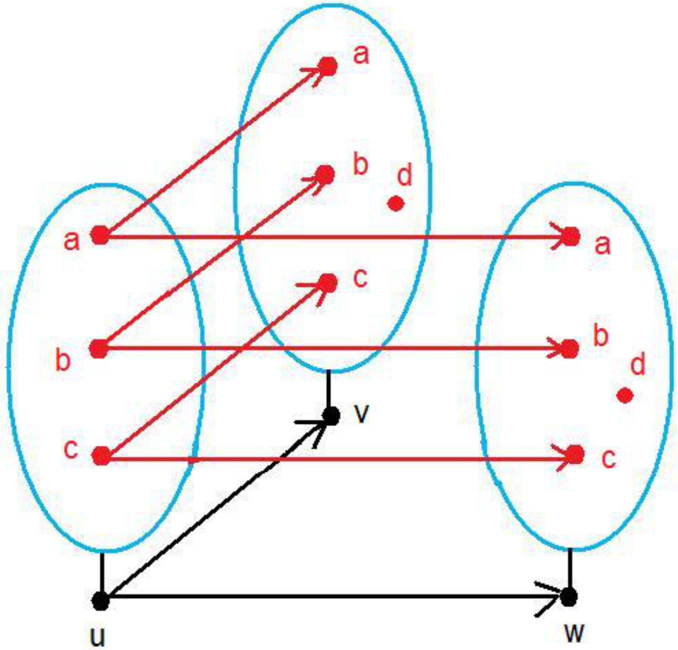
if $u R v$, then $D_u \subseteq D_v$

D_u is the domain at the world u

A Kripke model over Φ is a collection of classical models:

$M=(\Phi,\theta)$, where $\theta=(\theta_u)_{u \in W}$ is a valuation

$\theta_u(P)$ is an n-ary relation on D_u for each n-ary predicate letter P


a
b
d
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Kripke semantics-2

For a modal formula $A(x_1,\ldots, x_n)$ and $d_1,\ldots, d_n \in D_u$ consider a D_u-sentence $A(d_1,\ldots, d_n)$.

Def Forcing (truth) relation $M,u \vDash B$ between the worlds u and D_u-sentences B is defined by induction:

$M,u \vDash P(d_1,\ldots, d_n)$ iff $(d_1,\ldots, d_n) \in \theta_u(P)$

(for a proposition letter P: iff $\theta_u(P)=1$)

$M,u \vDash \Box B$ iff for any v, uRv implies $M,v \vDash B$

$M,u \vDash \forall x\, B$ iff for any $d \in D_u$ $M,u \vDash [d/x]B$

etc. (the other cases are clear)

Kripke semantics-3

Def (truth in a Kripke model; validity in a frame)

$M \vDash A(x_1,\ldots,x_n)$ iff for any $u \in W$

$$M,u \vDash \forall x_1 \ldots \forall x_n A(x_1,\ldots, x_n)$$

$\Phi \vDash A$ iff for any M over Φ, $M \vDash A$

Soundness theorem

$ML(\Phi) := \{A \mid \Phi \vDash A\}$ is an mpl (the *logic of* Φ)

Def The *logic of a class of frames* $\mathscr{C}$ is the intersection of the logics of frames from $\mathscr{C}$. A logic of a class of Kripke frames is called Kripke-complete.

Products and semiproducts of frames-1

In this talk we are interested only in products with universal frames.

Def. The *product* of Kripke frames

$$(W_1, R_1) \times (V, V\times V) := (W_1 \times V, R_h, R_v),$$

where

$$(x_1, y_1) R_h (x_2, y_2) \text{ iff } \quad x_1 R_1 x_2 \ \& \ y_1 = y_2$$

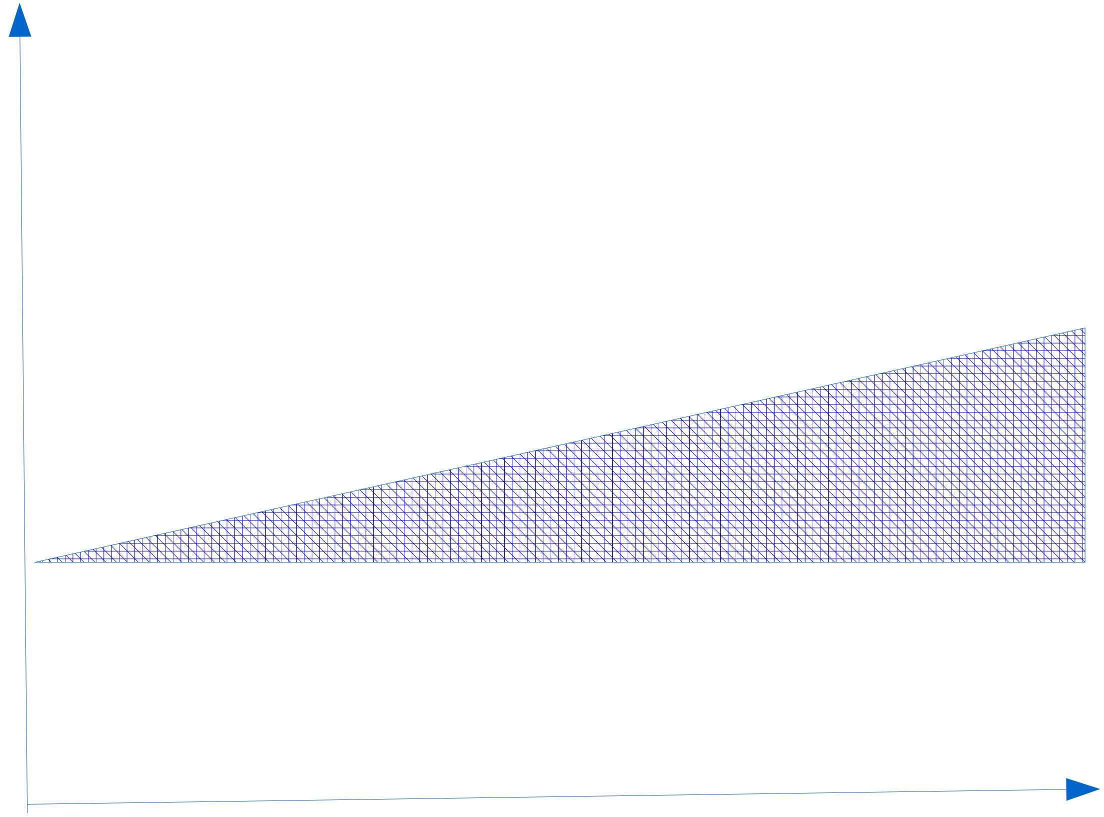
$$(x_1, y_1) R_v (x_2, y_2) \text{ iff } \quad x_1 = x_2$$

Def. A *semiproduct* (or an *expanding* product) is a subframe of a product, which is horizontably stable:

$$F = (F_1 \times F_2)|W, \text{ where } W \subseteq W_1 \times W_2,\ R_h(W) \subseteq W.$$

Products and semiproducts of frames-2

A semiproduct of a linear order (W,<) with a universal frame. It is the same as a predicate Kripke frame over (W,<):



Semiproducts and products of propositional modal logics

Def. The *semiproduct* of L_1 with S5

$L_1 \leftthreetimes \mathbf{S5} := \mathbf{ML}(\{F \mid F$ is a semiproduct of some

$F_1 \vDash L_1$ and a universal $F_2\})$

The *product* of L_1 with S5

$L_1 \times \mathbf{S5} := \mathbf{ML}(\{F \mid F$ is a product of some

$F_1 \vDash L_1$ and a universal $F_2\})$

Semiproducts and products-2

A semiproduct with a universal frame can be regarded as a predicate Kripke frame;

- a product as a predicate frame with a constant domain. Hence

Proposition $\mathbf{\Lambda \leftthreetimes S5 = \overline{Q\Lambda}\text{-}1,\ \Lambda \times S5 = \overline{Q\Lambda C}\text{-}1}$

Corollary $\mathbf{\Lambda \rfloor S5} \subseteq \mathbf{Q\Lambda}\text{-}1 \subseteq \mathbf{\Lambda \leftthreetimes S5}$

$\mathbf{[\Lambda, S5]} \subseteq \mathbf{Q\Lambda C}\text{-}1 \subseteq \mathbf{\Lambda \times S5}$

Def **Λ** is *quantifier-friendly* if $\mathbf{\Lambda \rfloor S5} = \mathbf{Q\Lambda}\text{-}1$.

- **Λ** is *Barcan-friendly* if $\mathbf{[\Lambda, S5]} = \mathbf{Q\Lambda C}\text{-}1$
- **Λ,S5** are *semiproduct-matching* if $\mathbf{\Lambda \leftthreetimes S5} = \mathbf{\Lambda \rfloor S5}$.
- **Λ,S5** are *product-matching* if $\mathbf{\Lambda \times S5} = \mathbf{[\Lambda, S5]}$

semiproduct-matching ⇒ *quantifier-friendly*,
product-matching ⇒ *Barcan-friendly*.

Semiproduct and products-3

Theorem 1A [MDML>>Gabbay & Sh 1998] If **Λ** is Horn axiomatizable and Kripke complete, then **Λ, S5** are product-matching.

Theorem 1B (MDML, Th.9.10). If **Λ** = **K, T, K4, S4, S5,** then **Λ, S5** are semiproduct-matching.

Def. A Horn sentence is a classical first-order sentence of the form $\forall x \forall y \forall \mathbf{z} \, (\varphi(x,y,\mathbf{z}) \to R(x,y))$, where φ is positive, $R(x,y)$ is atomic.

Semiproduct and products-4

A propositional modal logic is *Horn axiomatizable* if the class of its frames is definable by Horn sentences and modal variable-free formulas.

A typical example is the axiom 5: $\Diamond\Box p \to \Box p$ expressing *Euclideaness*: $\forall x \forall y \forall z\ (xRy \ \&\ xRz \to yRz)$.

Theorem B prompts that semiproduct-matching should be a rare property.

Incompleteness

Consider the logics

$\square$**T**:=**K**+$\square(\square p \to p)$ (frames: $\forall x \forall y\ (xRy \to yRy)$)

SL4 = **K** + $\square p \to \square\square p + \square p \leftrightarrow \Diamond p$

(the logic of the two-world frame

with the first world irreflexive and the second one reflexive)

_ Theorem 1C [Sh & Shkatov, in preparation]

If $\square$**T** $\subseteq$ **Λ** $\subseteq$ **SL4**, then

(1) **Λ** is not semiproduct-matching with **S5**

(2) **QΛ** is Kripke-incomplete

Quantifier-friendliness

Theorem 1A transforms as follows:

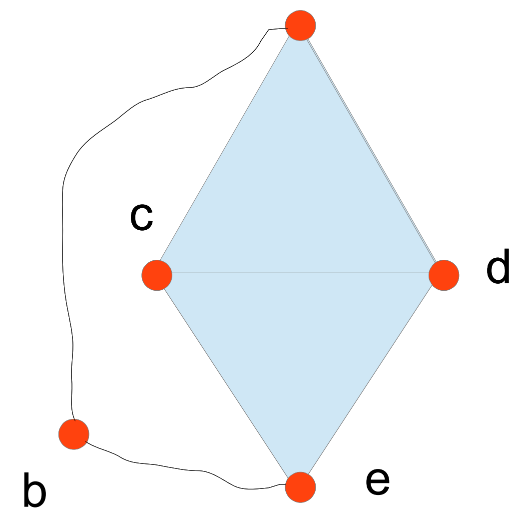
Theorem 2 [MDML>>Gabbay & Sh 1998] If **Λ** is Horn axiomatizable and Kripke complete, then **Λ** is quantifier-friendly.

For the proof we use s*implicial semantics* of first-order modal logics introduced by Dmitry Skvortsov in the early 1990s.

As we have seen, Kripke semantics does not work for **QΛ.** Other semantics (e.g. Ghilardi's functor semantics) may not work either.

Simplicial complexes

Geometric simplicial complex



Abstract simplicial complex

{acd, cde, ac, ad, cd, de, ce, ab, be, a,b,c,d,e}

$$X \in S \ \& \ Y \subset X \Rightarrow Y \in S$$

Simplicial sets
(J.P. May, 1967)

Δ is the category:

Ob **Δ** = ω,

Δ(m,n) = (non-strict) monotonic maps (m+1) → (n+1)

A *simplicial set* is a contravariant functor X: $\mathbf{\Delta}^{\circ} \rightsquigarrow$ **SET**

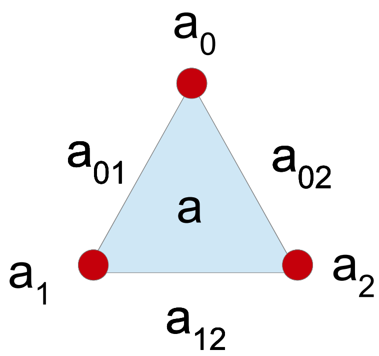
X(n) is the set of n-dimensional simplices

For every f ∈ **Δ**(m,n), X(f): X(n) → X(m) is a face map selecting an m-dimensional face of an n-dimensional simplex (it may be degenerate – if f is not injective)

Simplicial sets-2

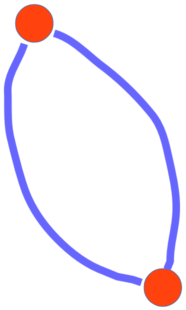
Example: If $a \in \mathbf{X}(2)$ is a triangle,

$f \in \mathbf{\Delta}(1,2)$, $f(0)=0$, $f(1)=2$, then $X(f)$ chooses the second side of a (it can be denoted by a_{02}) .



Two differences between simplicial complexes and simplicial sets:

- simplicial sets include degenerate simplices (such as a_{11}, a_{002})
- in simplicial sets two different simplices may have the same proper faces.



Simplicial frames-1

Introduced by D.Skvortsov (1990), an abstract (Skvortsov&Sh) in 1991; the paper in 1993.

In these publications simplicial frames we called 'Kripke metaframes'. Later the names were changed:

Kripke metaframes >> Simplicial frames

Cartesian metaframes >> Kripke metaframes

A *simplicial frame* is a modification of a simplicial set.

- $\mathbf{\Delta}$ is replaced by another category $\mathbf{\Sigma}$

Ob $\mathbf{\Sigma}$ = ω,

$\mathbf{\Sigma}_{mn}$ = *all* maps $I_m \to I_n$ (where $I_n=\{1,\dots,n\}$, $I_0=\varnothing$).

Let $\Sigma = \cup\{ \mathbf{\Sigma}_{mn} \mid m,n \geq 0\}$

- Accessibility relations are also involved

Simplicial frames-2

Roughly, a simplicial frame is a layered Kripke frame. The worlds are at level 0, individuals at level 1 (0–simplices), abstract n–tuples of individuals at level n ((n–1)–simplices).

Def A *simplicial frame* over a propositional Kripke frame F=(W,R) is **F** = (F, D, **R**, π), where

- $D=(D^n)_{n\geq 0}$, $\mathbf{R}=(R^n)_{n\geq 0}$, (D^n,R^n) is a propositional frame, $(D^0,R^0) = F$,
- $\pi = (\pi_\sigma)_{\sigma\in\Sigma}$, $\pi_\sigma : D^n \to D^m$ for $\sigma \in \Sigma_{mn}$

$\Sigma_{0n} =\{\varnothing_n\}$ (the empty map).

$\pi_{\varnothing_n}$ sends every absract n–tuple to "its possible world".

$D^n(u)$ denotes $(\pi_{\varnothing_n})^{-1}(u)$, the set of "n–tuples living in the world u",

Simplicial frames-3

A *metaframe* is a simplicial frame, in which n–tuples are real: $D^n(u)=(D^1(u))^n$

Definition A *valuation* in **F** is a function ξ such that $\xi_u(P) \subseteq D^n_u$ for every n–ary predicate letter P.

M=(**F**, ξ) is a *simplicial model* over **F**.

An *assignment* of length n at u is a pair (**x, a**), where **x** is a list of different variables of length n, $\mathbf{a} \in D^n(u)$. (We denote it by **a**/**x**.)

Simplicial frames-4

Definition (truth of a formula A in a simplicial model M at u under an assignment ($\mathbf{x}$, $\mathbf{a}$) involving the formula parameters)

This makes sense if $\mathbf{a}$ lives in u

Notation: M, $\mathbf{a}/\mathbf{x}$, u ⊨ A.

$M, \mathbf{a}/\mathbf{x}, u \models P(\mathbf{x}\cdot\sigma)$ iff $\pi_\sigma(\mathbf{a}) \in \xi_u(P)$, $(\mathbf{x}\cdot\sigma := (x_{\sigma(1)},\ldots,x_{\sigma(n)}))$,

$M, \mathbf{a}/\mathbf{x}, u \models \Box B$ (for $\mathbf{a} \in D^n(u)$) iff

$\forall v, \mathbf{b}$ $(uRv \;\&\; \mathbf{b} \in D^n(v) \;\&\; \mathbf{a}R^n\mathbf{b} \Rightarrow M, \mathbf{b}/\mathbf{x}, v \models B)$

$M, \mathbf{a}/\mathbf{x}, u \models \exists y\, B$ (for $y \notin \mathbf{x}$, $\mathbf{a} \in D^n(u)$) iff

$\exists \mathbf{c} \in D^{n+1}(u)$ $(\pi_{\delta_{n+1}}(\mathbf{c}) = \mathbf{a} \;\&\; M, \mathbf{c}/\mathbf{x}y \models B)$,

$M, \mathbf{a}/\mathbf{x}, u \models \exists x_i\, B$ iff $M, \pi_{\delta_i}(\mathbf{a})/(\mathbf{x}\cdot\delta_i), u \models \exists x_i\, B$, where δ_i is the monotonic inclusion map $I_n \to I_{n+1}$ skipping i.

Simplicial frames-5

Truth in a model:

$M \vDash A(x_1,\dots,x_n)$ iff for any $u \in W$ $M,u,/ \vDash \forall x_1 \dots \forall x_n A(x_1,\dots,x_n)$

Validity in a frame: $\Phi \vDash A$ iff for any M over Φ, $M \vDash A$

Strong validity in a frame: $\Phi \vDash^+ A$ iff for any n $\Phi \vDash A^n$.

Soundness theorem (Skvortsov,1991-93)

$\mathbf{ML}(\Phi) := \{A \in MF \mid \Phi \vDash^+ A\}$ is an mpl if Φ satisfies the conditions

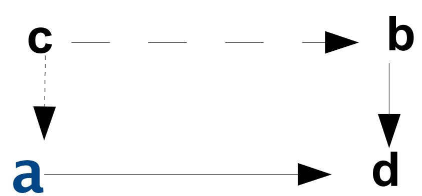
- $\pi_{\varnothing_n}$ is surjective,
- $\pi_{\sigma\cdot\tau} = \pi_\tau \cdot \pi_\sigma$; $\pi_{id(I_n)} = id(D^n)$. [id(X) is the identity map on X]
- for $\sigma \in \Sigma_{mn}$ $\pi_\sigma : (D^n, R^n) \to (D^m, R^m)$ is a morphism, i.e.,

$$\pi_\sigma(R^n(\mathbf{a})) = R^m(\pi_\sigma(\mathbf{a})) \text{ for any } \mathbf{a} \in D^n.$$

Simplicial frames-6

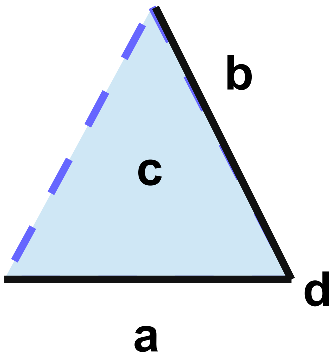
• (weak Kan condition) if $\pi_{\delta_{m+1}}(\mathbf{b}) = \pi_\sigma(\mathbf{a})=\mathbf{d}$, $\sigma \in \Sigma_{mn}$, then

for some $\mathbf{c} \in D^{n+1}$ $\pi_{\sigma_+}(\mathbf{c})=\mathbf{b}$ & $\pi_{\delta_{n+1}}(\mathbf{c}) = \mathbf{a}$.



($\sigma_+ \in \Sigma_{m+1,n+1}$ extends σ by $\sigma_+(m+1)=n+1$)

In particular, this means that two simplices with a common face are faces of a simplex of higher dimension:



In metaframes: $\mathbf{d}=a_{\sigma(1)}\ldots a_{\sigma(m)}$, $\mathbf{b}=\mathbf{d}b_{m+1}$; then $\mathbf{c}=\mathbf{a}b_{m+1}$

Simplicial frames-7

The method of proof of Theorem 2 (on quantifier-friendliness): for Horn-axiomatizable and complete **Λ**

if **Λ** $\nvDash$ A*, then **QΛ** $\nvDash$ A.

Suppose **Λ** $\nvDash$ **A***, then by completeness there is a frame $G=(W,R_1,R_2) \nvDash A^*$, $G \vDash$ **Λ.** We then construct a simplicial frame **F** $\nvDash$ A*, **F** $\vDash$ **QΛ.**

- Extract F_1, F_0 from G: $F_1=(W,R_1)$, $F_0=(W/R_2,R_0)$,

$u^{\sim}R_0v^{\sim}$ iff $\exists u'\in u^{\sim}\exists v'\in v^{\sim}u'R_1v'$

Put $\pi_{\varnothing 1}(u):=u^{\sim}$.

Simplicial frames-8

- The main construction: the *conglomerate* **F** over G.

  $(D^n,R^n) = (F_1)^n$ (in the standard model-theoretic sense) $\coprod$

  a disjoint union of several copies of $(F_1)^m$ for $m<n$.

  As Horn sentences respect model-theoretic products, it follows that $(D^n,R^n) \vDash \mathbf{\Lambda}$.

  Lemma (Skvortsov) If Φ is a simplicial frame, B a propositional formula, then $\Phi \vDash^+ B$ iff $(D^n,R^n) \vDash B$ for all n.

  Hence $\mathbf{F} \vDash^+ \mathbf{Q\Lambda}$.

  $\mathbf{F} \nvDash A$ follows easily from $G \nvDash A_*$. QED.

Some open problems

1. Describe **QΛ**-1 when **Λ** is not quantifier-friendly. What happens for **Λ** = **S4.1, S4.2, S4.3**?

2. If **Λ** is decidable, can **QΛ**-1 be undecidable? This question makes sense already for quantifier-friendly **Λ**.

3. The same questions for **QΛC**-1 and Barcan-friendliness.

THANK YOU!!!

Incompleteness-2

There is a continuum of logics $\Box\mathbf{T} \subseteq \mathbf{\Lambda} \subseteq \mathbf{SL4}$

Theorem $\Box\mathbf{T}$, $\mathbf{SL4}$ are quantifier-friendly.

Some examples of completions and semiproducts

Consider the logics

$\mathbf{\Lambda = \Box T, K5, K45, SL4, \Box S5}$

$\mathbf{K5} = \mathbf{K} + \Diamond\Box p \to \Box p$

$\mathbf{K45} = \mathbf{K} + \Diamond\Box p \to \Box p$

<u>Theorem</u>

- $\overline{\mathbf{Q\Lambda}} = \mathbf{Q\Lambda} + \Box\forall x(\Box P(x) \to P(x))$
- $\mathbf{\Lambda} \leftthreetimes \mathbf{S5} = \mathbf{\Lambda} \rfloor\, \mathbf{S5} + \Box\blacksquare(\Box p \to p)$
- These logics $\mathbf{\Lambda} \leftthreetimes \mathbf{S5}$ have the FMP

Remarks on 1-variable fragments

Remark 1 (folklore? Behmann 1922?)

Every monadic classical first-order formula is equivalent to a Boolean combination of 1-variable formulas. So every monadic classical first-order formula with one parameter is equivalent to a 1-variable formula.

However the complexity of monadic classical logic is higher than of **S5** (NTIME($2^{n/\log n}$) > NP).

Remarks on 1-variable fragments-2

<u>Def</u> (Wolter & Zakharyaschev) A first-order modal formula is *monodic* if in every its subformula $\Box A$ A has at most one parameter.

<u>Remark 2</u> Every monadic monodic first-order formula is equivalent (in **QK**) to a Boolean combination of 1-variable formulas. So every monadic monodic first-order formula with one parameter is equivalent to a 1-variable formula.

Again: the complexities must be different.