A new logic arising from a scattered Stone space

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TACL 2019, Nice, France
17–21 June 2019

Syntax and Topological Semantics

Signature

- Countably many propositional letters
- Classical connectives: $\neg$ and $\to$
- Modal connective: $\Box$
- Typical abbreviations: $\Diamond\varphi := \neg\Box\neg\varphi$, $\varphi \vee \psi := \neg\varphi \to \psi$, $\varphi \wedge \psi := \neg(\varphi \to \neg\psi)$, and $\top := p \vee \neg p$

Topological Interpretation

Given a space X:

- Letters $\Rightarrow$ subsets of X
- Classical connectives $\Rightarrow$ Boolean operations in $\wp(X)$
- Modal box $\Rightarrow$ interior operator $\mathbf{i}$ of X; hence, diamond $\Rightarrow$ closure operator $\mathbf{c}$ of X

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Topological Semantics and S4

Valid Modal formulas

Call a formula φ valid in X provided it evaluates to X for any interpretation of the letters; in symbols $X \Vdash \varphi$

Valid Formulas	Corresponding Property
$\Box\top \leftrightarrow \top$	$\mathbf{i}X = X$
$\Box p \to p$	$\mathbf{i}A \subseteq A$
$\Box p \to \Box\Box p$	$\mathbf{i}A \subseteq \mathbf{ii}A$
$\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$	$\mathbf{i}(A \cap B) = \mathbf{i}A \cap \mathbf{i}B$

The logic of X is $\mathrm{Log}(X) = \{\varphi \mid X \Vdash \varphi\}$

Theorem (McKinsey and Tarski 1944)

For any space X, $\mathrm{Log}(X)$ is a normal extension of S4

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Topological Semantics and S4-frames

Generalizing Kripke Semantics for S4

- An S4-frame is $\mathfrak{F} = (W, R)$ where R is a reflexive and transitive relation on W
- An R-upset in $\mathfrak{F}$ is $U \subseteq W$ such that $w \in U$ and wRv imply $v \in U$
- The set of R-upsets forms the Alexandroff topology τ_R on W

Theorem (folklore)

- For an S4-frame $\mathfrak{F} = (W, R)$, $\mathfrak{F} \Vdash \varphi$ iff $(W, \tau_R) \Vdash \varphi$
- A Kripke complete extension of S4 is topologically complete

Observation

Such topological completeness is almost never with respect to spaces satisfying higher separation axioms

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Known Topological Completeness Results

McKinsey and Tarski 1944

For a separable crowded metrizable space X, $\mathrm{Log}(X) = \mathrm{S4}$

S4

Known Topological Completeness Results

Rasiowa and Sikorski 1963

For a crowded metrizable space X, $\text{Log}(X) = \text{S4}$

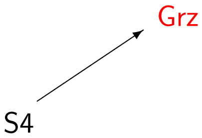
S4

Known Topological Completeness Results

Abashidze 1987 and Blass 1990 (independently)

For any ordinal space $\alpha \geq \omega^\omega$, $\mathsf{Log}(\alpha) = \mathsf{Grz}$

$\mathsf{Grz} := \mathsf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$



Known Topological Completeness Results

Abashidze 1987 (see also Bezhanishvili and Morandi 2010)

For an ordinal α such that $\omega^{n-1}+1 \leq \alpha \leq \omega^n$, $\mathsf{Log}(\alpha) = \mathsf{Grz}_n$

$$
\begin{aligned}
\mathsf{bd}_1 &:= \Diamond\Box p_1 \to p_1 \\
\mathsf{bd}_{n+1} &:= \Diamond(\Box p_{n+1} \wedge \neg \mathsf{bd}_n) \to p_{n+1} \\
\mathsf{Grz}_n &:= \mathsf{Grz} + \mathsf{bd}_n
\end{aligned}
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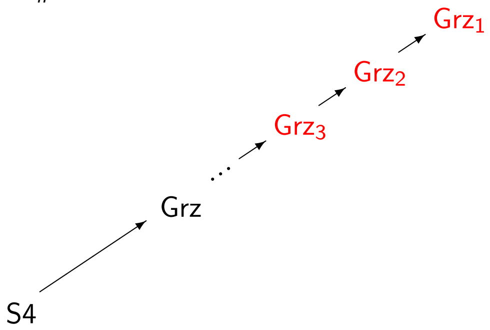
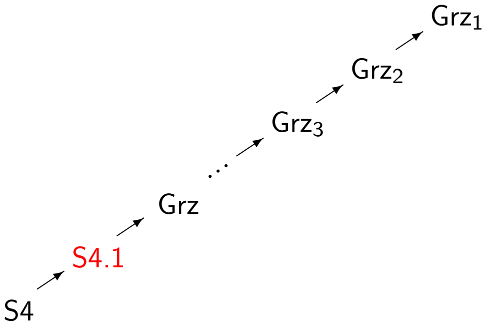


Known Topological Completeness Results

Bezhanishvili, Gabelaia, and L-B 2015

Metrizable spaces yield exactly these logics: S4, S4.1, Grz, or Grz_n

$\mathsf{S4.1} := \mathsf{S4} + \Box\Diamond p \rightarrow \Diamond\Box p$

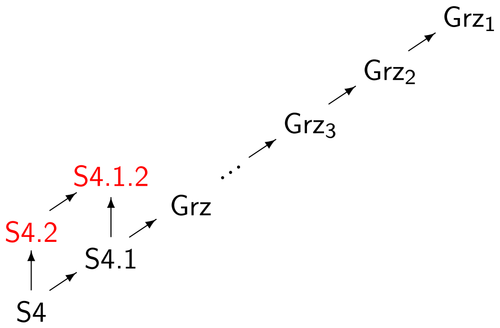


Known Topological Completeness Results

Bezhanishvili and Harding 2012

Each of the following logics arises from a Stone space

$$\mathsf{S4.2} \quad := \quad \mathsf{S4} + \Diamond\Box p \to \Box\Diamond p$$
$$\mathsf{S4.1.2} \quad := \quad \mathsf{S4} + \Box\Diamond p \leftrightarrow \Diamond\Box p$$

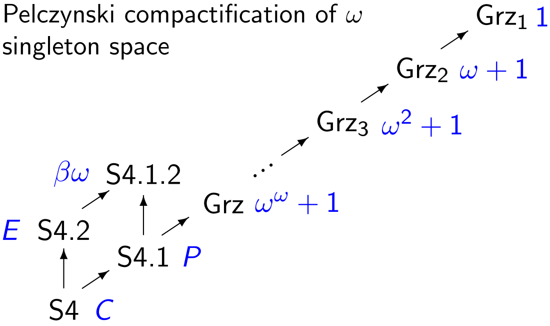


Known Topological Completeness Results

Examples

A Stone space giving rise to each logic below

C	$:=$	the Cantor space
E	$:=$	the Gleason cover of $[0,1]$
$\beta\omega$	$:=$	the Čech-Stone compactification of ω
P	$:=$	the Pelczynski compactification of ω
1	$:=$	the singleton space



Our Goal

Question posed in Bezhanishvili and Harding 2012

Is there a Stone space whose logic is not in the previous list?

Answer

Yes!

We build a space whose logic is strictly between Grz_3 and Grz_2

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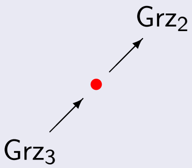

Mrowka Spaces

Recall

Call a family $\mathscr{R}$ of infinite subsets of ω almost disjoint provided $\forall R, Q \in \mathscr{R}$, if $R \neq Q$ then $R \cap Q$ is finite

Definition

A Mrowka space is $X := \omega \cup \mathscr{R}$ where $\mathscr{R}$ is almost disjoint and whose topology is generated by the basis consisting of:

- $O(n) := \{n\}$ for $n \in \omega$
- $O(R, F) := \{R\} \cup (R \setminus F)$ for $R \in \mathscr{R}$ where $F \subset \omega$ is finite

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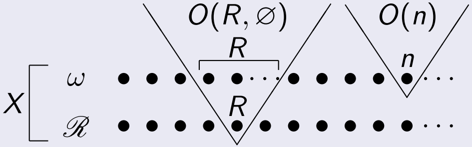

Properties of Mrowka Spaces

Let $X = \omega \cup \mathscr{R}$ be a Mrowka space

Theorem (Mrowka)

- ω is open and dense in X
- $\mathscr{R}$ is closed and discrete in X
- Each $O(R, F)$ is clopen in X
- Each $O(R, \varnothing)$ is homeomorphic to the one-point compactification of ω, which is homeomorphic to the ordinal space $\omega + 1$

Corollary

- X is a scattered locally compact Hausdorff space
- if $\mathscr{R}$ is infinite then X is not compact

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The Spaces of Interest I

Theorem (Mrowka)

There is an infinite almost disjoint family $\mathscr{R}$ such that the Čech-Stone compactification βX of the Mrowka space $X = \omega \cup \mathscr{R}$ is the one-point compactification αX of X

Convention for this talk

Any Mrowka space $X = \omega \cup \mathscr{R}$ is such that $\beta X = \alpha X := X \cup \{\infty\}$

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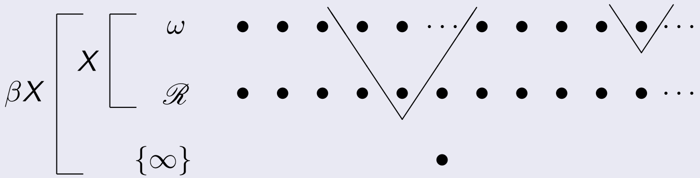

The Spaces of Interest II

Theorem

If X is a Mrowka space then the space $\beta X = X \cup \{\infty\}$ is a scattered Stone space of Cantor-Bendixson rank 3

Proof sketch

- Clearly βX is compact and Hausdorff
- Letting $\mathbf{d}$ be the derived set operator in βX, we have

$$\mathbf{ddd}(\beta X) = \mathbf{dd}(\mathscr{R} \cup \{\infty\}) = \mathbf{d}(\{\infty\}) = \varnothing$$

  Thus βX is scattered and of Cantor-Bendixson rank 3
- A compact scattered space is zero-dimensional

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Main Tools

Definitions and known results

Let X and Y be spaces

- Y is an interior image of X if there is $f : X \to Y$ which is onto such that $f^{-1}(\mathbf{c}_Y A) = \mathbf{c}_X f^{-1}(A)$ for each $A \subseteq Y$
- If Y is an interior image of X then $\text{Log}(X) \subseteq \text{Log}(Y)$
- If X is scattered then
  - $X \Vdash \Box(\Box(p \to \Box p) \to p) \to p$
  - $X \Vdash \text{bd}_n$ iff the Cantor-Bendixson rank of X is $\leq n$

Let $\mathfrak{F}$ be a finite rooted S4-frame

- Let $\chi_{\mathfrak{F}}$ denote the Jankov-Fine formula of $\mathfrak{F}$, which syntactically characterizes the structure of $\mathfrak{F}$
- $X \Vdash \neg\chi_{\mathfrak{F}}$ iff $\mathfrak{F}$ is not an interior image of any open subspace of X

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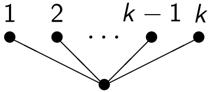
On Interior Images of βX

Let X be a Mrowka space such that $\beta X = X \cup \{\infty\}$,
$\mathfrak{F}$ be a finite partially ordered S4-frame and

for nonzero $k \in \omega$, let $\mathfrak{T}_k$ be the tree



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For nonzero $k \in \omega$, the tree $\mathfrak{T}_k$ is an interior image of βX

As Grz_2 is the logic of $\{\mathfrak{T}_k \mid k \in \omega \setminus \{0\}\}$,

Corollary

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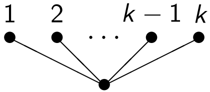

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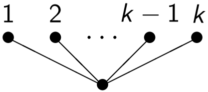

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ON INTERIOR IMAGES OF βX

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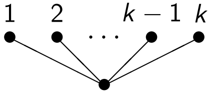
$\mathfrak{F}$ be a finite partially ordered S4-frame and

for nonzero $k \in \omega$, let $\mathfrak{T}_k$ be the tree



LEMMA

$\mathfrak{F}$ is an interior image of βX iff $\mathfrak{F}$ is an interior image of an open subspace of X

LEMMA

For nonzero $k \in \omega$, the tree $\mathfrak{T}_k$ is an interior image of βX

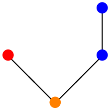
As Grz_2 is the logic of $\{\mathfrak{T}_k \mid k \in \omega \setminus \{0\}\}$,

COROLLARY

$\mathrm{Log}(\beta X) \subset \mathrm{Grz}_2$ (strict since $\beta X \not\Vdash \mathrm{bd}_2$ as βX is of C-B rank 3)

A Tree that is NOT an Interior Image

Let X be a Mrowka space such that $\beta X = X \cup \{\infty\}$ and

$\mathfrak{T}$ the tree 

Lemma

$\mathfrak{T}$ is not an interior image of βX

Proof (sketch)

Let $f : \beta X \to \mathfrak{T}$ be an onto interior map

- ∞ is the only preimage of the root
- Let A be the preimage of red and B the preimage of blue
- $\infty \in \mathbf{c}A \cap \mathbf{c}B$
- A and B are disjoint opens in X with $X = A \cup B$, hence completely separated in X giving $\mathbf{c}A \cap \mathbf{c}B = \varnothing$ in βX

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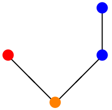
$\mathfrak{T}$ the tree 

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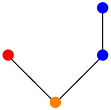
$\mathfrak{T}$ the tree 

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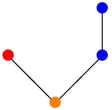
$\mathfrak{T}$ the tree 

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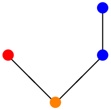
$\mathfrak{T}$ the tree 

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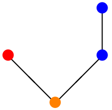
$\mathfrak{T}$ the tree 

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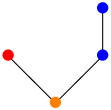
$\mathfrak{T}$ the tree 

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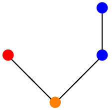
$\mathfrak{T}$ the tree 

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The Main Result

Let X be a Mrowka space such that $\beta X = X \cup \{\infty\}$

Theorem

$\mathsf{Grz}_3 + \neg\chi_{\mathfrak{T}} \subseteq \mathsf{Log}(\beta X) \subset \mathsf{Grz}_2$

Proof (sketch)

- As βX is scattered with Cantor-Bendixson rank 3, $\mathsf{Grz}_3 \subseteq \mathsf{Log}(\beta X)$
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The Unfinished Story

Let X and Y be Mrowka spaces such that $\beta X = X \cup \{\infty\}$ and $\beta Y = Y \cup \{\infty\}$

Open problems

- Is it the case that $\mathrm{Log}(\beta X) = \mathrm{Log}(\beta Y)$ when X and Y are not homeomorphic?
- If so, is $\mathrm{Log}(\beta X)$ finitely axiomatizable?
- If not:
  - How many logics arise in this manner?
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Thank You...

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