Semantic Analysis and Proof Theory for monotonic modal Logics

Jinsheng Chen

Delft University of Technology

Joint work with G. Greco, A. Palmigiano and A. Tzimolis

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Overview

Tools:

- Translation from monotonic modal logic E to a multimodal modal logic
- ALBA (Ackermann Lemma Based Algorithm)

Tools:

- Develop display calculi for E and its extensions
- Calculate first order correspondents of formulas over monotonic neighbourhood frames

Monotonic modal logic $\mathbf{L}_\nabla$

Syntax

Set of formulas in the language $\mathcal{L}_\nabla = (\wedge, \neg, \nabla)$

- containing all propositional tautologies
- closed under modus ponens, uniform substitution and

$$\frac{\varphi \to \psi}{\nabla\varphi \to \nabla\psi}$$

Semantics

Monotone neighbourhood frames: (W, ν) s.t.

- $\nu : W \to \mathcal{P}\mathcal{P}(W)$
- Closed under supersets:
  $\forall w \in W[X \in \nu(w) \;\&\; X \subseteq Y \Rightarrow Y \in \nu(w)]$

Translation

- $\mathcal{F} = (W, \nu) \rightsquigarrow \mathcal{F}' = (W, \mathcal{P}(W), R_{\nu^c}, R_\nu, R_\ni, R_{\not\ni})$, where , such that
  - $R_\nu \subseteq W \times \mathcal{P}(W)$, $wR_\nu X$ iff $X \in \nu(w)$
  - $R_{\nu^c} \subseteq W \times \mathcal{P}(W)$, $wR_{\nu^c} X$ iff $X \notin \nu(w)$
  - $R_\ni \subseteq \mathcal{P}(W) \times W$, $XR_\ni x$ iff $x \in X$
  - $R_{\not\ni} \subseteq \mathcal{P}(W) \times W$, $XR_{\not\ni} x$ iff $x \notin X$

Let $\mathcal{L}_3$ be the language containing $\wedge$, $\neg$, $\Box_\nu$, $\Box_\ni$, $\Box_{\nu^c}$, $\Diamond_{\not\ni}$. By monotonicity, we have

- $\mathcal{F}' \models \Diamond_\nu \Box_\ni \varphi \Leftrightarrow \mathcal{F}' \models \Box_{\nu^c} \Diamond_{\not\ni} \varphi$

Translation

Define the translations $\tau_1, \tau_2 : \mathcal{L}_1 \to \mathcal{L}_3$ as follows:

- $\tau_1(p) = \tau_2(p) = p$
- $\tau_1(\varphi \wedge \psi) = \tau_1(\varphi) \wedge \tau_1(\psi)$,$\tau_2(\varphi \wedge \psi) = \tau_2(\varphi) \wedge \tau_2(\psi)$
- $\tau_1(\neg\varphi) = \neg\tau_2(\varphi)$, $\tau_2(\neg\varphi) = \neg\tau_1(\varphi)$
- $\tau_1(\nabla\varphi) = \Diamond_\nu \Box_\ni \tau_1(\varphi)$, $\tau_2(\nabla\varphi) = \Box_{\nu^c} \Diamond_{\not\ni} \tau_1(\varphi)$

We have:

$$\mathcal{F} \models \varphi \vdash \psi \Leftrightarrow \mathcal{F}' \models \tau_1(\varphi) \vdash \tau_2(\psi)$$

- The context sensetive translation is for a better application of ALBA.

Display Calculi

Let $\mathcal{K}$ be the set of two-sorted normal Kripke frames $(X, Y, R_1, R_2, R_3, R_4)$, where $R_1, R_3 \subseteq X \times Y$ and $R_2, R_4 \subseteq Y \times X$. Hence $\mathcal{C}' \subseteq \mathcal{K}$. Then we have:

$$
\begin{array}{lll}
 & \mathcal{C} \models \varphi \vdash \psi & \\
\Leftrightarrow & \mathcal{C}' \models \tau_1(\varphi) \vdash \tau_2(\psi) & (\textit{translation}) \\
\Leftarrow & \mathcal{K} \models \tau_1(\varphi) \vdash \tau_2(\psi) & (\mathcal{C}' \subseteq \mathcal{K}) \\
\Leftrightarrow & G \vdash \tau_1(\varphi) \vdash \tau_2(\psi) &
\end{array}
$$

where G is a cut-free display calculus sound and complete with respect to K.

Extend *G* using ALBA

ALBA for proof theory provides the following theorem:

Theorem

Let $\mathfrak{A}$ be a normal distributive lattice extension and G a cut-free display calculus sound and complete respect to $\mathfrak{A}$. If $\varphi \leq \psi$ is a analytic inductive inequality then $\mathfrak{A} + \{\varphi \vdash \psi\}$ is sound and complete with respect to $G +$ ALBA$(\varphi \leq \psi)$.

Extend G using ALBA

Structural rule of $\varphi \vdash \psi$:

$$
\begin{array}{lll}
 & \mathcal{C} \models \forall \overrightarrow{p}[\varphi \vdash \psi] & \\
\Leftrightarrow & \mathcal{C}' \models \forall \overrightarrow{p}[\tau_1(\varphi) \vdash \tau_2(\psi)] & (\textit{translation}) \\
\Leftarrow & \mathcal{K} \models \forall \overrightarrow{p}[\tau_1(\varphi) \vdash \tau_2(\psi)] & (\mathcal{C}' \subseteq \mathcal{K}) \\
\Leftrightarrow & \mathcal{K}^+ \models \forall \overrightarrow{p}[\tau_1(\varphi) \leq \tau_2(\psi)] & (\textit{complex algebras}) \\
\Leftrightarrow & \mathcal{K}^+ \models \textit{ALBA}(\tau_1(\varphi) \leq \tau_2(\psi)) & (\textit{ALBA})
\end{array}
$$

Extend G using ALBA

$$\mathbf{C}.\nabla p \wedge \nabla q \rightarrow \nabla(p \wedge q)$$

$$\begin{array}{ll} & \mathcal{C} \models \forall p \forall q[\nabla p \wedge \nabla q \vdash \nabla(p \wedge q)] \\ \Leftrightarrow & \mathcal{C}' \models \forall p \forall q[\Diamond_\nu \Box_\ni p \wedge \Diamond_\nu \Box_\ni q \vdash \Box_{\nu^c} \Diamond_{\not\ni}(p \wedge q)] \\ \Leftarrow & \mathcal{K} \models \forall p \forall q[\Diamond_\nu \Box_\ni p \wedge \Diamond_\nu \Box_\ni q \vdash \Box_{\nu^c} \Diamond_{\not\ni}(p \wedge q)] \\ \Leftrightarrow & \mathcal{K}^+ \models \forall p \forall q[\Diamond_\nu \Box_\ni p \wedge \Diamond_\nu \Box_\ni q \leq \Box_{\nu^c} \Diamond_{\not\ni}(p \wedge q)] \\ \Leftrightarrow & \mathcal{K}^+ \models \forall x \forall y \forall z[\Diamond_{\not\ni}(\blacklozenge_\in x \wedge \blacklozenge_\in y) \leq z \Rightarrow \Diamond_\nu x \wedge \Diamond_\nu y \leq \Box_{\nu^c} z] \end{array}$$

Structural rule for **C**:

$$\frac{\overline{\Diamond}_{\not\ni}(\overline{\blacklozenge}_\in X \overline{\wedge} \overline{\blacklozenge}_\in Y) \leq Z}{\overline{\Diamond}_\nu X \overline{\wedge} \overline{\Diamond}_\nu Y \leq \overline{\Box}_{\nu^c} Z} R$$

$\mathcal{K} + \mathbf{C}$ is sound and complete with respect to $G + R$.

$$E + \mathbf{C} \vdash \varphi \rightarrow \psi \Leftarrow G + R \vdash \tau_1(\varphi) \vdash \tau_2(\psi)$$

Correspondce Theory

ALBA for correspondence theory provides the following theorem:

Theorem

Let $\mathfrak{A}$ be a normal distributive lattice extension. If $\mathfrak{A}$ is complete and atomic and $\varphi \leq \psi$ be an inductive inequality then

$$\mathfrak{A} \models \forall \overrightarrow{p}[\varphi \leq \psi] \Leftrightarrow \mathfrak{A} \models \forall \overrightarrow{i} \forall \overrightarrow{m}\ ALBA(\varphi \leq \psi)$$

where $\overrightarrow{i}$ is a sequence of nomials, $\overrightarrow{m}$ is a sequence of conomials and $ALBA(\varphi \leq \psi)$ is a quasi-inequality not containing proposition letters.

- In complex algebras, nomials are singleton $\{x\}$ and conomials are $W \setminus \{x\}$.

Correspondence Theory

First-order correspondent of $\varphi \vdash \psi$ over $\mathcal{C}$:

$$
\begin{array}{lll}
 & \mathcal{C} \models \forall \overrightarrow{p}[\varphi \vdash \psi] & \\
\Leftrightarrow & \mathcal{C}' \models \forall \overrightarrow{p}[\tau_1(\varphi) \vdash \tau_2(\psi)] & (\textit{translation}) \\
\Leftrightarrow & \mathcal{C}'^{+} \models \forall \overrightarrow{p}[\tau_1(\varphi) \leq \tau_2(\psi)] & (\textit{complex algebras}) \\
\Leftrightarrow & \mathcal{C}'^{+} \models \forall \overrightarrow{i} \forall \overrightarrow{m}\ ALBA(\tau_1(\varphi) \leq \tau_2(\psi)) & (\textit{ALBA}) \\
\Leftrightarrow & \mathcal{C}' \models FO(ALBA(\tau_1(\varphi) \leq \tau_2(\psi))) & \\
\Leftrightarrow & \mathcal{C} \models FO(ALBA(\tau_1(\varphi) \leq \tau_2(\psi))) &
\end{array}
$$

$$C.\nabla p \wedge \nabla q \rightarrow \nabla(p \wedge q)$$

$$
\begin{array}{ll}
& \mathcal{C} \models \forall p \forall q[\nabla p \wedge \nabla q \vdash \nabla(p \wedge q)] \\
\Leftrightarrow & \mathcal{C}' \models \forall p \forall q[\Diamond_\nu \Box_\ni p \wedge \Diamond_\nu \Box_\ni q \vdash \Box_{\nu^c} \Diamond_{\not\ni}(p \wedge q)] \\
\Leftrightarrow & \mathcal{C}'^+ \models \forall p \forall q[\Diamond_\nu \Box_\ni p \wedge \Diamond_\nu \Box_\ni q \leq \Box_{\nu^c} \Diamond_{\not\ni}(p \wedge q)] \\
\Leftrightarrow & \mathcal{C}'^+ \models \forall i_1 \forall i_2[\Diamond_\nu i_1 \wedge \Diamond_\nu i_2 \leq \Box_{\nu^c} \Diamond_{\not\ni}(\blacklozenge_\in i_1 \wedge \blacklozenge_\in i_2)] \\
\Leftrightarrow & \mathcal{C}'^+ \models \forall X \forall Y[\Diamond_\nu \{X\} \wedge \Diamond_\nu \{Y\} \leq \Box_{\nu^c} \Diamond_{\not\ni}(\blacklozenge_\in \{X\} \wedge \blacklozenge_\in \{Y\})] \\
\Leftrightarrow & \mathcal{C}' \models \forall X \forall Y[\{w : wR_\nu X)\} \cap \{w : wR_\nu Y)\} \subseteq \\
& \qquad \{w : \forall Z(xR_{\nu^c} Z \Rightarrow \exists(x \in X \cap Y(ZR_{\not\ni} x))\} \\
\Leftrightarrow & \mathcal{C} \models \forall w \forall X \forall Y[X \in \nu(w) \ \&\ Y \in \nu(w) \Rightarrow X \cap Y \in \nu(w)]
\end{array}
$$

Conclusion

- We use a modified translation and ALBA (Ackermann Lemma Based Algorithm) to investigate the proof theory and corrspondence theory of monotonic modal logic and its extension.
- We can also use this method for those non-normal logics which can be translated into normal logics, like conditional logic.