Point-free theories of space and time: topological models, representation theory and duality

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Outline

1. Introduction
   - Region-base theory of space and time - Whitehead's view and a short description of the field
   - Contact and precontact algebras
2. Dynamic contact algebras (DCA)
   - The main idea of the definition
   - Dynamic contact algebras, abstract definition
   - Abstract points in DCA
3. Topological models and duality
   - Dynamic Mereotopological Spaces (DMS)
   - Duality theory
4. Literature
   - Survey papers
   - Some other papers

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Point-free theory of space and time

The roots of point-free theory of space and time go to some ideas of Whitehead formulated in an early form in his book

- **The Organization of Thought (1917) (page 195)**

and developed later on in the books

- **The Concept of Nature, Cambridge Univ. Press, Cambridge (1920)**,
- **Science and the Modern World, The Great Books of the Western World, 55, The MacMillan, New Work (1925)**,
- **Process and Reality: An Essay in Cosmology, Macmillan Co., New York (1929).**

Point-free theory of space and time: **SUMMARY**

These ideas can be summarized as follows:

(1) Space should be extracted from some relations between existing things in reality.

(2) Points (lines, surfaces) do not have separate existing in reality and should not be put as primitive notions in the theory of space. So,

(3) The theory of space should be "point-free".

(4) Instead, some more realistic notions like **region** (as abstraction of real thing) and some relations between regions like "**part-of**" and "**contact**" should be used. So,

(5) The theory of space should be "**region-based**".

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The same things have to be said for time:

(6) Time points (moments of time) also do not have separate existing in reality and should be excluded as a primitive notion of the theory.

(7) Time should be extracted from some spatio-temporal relations between changing things in reality. Hence

(8) The **theory of time should** also be "**point-free**". And finally

(9) Like in relativity theory the theory of **time** should not be separated from the theory of **space** and they both should be developed **together**.

Mereotopology and contact algebras

• Contemporary theory of space in the style of Whitehead is called "(R)egion (B)ased (T)heory of (S)pace **(RBTS).** It is a point-free theory based on the notion of **region** and some relations taken from **MEREOLOGY** like **part-of** and **overlap** and on some relations taken from **TOPOLOGY** like **contact.** For these reasons **RBTS** is also called **MEREOTOPOLOGY**.

• One algebraic formulation of mereotopology is based on some notions of **CONTACT ALGEBRA** (to be introduced later in the talk).

Dynamic mereotopology and dynamic contact algebras

• Integrated theory of space and time incorporates RBTS and mereotopology and can be considered as **DYNAMIC MEREOTOPOLOGY**.

• An algebraic formulation of of dynamic mereotopology is based on some notions of **DYNAMIC CONTACT ALGEBRA** (to be introduced later on in the talk).

THE AIM OF THIS TALK

The aim of this talk is,

• **First,** to introduce in some details one (abstract) version of **dynamic contact algebra** (DCA), extracted from a specific model called **SNAPSHOT MODEL,**

• **Second,** to introduce **topological models** for this version of DCA and the expected representation theory,

• **Third** to extend the representation theory to several kinds of **topological duality.**

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Definition of contact and precontact algebras

Let $\underline{B} = (B, 0, 1, ., +, *, C)$ be a non-degenerate Boolean algebra and C – a binary relation in B. C is called a contact relation in $\underline{B}$ if it satisfies the axioms

$(C1)$ If xCy, then $x, y \neq 0$,

$(C2)$ If xCy and $x \leq x'$ and $y \leq y'$, then $x'Cy'4$,

$(C3')$ If $xC(y + z)$, then xCy or xCz,

$(C3'')$ If $(x + y)Cz$ then xCz or yCz,

$(C4)$ If $x.y \neq 0$, then xCy

$(C5)$ If xCy, then yCx.

If C is a contact relation in $\underline{B}$ then the pair (B, C) is called a contact algebra, CA.

If we omit the axioms $(C4)$ and $(C5)$ then C is called a precontact relation and the pair (B, C) - a precontact algebra.

The elements of B are called regions and 0 a zero-region.

Topological example of CA

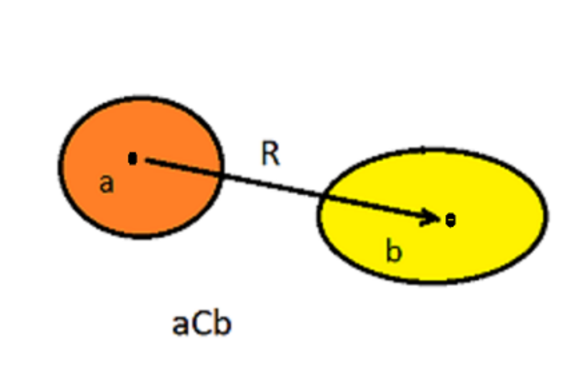
The CA of regular closed sets Let X be an arbitrary topological space. A subset a of X is *regular closed* if $a = Cl(Int(a))$. The set of all regular closed subsets of X will be denoted by $RC(X)$. It is a well-known fact that regular closed sets with the following operations form a Boolean algebra.

- $a + b = a \cup b$,
- $a.b = Cl(Int(a \cap b))$,
- $a^* = Cl(X \setminus a)$, and
- $0 = \varnothing$, $1 = X$

If we define **contact** by

- aCb iff $a \cap b \neq \varnothing$ (a and b have a common point), then $RC(X)$ becomes a contact algebra.
- **Every contact algebra is representable as a subalgebra of $RC(X)$ of some topological space X** (Dimov, Vakarelov).

Relational contact



Relational contact and precontact algebras

Let X be a non-empty set and R be a binary relation in X. Let $\mathbf{B}(X)$ be the Boolean algebra of all subsets of X. For $a, b \in \mathbf{B}(X)$ define aC_Rb iff $(\exists x \in a)(\exists y \in b)(xRy)$. Then C_R is a precontact relation in $\mathbf{B}(X)$. More over:

• R is transitive relation iff C_R satisfies the Efremovich axiom (CE) $x\overline{C}y \rightarrow (\exists z \in B)(x\overline{C}z \text{ and } z^*\overline{C}y)$

• R is reflexive and symmetric relation iff $(\mathbf{B}(X), C_R$ is a contact algebra

• **Every contact and precontact algebra has a relational representation of the above type** (Dunch, Vakarelov)

Abstract points of contact algebras: clans and clusters

DEFINITION. A set Γ of regions of B is called a **clan** if:
(Clan 1) $1 \in \Gamma$, $0 \notin \Gamma$,
(Clan 2) $a + b \in \Gamma$ iff $a \in \Gamma$ or $b \in \Gamma$,
(Clan 3) If $a, b \in \Gamma$, then aCb.

• Every clan can be extended into a **maximal clan.** Every ultrafilter in B is a clan in B.

• Γ is a **cluster** if it is a clan such that if $a \notin \Gamma$, then there exists $b \in \Gamma$ such that $a\overline{C}b$.
• If the contact relation C in B satisfies the Efremovich axiom
(CE) $a\overline{C}b \Rightarrow (\exists c)(a\overline{C}c$ and $c^{*}\overline{C}b$,
then Γ is a cluster in B iff Γ is a maximal clan in B.

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Description of the main idea in defining DCA

(1) First we introduce a concrete point-based model of dynamic space.

(2) This model is based on the notion of **TIME STRUCTURE** and

(3)the so called **SNAPSHOT CONSTRUCTION** of the algebra of **dynamic regions** and some basic spatio-temporal predicates between them: **space contact**, **time contact** and **precedence**.

(4) We introduce the abstract definition of DCA taking as axioms some true sentences in the snapshot model.

Time structures

DEFINITION.Time structure - $(T, \prec)$, $T \neq \varnothing$ is set of points of time and $\prec$ is a binary relation in T called before-after relation (or time order). It may satisfy some of the following axioms called

PROPERTIES OF TIME:

- **(RS)** *Right seriality* $(\forall m)(\exists n)(m \prec n)$,
- **(LS)***Left seriality* $(\forall m)(\exists n)(n \prec m)$,
- **(Up Dir)** *Updirectedness* $(\forall i, j)(\exists k)(i \prec k$ and $j \prec k)$,
- **(Down Dir)** *Downdirectedness* $(\forall i, j)(\exists k)(k \prec i$ and $k \prec j)$,
- **(Dens)** *Density* $i \prec j \rightarrow (\exists k)(i \prec k$ and $k \prec j)$,
- **(Ref)** *Reflexivity* $(\forall m)(m \prec m)$,
- **(Lin)** *Linearity* $(\forall m, n)(m \prec n$ or $n \prec m)$,
- **(Tri)** *Trichotomy* $(\forall m, n)(m = n$ or $m \prec n$ or $n \prec m)$,
- **(Tr)** *Transitivity* $(\forall ijk)(i \prec j$ and $j \prec k \rightarrow i \prec k)$.

”Snapshot” construction (1)

Suppose that we want to describe a dynamic environment consisting of regions changing in time. **One way to do this is to make a video.**

Formally this is the following:

- First we suppose that we are given a time structure $\underline{T} = (T, \prec)$ and want to know what is the spatial configuration of regions at each moment of time $m \in T$. We assume that for each $m \in T$ the spatial configuration of the regions forms a contact algebra $(\underline{B}_m, C_m)$, (**coordinate contact algebra**) which is considered as a **”snapshot”** of this configuration.
- We identify a given changing region a with the series $< a_m >_{m \in T}$ of snapshots and call such a series a **dynamic region.** We denote by $B(\underline{T})$ the set of all dynamic regions.

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- We identify a given changing region a with the series $< a_m >_{m \in T}$ of snapshots and call such a series a **dynamic region.** We denote by $B(\underline{T})$ the set of all dynamic regions.

"Snapshot" construction (2)

- If $a =< a_m >_{m \in T}$ is a given dynamic region then a_m can be considered as a **at the time point** m,
- $a_m \neq 0_m$ means that a **exists at the time point** m,
- $a_m C_m b_m$ means that a **and** b **are in a contact at the moment** m.
- We assume that the set $B(\underline{T})$ is a Boolean algebra with Boolean constants and operations defined coordinatewise:
- $1 =< 1_m >_{m \in T}$, $0 =< 0_m >_{m \in T}$,
- Boolean ordering $a \leq b$ iff $(\forall m \in T)(a_m \leq_m b_m)$ and
- Boolean operations are defined "coordinatewise":
  $a + b =_{def} < a_m(+_m)b_m >_{m \in T}$, $a.b =_{def} < a_m(._m)b_m >_{m \in T}$,
  $a^* =_{def} < a^*_m >_{m \in T}$.

"Snapshot" construction (3)

- Note that the Boolean algebra $B(\underline{T})$ is a subalgebra of the Cartesian product $\prod_{m\in T} B_m$ of the contact algebras $(\underline{B}_m, C_m)$, $m \in T$.
- A model which coincides with the Cartesian product is called a **full model.**
- $B(\underline{T})$ is called a **rich model** if it contains all dynamic regions a such that for all $m \in T$ we have $a_m = 0_m$ or $a_m = 1_m$.
  Obviously full models are rich.

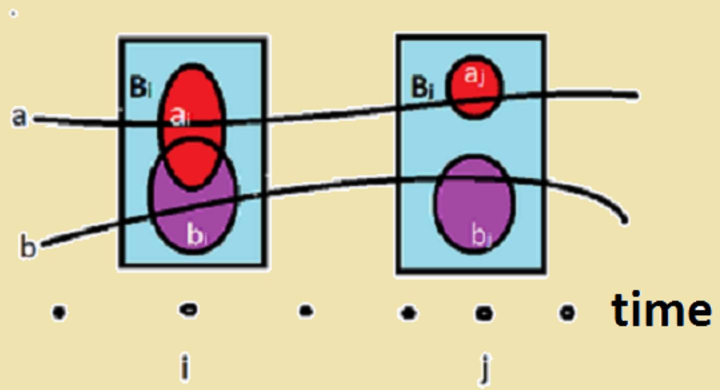
Basic spatio-temporal relations

Let $a =< a_m >_{m \in T}$ and $b =< b_n >_{n \in T}$ be two dynamic regions. We define:

SPACE CONTACT $aC^s b$ iff $(\exists m \in T)(a_m C_m b_m)$.

TIME CONTACT $aC^t b$ iff $(\exists m \in T)(a_m \neq 0_m$ and $b_m \neq 0_m)$. It can be considered also as a kind of **simultaneity relation**

PRECEDENCE $a\mathcal{B}b$ iff $(\exists m, n \in T)(m \prec n$ and $a_m \neq 0_m$ and $b_n \neq 0_n)$.


a
b
Bi
ai
bi
Bj
aj
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time
i
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Standard dymamic Contact algebra (DCA)

Definition

Let $B = B(\underline{T})$ be a rich subalgebra of the Cartesian product $\prod_{m \in T} B_m$ equipped with the relations C^t, C^s and $\mathcal{B}$, Then $(B, C^t, C^s, \mathcal{B})$ is called a **standard dynamic contact algebra** (standard DCA).

Correspondence table (1)

- **(RS)** *Right seriality* $(\forall m)(\exists n)(m \prec n) \Longleftrightarrow$
  (rs) $a \neq 0 \rightarrow a\mathcal{B}1$,
- **(LS)** *Left seriality* $(\forall m)(\exists n)(n \prec m) \Longleftrightarrow$
  (ls) $a \neq 0 \rightarrow 1\mathcal{B}a$,
- **(Up Dir)** *Updirectedness* $(\forall i, j)(\exists k)(i \prec k$ and $j \prec k) \Longleftrightarrow$
  (up dir) $a \neq 0 \wedge b \neq 0 \rightarrow a\mathcal{B}p \vee b\mathcal{B}p^*$,
- **(Down Dir)** *Downdirectedness* $(\forall i, j)(\exists k)(k \prec i$ and $k \prec j)$ $\Longleftrightarrow$
  (down dir) $a \neq 0 \wedge b \neq 0 \rightarrow p\mathcal{B}a \vee p^*\mathcal{B}b$,
- **(Dens)** *Density* $i \prec j \rightarrow (\exists k)(i \prec k \wedge k \prec j) \Longleftrightarrow$
  (dens) $a\mathcal{B}b \rightarrow a\mathcal{B}p$ or $p^*\mathcal{B}b$,

Correspondence table (2)

- **(Ref)** *Reflexivity* $(\forall m)(m \prec m) \Longleftrightarrow$
  (ref) $aC^t b \rightarrow a\mathcal{B}b$,
- **(Lin)** *Linearity* $(\forall m, n)(m \prec n \vee n \prec m) \Longleftrightarrow$
  (lin) $a \neq 0 \wedge b \neq 0 \rightarrow a\mathcal{B}b \vee b\mathcal{B}a$,
- **(Tri)** *Trichotomy* $(\forall m, n)(m = n$ or $m \prec n$ or $n \prec m) \Longleftrightarrow$
  (tri) $(aC^t c$ and $bC^t d$ and $c\overline{C}^t d) \rightarrow (a\mathcal{B}b$ or $b\mathcal{B}a)$,
- **(Tr)** *Transitivity* $i \prec j$ and $j \prec k \rightarrow i \prec k \Longleftrightarrow$
  (tr) $a\overline{\mathcal{B}}b \rightarrow (\exists c)(a\overline{\mathcal{B}}c \wedge c^*\overline{\mathcal{B}}b)$.

The conditions denoted by (xxx) using small letters are called **time axioms**

Lemma

Correspondence Lemma
The left hand side of an $\Leftrightarrow$ is true in the time structure $(T, \prec)$ iff the right side is true in the standard DCA

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Abstract definition of DCA

DEFINITION. The system $A = (B_A, C^t_A, C^s_A, \mathcal{B}_A)$ is called a dynamic contact algebra (DCA) if B_A is a Boolean algebra and the relations $C^t_A, C^s_A, \mathcal{B}_A$ satisfy the following axioms:

- (CC^t) C^t is a **contact relation** in B, called **time contact** which satisfies the Efremovich axiom
- (C^tE) $a\overline{C}^t_A b \Rightarrow (\exists c)(a\overline{C}_A c$ and $c^*\overline{c}^t_A b)$,
- (CC^s) C^s is a **contact relation**, called **space contact**, satisfying in addition the following axiom
- (C^sC^t) $aC^s_A b \Rightarrow aC^t_A b$,
- $PC\mathcal{B})$ $\mathcal{B}$ is a **precontact relation**, called **precedence** and satisfying in addition
- $(C^t\mathcal{B})$ $(a\overline{\mathcal{B}}_A b \Rightarrow (\exists c)(a\overline{C}^t_A c$ and $c^*\mathcal{B}_A b)$,
- $(\mathcal{B}C^t)$ $(a\overline{\mathcal{B}}_A b \Rightarrow (\exists c)(a\overline{\mathcal{B}}_A c$ and $c^*\overline{C}^t_A b)$.

We also consider DCA satisfying some of the 9-th time axioms

Theorem

All axioms of DCA $A = (B_A, C^t_A, C^s_A, \mathcal{B}_A)$ are true in the snapshot model, i.e. the standard DCA over a snapshot model is indeed a DCA.

Theorem

Representation theorem for DCA-s over snapshot models*.*
(1) Every DCA $A = (B_A, C^t_A, C^s_A, \mathcal{B}_A)$ is representable over a snapshot model over a certain time structure $(T, \prec)$.
(2) A satisfies some of the time axioms iff the time structure $(T, \prec)$ satisfies the corresponding time conditions listed in the Correspondence table.

DCA as a generalization of contact algebra

- Let $A = (B_A, C_A)$ be a contact algebra. Let
- $C^s_A = C_A$,
- $aC^t_A b \Leftrightarrow a \neq 0$ and $b \neq 0$, and
- $\mathcal{B}_A = C^t_A$.

Then it is easy to show that A with thus defined relations is a DCA. This example shows that contact algebras are special cases of DCA-s and that DCA is a generalization of contact algebra.

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t-clans, s-clans and t-clusters in DCA

Let $A = (B_A, C^t_A, C^s_A, \mathcal{B}_A)$ be a DCA.

• $X^t_A$is the set of clans with respect to the contact C^t_A called **t-clans**.

• X^s_A is the set clans with respect to the contact C^s_A called **s-clans** . **Every s-clan is a t-clan.**

• T_A is the set of maximal clans with respect to C^t_A called **t-clusters** . **Every t-clan is contained in a unique t-cluster**, so

• there exists a function $\gamma_A : X^t_A \to T_A$ such that for any $\Gamma \in X^t_A$ $\Gamma \subseteq \gamma_A(\Gamma)$ and if $\Gamma \in T_A$, then $\Gamma = \gamma_A(\Gamma)$.

• The set X^t_A is equipped with a **before-after** relation $\prec_A$: $\Gamma \prec_A \Delta$ iff $(\forall a \in \Gamma)(\forall b \in \Delta)(a\mathcal{B}_A b)$.

• $(S^t_A, S^s_A, T_A, \gamma_A, \prec_A)$ is called the **Clan-structure of A.**

• $(T_A, \prec_A)$ is called the **Time structure of A.**

Clan characterization of time contact space contact and precedence

Lemma

(i) aC^s_Ab iff there exists a s-clan Γ such that $a, b \in \Gamma$,
(ii) aC^t_Ab iff there exists a t-clan (t-custer) Γ such that $a, b \in \Gamma$,
(iii) $a\mathcal{B}_Ab$ iff there exist t-clusters Γ, Δ such that $\Gamma \prec \Delta$, $a \in \Gamma$ and $b \in \Delta$.

t-clusters are considered as **TIME POINTS**, t-clans are considered as **PARTIAL TIME POINTS**, s-clans are considered as **SPACE POINTS**.

Lemma

Correspondence Lemma *Consider the Correspondence table and let (Θ) be any of the 9 conditions of time and (θ) be the corresponding time axiom. Then the following conditions are equivalent:*
(i) (Θ) is true in the time structure $(T_A, \prec_A)$
(ii) (θ) is true in the algebra A.

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Dynamic Mereotopological Spaces (DMS)

A system $S = (X^t_S, X^s_S, T_S, \gamma_S, \prec_S, M_S)$ is called Dynamic Mereotopological Space (DMS, DM-space) if the next list of axioms are satisfied. **Intuitively DMS is abstracted from the clan-structure of DCA by introducing in it a topology.**

The axioms of DMS:

• (1) X^t_S is a topological space, the elements of X^t_S are called **partial time points of S.**

• (2) M_S is a subalgebra of the set $RC(X^t_S)$ of regular closed sets of X^t_S and M_S is a closed base of the topology of X^t_S.

• (3) $X^s_S \subseteq X^t_S$, $T_S \subseteq X^t_S$. The elements of X^s_S are called **space points of S**. Every space point is a partial time point.

• (4) the binary relation $\prec_S$ in X^t_S is called **before-after relation.** The elements of the set T_S are called **time points of S** and the subsystem $(T_S, \prec_S)$ is called the **time structure of** S.

• (5) $\gamma_S : X^t_S \to TS$ is a function such that for $x \in T_S$ we have $\gamma_S(x) = x$.
• (6) **Definitions**: Let for $a, b \in RC(X^t_S)$ define:
$aC^t_S b$ iff $a \cap b \neq \varnothing$, **time contact**,
$aC^s_S b$ iff $a \cap b \cap X^s_S \neq \varnothing$, **space contact**,
$a\mathcal{B}_S b$ iff there exist $x, y \in T_S$ such that $x \prec_S y$, $x \in a$ and $y \in b$ - **precedence**,
$RC(S) =_{def} (RC(X^t_S), C^t_S, C^s_S, \mathcal{B}_S)$ - **regular-sets algebra.**
For $x \in X^t_S$ define $\rho_S(x) = \{a \in M_S : x \in a\}$.
• (7) The system $S^+ = (M_S, C^t_S, C^s_S, \mathcal{B}_S)$ with the above defined relations restricted to M_S is a DCA. S^+ is called the **canonical DCA of** S or the **dual of** S.
• (8) If $x \prec y$ iff $(\forall a, b \in M_S)(x \in a, y \in b \Rightarrow a\mathcal{B}_S b$.
• (9) If $x \prec_S y$, then $\gamma_S(x) \prec_S \gamma_S(y)$.
• (10) If $a \in RC(X^t_S)$ and $x \in a$, then $\gamma_S(x) \in a$.
• (11) If $x \in T_S$ then $\rho_S(x)$ is a t-cluster in S^+,

DM-compactness

Definition

(1) S is called a **T0 space** if X^t_S is a T0 topological space.
(2) A t-clan (s-clan, t-cluster) Γ of S^+ is called a **point t-clan** (s-clan, t-cluster) if there is a point $x \in X^t_S$ ($x \in X^s_S$, $x \in T_S$) such that $\Gamma = \rho_S(x)$.
(3) S is a **DM-compact** if every t-clan (respectively s-clan, t-cluster) is a point t-clan (respectively s-clan, t-cluster).

Some comments

• Having in mind that DCA is a generalization of CA we can see that the notion of **dynamic mereotopological space** is a generalization of the notion of **mereotopological space**, introduced by Robert Goldblatt for a kind of duality theory for contact algebras and mereotopological spaces.

• The notion of **DM-compactness** is a generalization of the notion of **mereocompactnes** introduced by Goldblatt for mereotopological spaces. DM-compactnes implies compactness of the space X_S^t

• **Our aim is to extend the Goldblatt's duality to a duality for DCA , and DMS** (both with corresponding kinds of morphisms). We do this for one type of morphisms and indicate that the **proof goes uniformly also for several other kinds of morphisms** obtaining in this way a class of duality theorems.

Properties of T0 and DM-compact DMS

T0 and DM-compact DM-spaces have good properties as the next two theorems show.

Correspondence Theorem for DMS

Let $S = (X^t_S, X^s_S, T_S, \gamma_S, \prec_S), M_S)$ be a T0 and DM-compact DMS, $RC(S)$ be its regular-sets algebra and $(T_S, \prec_S)$ be its time structure. Then:

(I) $RC(S)$ is a DCA.

(II) Consider the Correspondence table and let (Θ) be any of the 9 conditions of time discussed in snapshot models and let (θ) be the corresponding time axiom. Then the following conditions are equivalent:

(i) (Θ) is true in the time structure $(T_S, \prec_S)$,

(ii) (θ) is true in the algebra S^+ (the dual DCA of S),

(iii) (θ) is true in the algebra $RC(S)$.

Canonical DMS of a DCA

Let $A = (B_A, C^t_A, C^s_A, \mathcal{B}_A)$ be a DCA. We associate to DCA in a canonical way a DM-space - the **canonical space of A** (the **dual space** of A):

$A_+ = (X^t_A, X^s_A, T_A, \gamma_A, \prec_A, M_A)$,

where M_A is defined as follows and is used to introduce a topology in the set X^t_A considering it as a basis of the closed sets in the topology:

- For $a \in B_A$ let $f_A(a) = \{\Gamma \in X^t_A : a \in \Gamma\}$ and put
- $M_A = \{f_A(a) : a \in B_A\}$.
- We define the algebra $(A_+)^+$ - **"the dual of A_+"** as follows:

$(A_+)^+ = (M_A, C^t_{A_+}, C^s_{A_+}, \mathcal{B}_{A_+})$.

- $RC(A_+) =_{def} (RC(X^t_A), C^t_{A_+}, C^s_{A_+}, \mathcal{B}_{A_+})$.

Topological representation theorem for DCA

having in mind the above notations we have the following theorem.

Theorem

Topological representation theorem for DCA.
Let $A = (B_A, C^t_A, C^s_A, \mathcal{B}_A)$ *be a DCA. Then*

(i) A_+ *is a DM-space which is T0 and DM -compact.*

(ii) f_A *is an isomorphism from A onto the algebra* $(A_+)^+$*, which is a subalgebra of* $RC(A_+)$*.*

(iii) f_A *is an isomorphic embedding from A into the algebra* $RC(A_+)$

(iv) *Let* (θ) *be any of the 9 time axioms. Then* (θ) *is true in A iff* (θ) *is true in* $RC(A_+)$*.*

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DCA-morphisms and the category **DCA**

Let A, A' be two DCA. A DCA-morphism $f : A \to A'$ is a function $f : B_A \to B_{A'}$ if the following conditions are satisfied:

• (1) f is a Boolean homomorphism from A to A' respecting the relations R^t_A, R^s_A and $\mathcal{B}_A$, namely:

• (2) For $a, b \in B_A$: if $f(a)C^t_{A'}f(b)$, then $aC^t_A b$,

• (3) For $a, b \in B_A$: if $f(a)C^s_{A'}f(b)$, then $aC^s_A b$,

• (4) For $a, b \in B_A$: if $f(a)\mathcal{B}_{A'}f(b)$, then $aC^{\mathcal{B}}_A b$,

It can be easily shown that compositions of DCA-morphisms is a DCA-morphism and that the identity mapping is a DCA-morphism. So the class of all DCA with such morphisms is a category denoted by **DCA**.

DMS-morphisma and the category **DMS**

Let S and S' be two DM spaces. A DMS-morphism $\theta : S \to S'$ is a function $\theta : X^t_S \to X^t_{S'}$ satisfying the following conditions:

- (1) if $x \in X^s_S$, then $\theta(x) \in X^s_{S'}$,
- (2) if $x \prec_S t$, then $\theta(x) \prec_{S'} \theta(y)$,
- (3) If $a \subseteq X^t_{S'}$ and $a \in M_{S'}$ then $\theta^+ =_{def} \theta^{-1}(a) =_{def} \{x \in X^t_S : \theta(x) \in a\} \in M_S$ and the mapping $a \mapsto \theta^{-1}(a)$ is a Boolean homomorphism from $M_{S'}$ to M_S. Note that **every DMS morphism is a continious mapping.**

- It is obvious that the identity is a DMS-morphism and that composition of two DMS-morphisms is a DMS-morphism, so the class **DMS** of all DM-spaces with thus defined morphisms is a category.

- We denote by **DMS*** the full subcategory of all T0 and DM-compact DM-spaces.

Duality Theorem

THE MAIN THEOREM. *The category **DCA** of dynamic contact algebras is dually equivalent to the category **DMS*** of T0 and DMS-compact dynamic mereotopological spaces.*

PROOF(sketch). We define two contravariant functors Φ : **DCA** $\rightarrow$ **DMS** and Ψ : **DMS** $\rightarrow$ **DCA** as follows

• If A is DCA and f is a DCA morphism, then $\Phi(A) = A_+$ and $\Phi(f) = f^+ = f^{-1}$

• If S is DMS and θ is a DMS morphism, then $\Psi(S) = S^+$ and $\Psi(\theta) = \theta^+ = \theta^{-1}$.

• It can be proved that the compositions $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are naturally isomorphic to the corresponding identity functors, which proves the intended duality theorem.■

Note that if we replace **DMS*** with **DMS** the duality theorem will not be true.

Comments

• Similar duality theorem can be obtained for other types of morphisms for DCA and DMS. For instance:
For DCA: add condition (5): if Γ is a t-cluster in the algebra A' then $f^{-1}(\Gamma) = \{a \in B_A : f(a) \in \Gamma\}$ is a t-cluster in the algebra A.
For DMS: add condition (4): if $x \in T_S$, then $\theta(x) \in T_{S'}$.
There are also other possibilities.

• Other types of duality theorems for DCA and DMS can be proved transferring the metods and constructions used for a duality for precontact (contact) algebras and precontact (contact) spaces developed in the paper

• G. Dimov E. Ivanova-Dimova and D. Vakarelov, A generalization of the Stone Duality Theorem, Topology and its Applications, 2017.

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THANKS