Relational semantics for extended contact algebras

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Contact algebra

Contact algebra is one of the main tools in RBTS.

Definition (Dimov and Vakarelov, 2006)

Contact algebra is a Boolean algebra $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, C)$ with additional binary relation C, called *contact*, satisfying the following axioms:

(C1) If aCb, then $a \neq 0$ and $b \neq 0$,
(C2) If aCb and $a \leq a'$ and $b \leq b'$, then $a'Cb'$,
(C3) If $aC(b + c)$, then aCb or aCc,
(C4) If aCb, then bCa,
(C5) If $a \cdot b \neq 0$, then aCb.

Topological contact algebra (Dimov and Vakarelov, 2006)

Let X be a topological space and $a \subseteq X$. We say that a is a regular closed set if $a = Cl(Int(a))$. It is a well known fact that the set $RC(X)$ of all regular closed subsets of X is a Boolean algebra with respect to the relations, operations and constants defined as follows:

$a \leq b$ iff $a \subseteq b$,
$0 = \emptyset$, $1 = X$,
$a + b = a \cup b$,
$a \cdot b = Cl(Int(a \cap b))$,
$a^* = Cl(-a)$, where $-a = X - a$.

If we define a contact C by

aCb iff $a \cap b \neq \emptyset$,

then we obtain the standard topological model of contact algebra.

Relational contact algebra (Vakarelov, 2007)

Let (W, R) be a relational system, where W is a nonempty set and R is a reflexive and symmetric binary relation in W and let B be a set of subsets of W closed under union, intersection and complement, containing $\emptyset$ and W. We define $0 = \emptyset$, $1 = W$.
For arbitrary $a, b \in B$ we define:
$a \leq b$ iff $a \subseteq b$
$a \cdot b = a \cap b$
$a + b = a \cup b$
$a^* = W - a$.

We define a contact relation between a and b as follows

(Def C_R) aC_Rb iff $\exists x \in a$ and $\exists y \in b$ such that xRy.

The obtained structure $\underline{B} = (B, \leq, \cdot, +, 0, 1, *, C_R)$ is called *relational contact algebra over* (W, R).

The predicate *internal connectedness* (c^o)

Let X be a topological space and x be its regular closed subset. "x is internally connected" means that the interior of x is a connected topological space in the subspace topology.

Proposition (T.I., 2015)

The predicate internal connectedness cannot be defined in the language of contact algebras.

Because of this we add to the language a new ternary predicate symbol $\vdash$ (*extended contact* or *covering*). By it we can already define c^o.

Proposition (T.I. and Vakarelov, 2015)

Let X be a topological space. For every a in $RC(X)$, $c^o(a)$ iff $\forall b \forall c(b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \nvdash a^*)$.

The relation extended contact ($\vdash$)

Another motivation for considering the predicate symbol $\vdash$ is that by it we can define the property of two regions their intersection to be a region:

Proposition (Vakarelov, 2016)

"$a \cap b$ is regular closed" iff $a, b \vdash a \cdot b$

Extended contact gives also the possibility to define the relation of contact: aCb iff $a, b \nvdash 0$.

The predicate internal connectedness (c^o)

By the predicate internal connectedness we can define the property "existing of cavities in a physical body".

Proposition (Vakarelov, 2016)

"a has cavities" iff "the complement of a is not connected", i.e. "a^* is not internally connected".

The predicate internal connectedness (c^o)

We cannot define "**a** has cavities" iff "the complement of **a** is not connected" because we do not have an operation complement and the complement of *a* is not always a region (is not a regular closed set - does not coincide with the closure of its interior). Because of this we define "**a** has cavities" iff "**a*** is not internally connected".

If we define "**a** has cavities" iff "**a*** is not connected", this is wrong - if the cavity in the ball **a** touches its boundary, **a*** is connected (and at the same time is not internally connected).

Because of these reasons we need the predicate "internal connectedness" instead of "connectedness" for defining the property "existing of cavities in a physical body".

Extended contact algebra

Definition (T.I., 2015)

Extended contact algebra (ExtCA for short) is a structure $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$, where $(B, \leq, 0, 1, \cdot, +, *)$ is a nondegenerate Boolean algebra, $\vdash$ is a ternary relation in B such that the following axioms are true:

(1) $a, b \vdash c \rightarrow b, a \vdash c$,
(2) $a \leq c \rightarrow a, b \vdash c$,
(3) $a, b \vdash x,\ a, b \vdash y,\ x, y \vdash c \rightarrow a, b \vdash c$,
(4) $a, b \vdash c \rightarrow a \cdot b \leq c$,
(5) $a, b \vdash c \rightarrow a + x, b \vdash c + x$,

C is a binary relation in B such that for any $a, b \in B$:
$aCb \leftrightarrow a, b \nvdash 0$.

c^o is a unary predicate in B such that for any $a \in B$:
$c^o(a) \leftrightarrow \forall b \forall c (b \neq 0, c \neq 0, a = b + c \rightarrow b, c \nvdash a^*)$.

Extended contact algebra

Topological ExtCA - topological contact algebra with added relations $\vdash$ and c^o, where
$a, b \vdash c$ means that the intersection of **a** and **b** is included in **c**.

Theorem (Topological representation theorem (T.I.,2015))

Every ExtCA is isomorphically embedded in a topological ExtCA over some compact, semiregular, T_0 topological space.

A quantifier-free logic for ExtCA

We consider a quantifier-free first-order language $\mathcal{L}$ with equality which has:

- constants: 0, 1
- functional symbols: $+$, $\cdot$, $*$
- predicate symbols: $\leq$, $\vdash$, C, c^o

A quantifier-free logic for ExtCA

We consider the logic L which has the following:

axioms:

- the axioms of the classical propositional logic
- the axioms of Boolean algebra
- the axioms of ExtCA concerning the relations extended contact and contact
- the axiom schema:
  (Ax c^o) $c^o(p) \wedge q \neq 0 \wedge r \neq 0 \wedge p = q + r \rightarrow q, r \nvdash p^*$

rules:

- MP
- (Rule c^o) $\frac{\alpha \rightarrow (p \neq 0 \wedge q \neq 0 \wedge a = p + q \rightarrow p, q \nvdash a^*) \text{ for all variables } p,\ q}{\alpha \rightarrow c^o(a)}$, where α is a formula, a is a term.

A quantifier-free logic for ExtCA

Theorem (Completeness theorem (T.I., 2016))

For every formula α in the language of ExtCAs the following conditions are equivalent:
(i) α is a theorem of L;
(ii) α is true in all ExtCAs.

Theorem (T.I., 2016)

The logic L is decidable.

A quantifier-free logic for ExtCA

We consider a nicer logic $\mathbb{L}$.

axioms:

- the axioms of the classical propositional logic
- the axioms of Boolean algebra
- the axioms of ExtCA concerning the relations extended contact and contact
- the axiom schemes:
  (Ax c^o) $c^o(p) \wedge q \neq 0 \wedge r \neq 0 \wedge p = q + r \rightarrow q, r \not\vdash p^*$
  (Ax c^o 1) $c^o(0)$
  (Ax c^o 2) $\neg c^o(p + q) \rightarrow \neg c^o(p) \vee \neg c^o(q)$
  (Ax c^o 3) $c^o(p + q) \rightarrow c^o(p) \wedge c^o(q)$

rule: MP

A quantifier-free logic for ExtCA

Proposition (T.I., 2018)

The logics L and $\mathbb{L}$ have the same theorems.

Relational representation of ExtCAs

We consider two languages:

- with predicate symbols $\leq, \vdash, C$
- with predicate symbols $\leq, \vdash, C, c^o$ (the language of ExtCAs)

Relational representation of weak ExtCAs

First we consider the language of ExtCAs without internal connectedness.

Definition (Weak ExtCA (Balbiani, 2017))

Weak ExtCA is a structure of the form $(B, 0, *, +, \vdash)$, where $(B, 0, *, +)$ is a non-degenerate Boolean algebra and $\vdash$ is a ternary relation on B such that for all $a, b, d, e, f \in B$,
$(WExtCA_1)$ if $a \leq d$, $b \leq e$ and $d, e \vdash f$ then $a, b \vdash f$,
$(WExtCA_2)$ if $a = 0$ or $b = 0$ then $a, b \vdash f$,
$(WExtCA_3)$ if $a, b \vdash f$ and $d, e \vdash f$ then $a \cdot d$, $b + e \vdash f$ and $a + d, b \cdot e \vdash f$,
$(WExtCA_4)$ if $a, b \vdash d$ and $d \leq f$ then $a, b \vdash f$.

Relational representation of weak ExtCAs

Definition (Balbiani, 2017)

A *parametrized frame* is a structure of the form (W, R), where W is a nonempty set and R is a function associating to each subset of W a binary relation on W.

Let $\vdash$ be the ternary relation on W's powerset defined by

$A, B \vdash D$ iff for all $s \in A$, for all $t \in B$ and for all $U \subseteq W$, if $D \subseteq U$ then not $R(U)(s, t)$

Proposition (Balbiani, 2017)

The Boolean algebra of all subsets of W together with this relation is a weak ExtCA.

Theorem (Representation theorem (Balbiani, 2017))

Let $(B, 0, *, +, \vdash)$ *be a weak ExtCA. There exists a parametrized frame* (W, R) *and an embedding of* $(B, 0, *, +, \vdash)$ *in* $(\mathcal{P}(W), \emptyset, ^{-}, \cup, \vdash)$.

Proof sketch

Let (W, R) be the structure where

- W is the set of all maximal filters in the Boolean algebra $(B, 0, \star, +)$,
- R is the function associating to each subset U of W the binary relation $R(U)$ on W defined by $R(U)(s, t)$ iff for all $a, b, d \in B$, if $a \in s$, $b \in t$ and $a, b \vdash d$ then there exists $e \in B$ such that $d \not\leq e$ and for all $u \in U$, $e \in u$.

(W, R) is a parametrized frame.

For any $a \in B$,
$h(a) = \{s \in W : \ a \in s\}$.

Relational representation of ExtCAs

Definition (T.I., 2017)

An *equivalence frame of type 1* is a structure of the form (W, R), where W is a nonempty set and R is an equivalence relation on W.

Definition (ExtCA over (W, R) (T.I., 2017))

$\underline{B} = (\mathcal{P}(W), \subseteq, \cup, \cap, \emptyset, W, \star, \vdash, C)$, where $\star$ denotes the set theoretical complement and for any subsets of W a, b, and c:

- $a, b \vdash c$ iff $\Big((\exists A \in a)(\exists B \in b)ARB \rightarrow (\exists C \in c)ARC\Big)$ and $a \cap b \subseteq c$,
- aCb iff $a, b \nvdash \emptyset$,

Let (W, R) be an equivalence frame of type 1 and α be a formula in the language of contact algebras with added predicate symbol $\vdash$. We say that α is true in (W, R), if α is true in the ExtCA over (W, R).

Relational representation of ExtCAs

Theorem (Representation theorem (T.I., 2017))

Let $\underline{B}$ be a finite ExtCA. Then in the considered language $\underline{B}$ is isomorphically embedded in the ExtCA over some equivalence frame of type 1 (W, R).

Proof sketch

$\underline{B}$ is isomorphic to a substructure $\underline{B_1}$ of the ExtCA $\underline{RC(X)}$ of the regular closed subsets of some finite topological space X.

Let W_1 be the set of all atoms of $\underline{B_1}$.

We consider the relational structure (W, R), where
$W \stackrel{def}{=} \{(A, B) \in W_1 \times X : B \in A\}$

Proof sketch

and for any $(A_1, B_1), (A_2, B_2) \in W$:
$(A_1, B_1)R(A_2, B_2) \overset{def}{\leftrightarrow} B_1 = B_2$.

R is an equivalence relation.

We define a mapping $h : B_1 \to \mathcal{P}(W)$ in the following way:
$h(a) \overset{def}{=} \{(A, B) \in W : A \subseteq a\}$ for any $a \in B_1$.

Let $\mathbb{L}_1$ be the logic, obtained from $\mathbb{L}$ by removing axioms (Ax c^o), (Ax c^o 1), (Ax c^o 2) and (Ax c^o 3). This logic is called *extended contact logic*.

Theorem (Completeness theorem with respect to relational semantics (T.I., 2017))

For every formula α in the considered language the following conditions are equivalent:
i) α is a theorem of $\mathbb{L}_1$;
ii) α is true in all equivalence frames of type 1 (W, R).

Relational representation of ExtCAs

Now we consider the language of ExtCAs - we need two equivalence relations instead of one.

Definition (T.I., 2017)

An *equivalence frame of type 2* is a structure of the form (W, R_1, R_2), where W is a nonempty set and R_1 and R_2 are equivalence relations on W.

Definition (ExtCA over (W, R_1, R_2) (T.I., 2017))

$\underline{B} = (\mathcal{P}(W), \subseteq, \cup, \cap, \emptyset, W, \star, \vdash, C, c^o)$, where $\star$ denotes the set theoretical complement and for any subsets of W a, b and c:

- $a, b \vdash c$ iff $\forall A, A_1, B, B_1 \Big(AR_1A_1 \in a, BR_1B_1 \in b, AR_2B \rightarrow (\exists C, C_1)(CR_1C_1 \in c, AR_2C) \Big)$ and $a \cap b \subseteq c$,
- aCb iff $a, b \nvdash \emptyset$,
- $c^o(a)$ iff $(\forall b, c \subseteq W)(b \neq \emptyset, c \neq \emptyset, a = b \cup c \rightarrow b, c \nvdash a^\star)$.

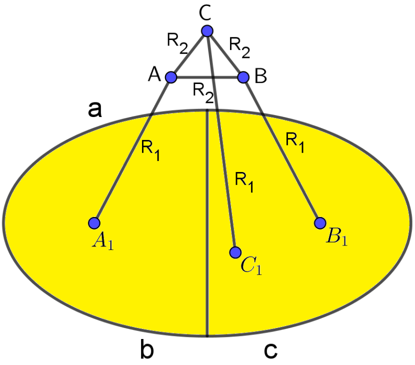
Relational representation of ExtCAs

Let (W, R_1, R_2) be an equivalence frame of type 2 and α be a formula in the language of ECAs. We say that α is true in (W, R_1, R_2) if α is true in the ExtCA over (W, R_1, R_2).

Internal connectedness in a relational ExtCA

It turns out that the internal connectedness in a relational ExtCA means the following:

$c^o(a)$ iff $(\forall b, c \subseteq W)(b, c \neq \emptyset$ and $a = b \cup c \rightarrow$
$b \cap c \neq \emptyset$ or
$(\exists A, A_1, B, B_1)(AR_1A_1 \in b,\ BR_1B_1 \in c,\ AR_2B,$
$(\forall C, C_1)(AR_2C, BR_2C, CR_1C_1 \rightarrow C_1 \in a)))$



Relational representation of ExtCAs

Theorem (Representation theorem (T.I., 2017))

Let $\underline{B}$ be a finite ExtCA. Then $\underline{B}$ is isomorphically embedded in the ExtCA over some equivalence frame of type 2 (W, R_1, R_2).

Proof sketch

$\underline{B}$ is isomorphic to a substructure $\underline{B_1}$ of the ExtCA $\underline{RC(X)}$ of the regular closed subsets of some finite topological space X.

Let W_1 be the set of all atoms of $\underline{B_1}$.

We consider the relational structure (W, R_1, R_2), where
$W \stackrel{def}{=} \{(A, B) \in W_1 \times X : \; B \in A\}$

Proof sketch

and for any $(A_1, B_1), (A_2, B_2) \in W$:

$(A_1, B_1) R_1 (A_2, B_2) \overset{def}{\leftrightarrow} A_1 = A_2$;

$(A_1, B_1) R_2 (A_2, B_2) \overset{def}{\leftrightarrow} B_1 = B_2$.

R_1 and R_2 are equivalence relations.

We define a mapping $h : B_1 \to \mathcal{P}(W)$ in the following way:

$h(a) \overset{def}{=} \{(A, B) \in W : A \subseteq a\}$ for any $a \in B_1$.

Theorem (Completeness theorem with respect to relational semantics (T.I., 2017))

For every formula α in the language of ExtCAs the following conditions are equivalent:
i) α is a theorem of $\mathbb{L}$;
ii) α is true in all equivalence frames of type 2 (W, R_1, R_2).

Future plans

- topological representation theorems for ExtCAs (eventually with added axioms) in T_1; T_2; weakly regular; connected; κ-normal and Euclidean spaces
- generalization of the considered relational representation theorems for all ExtCAs (not only for weak ExtCAs or for finite ExtCAs)
- complexity of the logic for ExtCAs
- logic for WExtCAs; decidability/complexity

Thank you very much!