Computing the validity degree in Łukasiewicz logic

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Outline

Setting: propositional (infinite-valued) Łukasiewicz logic.

Array of complexity results for decision problems.

Algebraic method: the standard MV-algebra.

Validity degree is an optimization problem.

Complete in $\mathrm{FP}^{\mathrm{NP}}$ under metric reductions:

- upper bound (oracle computation);
- lower bound (metric reduction).

Standard MV-algebra

Language: $\{\oplus, \neg\}$.

$[0,1]_Ł = \langle [0,1], \oplus, \neg \rangle$, with

$$x \oplus y = \min(1, x + y)$$
$$\neg x = 1 - x$$

Denote f_φ the function defined by the term φ in $[0,1]_Ł$.

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- $x \odot y$ is $\neg(\neg x \oplus \neg y)$;
- $x \to y$ is $\neg x \oplus y$;
- $x \vee y$ is $(x \to y) \to y$;
- $x \equiv y$ is $(x \to y) \odot (y \to x)$.

Moreover, x^n is $\underbrace{x \odot \cdots \odot x}_{n \text{ times}}$; analogously for nx.

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The algebra $[0,1]_Ł$ captures theorems and provability from finite theories in propositional Łukasiewicz logic.

In particular, $[0,1]_Ł$ provides a semantic method of investigating computational properties of propositional infinite-valued Łukasiewicz logic.

McNaughton functions

A function $f\colon [0,1]^n \to [0,1]$ is a McNaughton function if

- f is continuous
- f is piecewise linear: there are finitely many linear polynomials $\{p_i\}_{i\in I}$, with $p_i(\bar{x}) = \Sigma_{j=1}^{n} a_{ij}\, x_j + b_i$, such that for any $\bar{x} \in [0,1]^n$ there is an $i \in I$ with $f(\bar{x}) = p_i(\bar{x})$
- the polynomials p_i have integer coefficients $\bar{a}_i, b_i$.

Theorem [McNaughton 1951]

Term-definable functions of $[0,1]_{Ł}$ coincide with McNaughton functions.

Tautologies in standard MV-algebra

Consider MV-term $\varphi(x_1, \dots, x_n)$.

f_φ introduces a polyhedral complex C on its domain (i.e., $\bigcup C = [0,1]^n$) s.t. restriction of f_φ to each (n-dimensional) cell of C is a linear polynomial.

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Minimum (maximum) of f_φ on $[0,1]^n$ is attained at a vertex of a cell in C.

Vertices of cells of C occur as solutions of systems of linear equations, with integer coefficient bounded by $\sharp\varphi$ (the number of occurrences of variables in φ).

All vertices of n-dimensional cells of C are rational vectors $(p_1/q_1, \dots, p_n/q_n)$ with

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Tautologous terms of the standard MV-algebra are in coNP.

[Mundici 1987; Aguzzoli and Ciabattoni 2000; Aguzzoli 2006]

Standard RMV-algebra and validity degree

Language: MV, expanded with constants for rationals in $[0, 1]$.

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Let φ be an RMV-term and T a set thereof.
The validity degree of φ under T is

$$\|\varphi\|_T = \inf\{v(\varphi) \mid v \text{ model of } T\}.$$

Corresponding syntactic notion is $|\varphi|_T = \sup\{r \mid T \vdash_{\mathrm{RPL}} r \to \varphi\}$.

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Pavelka completeness:

$$|\varphi|_T = \|\varphi\|_T$$

For T finite, write τ instead of T.

- $|\varphi|_\tau = 1$ implies φ is provable from τ;
- $|\varphi|_\tau$ is rational.

[Hájek 1998]

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Then $y_0 \equiv \neg x$; $y_1 \equiv y_0^2$; $y_2 \equiv y_1^2$; $\dots$; $x \equiv \Pi_{n_i=1} y_i$ implicitly defines $1/n$ in variable x.

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Lemma: $\|\varphi\|_\tau = \|\varphi^\star\|_{\tau^\star \odot \delta_{\tau \odot \varphi}}$.

[Hájek 1998]

Two optimization problems in $[0,1]_Ł$

- **MAX value**

Instance: (R)MV-term φ.
Output: MAX(φ) (maximal value of f_φ in $[0,1]_Ł$).

GenSAT: for φ, c, d (with $c, d \in N$), is $f_\varphi(\bar{a}) \geq c/d$ for some $\bar{a} \in [0,1]^n$?
This is NP-complete.
[Mundici, Olivetti 1998]

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- **Validity Degree**

Instance: (R)MV-terms τ and φ.
Output: $\|\varphi\|_\tau$ (minimal value of f_φ on the 1-set of f_τ) in $[0,1]_Ł$.

where the 1-set of f_τ is $\{\bar{a} \in R^n \mid f_\tau(\bar{a}) = 1\}$.

Finite consequence in RMV: for τ, φ, is it the case that $\tau \models_{RMV} r \to \varphi$?
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Function problems such as these sometimes called "evaluation" or "cost" problems.

Non-approximability of MAX value

Work in MV-language.

Theorem

Let $\delta < 1/2$ be a positive real. Suppose ALG is a poly-time algorithm computing, for MV-term φ, a real number $ALG(\varphi)$ satisfying $|ALG(\varphi) - MAX(\varphi)| \leq \delta$. Then $P = NP$.

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Proof: solve Boolean SAT using ALG.

Instance: Boolean formula φ, given as $\{\odot, \vee\}$-combination of literals.
Then f_φ in $[0,1]_{Ł}$ is a convex function.

– φ satisfiable in $\{0,1\}$ implies φ satisfiable in $[0,1]_{Ł}$.

– φ not satisfiable in $\{0,1\}$: then f_φ is identically 0.

So $\varphi \in \mathrm{SAT}(\{0,1\})$ iff $\mathrm{MAX}(\varphi) = 1$ iff $\mathrm{ALG}(\varphi) > 1/2$.

[H., Savický 2016]

Computing MAX value: oracle computation

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MAX(φ) is attained at a vertex of a polyhedral decomposition of the domain, with rationalcoordinates with denominators of (binary) length bounded by $n \log(\sharp\varphi/n)$.

Oracle: GenSAT (given φ and a rational $r \in [0, 1]$, is MAX(φ) $\geq r$?) This is NP-c.

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Binary search within rationals on $[0,1]$ with denominators up to $N = (\sharp\varphi/n)^{n^2}$.
Minimal distance of any two such distinct numbers: $\left|\frac{p_1}{q_1} - \frac{p_2}{q_2}\right| \geq \frac{1}{N^2}$

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If not, let $a := 0$ and $b := 1$ and $k := 0$.
Repeat ++k; $\mathrm{MAX}(\varphi) \geq (a+b)/2$? $\begin{cases} \text{Y } a & := (a+b)/2; \\ \text{N } b & := (a+b)/2; \end{cases}$ until $2^k > N^2$.

This yields interval $[m/2^k, (m+1)/2^k)$ for some m, of length $1/2^k$, with exactly one rational with denominator up to N.

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MAX value is in $\mathrm{FP}^{\mathrm{NP}}$.

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Output: $\|\varphi\|_{\tau}$.

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Recall $\|\varphi\|_\tau = \|\varphi^\star\|_{\tau^\star \odot \delta_{\tau\odot\varphi}}$
with MV-terms $\varphi^\star$, $\tau^\star$ and $\delta_{\tau\odot\varphi}$.

So $\|\varphi\|_\tau$ is a rational p/q, with $q \leq N = (\sharp\{\varphi^\star, \tau^\star, \delta_{\tau\odot\varphi}\}/n)^{n^2}$,
where n is the number of variables in $\{\varphi^\star, \tau^\star, \delta_{\tau\odot\varphi}\}$
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The minimum of $f^\star_\varphi$ on the (compact) 1-region of $f_{\tau^\star \odot \delta_{\tau\odot\varphi}}$
is attained at a vertex of the common refinement of complexes of f_φ and $f_{\tau^\star \odot \delta_{\tau\odot\varphi}}$.
Then use Aguzzoli's bounds on denominators.

Validity Degree in $\mathrm{FP}^{\mathrm{NP}}$.
("Upper bound.")

Let $f, g : \Sigma^* \to N$.
A metric reduction of f to g is a pair (h_1, h_2) of p-time functions
(with $h_1 : \Sigma^* \to \Sigma^*$ and $h_2 : \Sigma^* \times N \to N$)
such that $f(x) = h_2(x, g(h_1(x)))$ for each $x \in \Sigma^*$.

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Let $z : N \to N$. $\mathrm{FP}^{\mathrm{NP}}[z(n)]$ is the class of functions computable in P-time with NP oracle with at most $z(|x|)$ oracle calls for input x.
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Theorem [Krentel 1988]

Assume $P \neq NP$. Then $FP^{NP}[O(\log\log n)] \neq FP^{NP}[O(\log n)] \neq FP^{NP}[n^{O(1)}]$.

In particular, there are no metric reductions from $\mathrm{FP}^{\mathrm{NP}}$-complete problems to problems in $\mathrm{FP}^{\mathrm{NP}}[O(\log n)]$.

[Krentel: Complexity of optimization problems, 1988]

Weighted MaxSAT problem

- Weighted MaxSAT

Instance: Boolean CNF formula $C_1 \wedge \cdots \wedge C_n$ (k variables) with weights $w_1, \ldots, w_n$.
Output: $\max_e \Sigma_i w_i e(C_i)$(max sum of weights of true clauses over all assignments to φ).

Theorem [Krentel 1988]

Weighted MaxSAT is complete in FP^{NP} (under metric reductions).

Computing the Validity Degree: lower bound

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Proof: reduce weighted MaxSAT to Validity Degree.
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It is easy to:
– switch min and max (using $\neg$);
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– make assignments Boolean (adding $x_i \vee \neg x_i$ for each $i \in \{1, \dots, k\}$)
– implicitly condition each w_i' with C_i under v:

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- $y_i \to b$ and $wy_i \equiv C_i$ for each $i \in \{1, \ldots, n\}$; then
  - $v(C_i) = 0$ implies $v(y_i) = 0$
  - $v(C_i) = 1$ implies $v(y_i) \geq 1/w$

  and so $v(y_i) = v(C_i)1/w$ for any model e of T;
- $z_i \equiv w_i y_i$;

which yields $v(z_i) = v(C_i)w'_i$ for any model v of T and any i.

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Theorem

Validity Degree is FP^{NP}-complete (under metric reductions).

Proof: reduce weighted MaxSAT to Validity Degree.
Maximize $\Sigma_i w_i e(C_i)$ over all assigments e.

It is easy to:
– switch min and max (using $\neg$);
– scale weights: take $w = \Sigma_i w_i$ and replace w_i with $w_i' = w_i/w$ (and de-scale again);

Build a theory T (or τ) to
– make assignments Boolean (adding $x_i \vee \neg x_i$ for each $i \in \{1, \dots, k\}$)
– implicitly condition each w_i' with C_i under v:

- $b \equiv (\neg b)^{w-1}$ (implicitly defines $1/w$);
- $y_i \to b$ and $wy_i \equiv C_i$ for each $i \in \{1, \dots, n\}$; then
  - $v(C_i) = 0$ implies $v(y_i) = 0$
  - $v(C_i) = 1$ implies $v(y_i) \geq 1/w$

  and so $v(y_i) = v(C_i)1/w$ for any model e of T;
- $z_i \equiv w_i y_i$;

which yields $v(z_i) = v(C_i)w_i'$ for any model v of T and any i.

Finally, let Φ be $\neg(z_1 \oplus z_2 \oplus \dots \oplus z_n)$. Compute $m = \|\Phi\|_T$ and return $(1 - m)w$.

Concluding remarks

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The Validity Degree problem, native to many-valued logic,
sits among other optimization problems in $\mathsf{FP}^{\mathsf{NP}}$.

Metric reductions are natural (many-one) reductions for optimization problems.
Between some pairs of problems, such reductions cannot exist unless P equals NP.
In the sense of metric reductions,
Validity Degree ranks among "hardest" (i.e., complete) $\mathsf{FP}^{\mathsf{NP}}$-problems.