Quantale semantics of Lambek calculus with subexponentials

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On categorial grammars

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The	Thames	nocturne	of	blue	and	gold
$(np/n)/np$	np	n	$np\backslash(np/ad)$	ad	$ad/(ad\backslash ad)$	ad

Changed	to	Harmony	in	grey	
$np\backslash(s/p)$	p/np	np	$np\backslash(np/ad)$	ad	$\to s$

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The basic reduction rules are:

- $A, A\backslash B \to B$
- $B/A, A \to B$

On categorial grammars

References:

- Lambek, J. (1958). The mathematics of sentence structure. American mathematical monthly, vol. 65, No. 3, 154-170, pp 154-170.
- Pentus, M. (1993). Lambek grammars are context free. Proceedings LICS.

Lambek calculus

Basic Lambek calculus

$$\frac{}{A \to A}\ \text{ax}$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, \Gamma, A \backslash B, \Theta \to C}\ \backslash \to \qquad \frac{A, \Pi \to B}{\Pi \to A \backslash B}\ \to \backslash, \Pi \text{ is non-empty}$$

$$\frac{\Gamma \to A \qquad \Delta, B, \Theta \to C}{\Delta, B / A, \Gamma, \Theta \to C}\ / \to \qquad \frac{\Pi, A \to B}{\Pi \to B / A}\ \to /, \Pi \text{ is non-empty}$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \bullet B, \Delta \to C}\ \bullet \to \qquad \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A \bullet B}\ \to \bullet$$

On Lambek calculus models

Completeness theorems (some of them)

- L is a logic of residual semigroups.
- L^* (the calculus without Lambek restriction) is a logic of residual monoids.
- L is complete w.r.t. to L-models, i.e. residual semigroup on subsets of a free semigroup.

See:

- Buszkowski W., (1986). Completeness Results for Lambek Syntactic Calculus. Zeitschrift fur mathematische Logik und Grundlagen der Mathematik, vol. 32, pp 13-28.
- Pentus M., (1995). Models for the Lambek calculus. Annals of Pure and Applied Logic, vol. 75, No 1-2, pp 179-213.

Additive connections

Additive $\wedge$ and $\vee$

$$\frac{\Gamma, A_i, \Delta \to B}{\Gamma, A_1 \wedge A_2, \Delta \to B} \wedge, i = 1, 2 \to \qquad \frac{\Gamma \to A \qquad \Gamma \to B}{\Gamma \to A \wedge B} \to \wedge$$

$$\frac{\Gamma, A, \Delta \to C \qquad \Gamma, B, \Delta \to C}{\Gamma, A \vee B, \Delta \to C} \vee \to \qquad \frac{\Gamma \to A_i}{\Gamma \to A_1 \vee A_2} \to \vee, i = 1, 2$$

Semantics of Lambeks calculus with additives

Lambek calculus with additives is incomplete w.r.t L-models.

Distributivities

- $\vdash A \vee (B \wedge C) \to (A \vee B) \wedge (A \vee C)$
- $\nvdash (A \vee B) \wedge (A \vee C) \to A \vee (B \wedge C)$
- $\nvdash A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
- $\vdash (A \wedge B) \vee (A \wedge C) \to A \wedge (B \vee C)$

Semantics of Lambeks calculus with additives

Quantale

A quantale is a triple $\mathcal{Q} = \langle Q, \bigvee, \cdot \rangle$, where $\langle Q, \bigvee \rangle$ is a complete join-semilattice and $\langle Q, \cdot \rangle$ is a semigroup such that for each indexing set I:

1. $a \cdot \bigvee\limits_{i \in I} b_i = \bigvee\limits_{i \in I} (a \cdot b_i)$
2. $\bigvee\limits_{i \in I} a_i \cdot b = \bigvee\limits_{i \in I} (a_i \cdot b)$

A quanlate is called unital, if $\langle A, \cdot \rangle$ is a monoid.

See:

- Abramsky S., Vickers S., (1993). Quantales, observational logic and process semantics, Math. Struct. in Comput. Sci. 3., pp 161-227.
- Eklund, P., Gutiérrez Garcia, J., Höhle, U., Kortelainen, J., (2018). Semigroups in Complete Lattices. Quantales, Modules and Related Topics.
- Yetter D., (1990). Quantales and (noncommutative) linear logic, Journal of Symbolic Logic, 55, pp 41-64.

Semantics of Lambeks calculus with additives

Relational (unital) quantale

Let $R \subseteq A \times A$ be a transtivite relation on A. Then relational quantale on A is a triple $\mathcal{Q} = \langle \mathcal{P}(R), \bigvee, \mathcal{I} \rangle$:

- $\langle \mathcal{P}(R), \bigvee, \subseteq \rangle$ is a complete semilattice
- Multiplication is defined as $R_1 \circ R_2 = \{\langle a, c \rangle \mid \exists b \in A, \langle a, b \rangle \in R_1 \text{ and } \langle b, c \rangle \in R_2\}$
- $\mathcal{I}$ is a neutral
- For each indexing set J, $S \circ \bigvee_{j \in J} S_j = \bigvee_{j \in J}(S \circ S_j)$ and $\bigvee_{j \in J} S_j \circ S = \bigvee_{j \in J}(S_j \circ S)$.

Theorem [Brown and Gurr, '93]

Any quantale is isomorphic to some relational quantale on its underlying set

See:

- Brown C., Gurr D., (1993)., A representation theorem for quantales, Journal of Pure and Applied Algebra, vol. 85., pp 27-42.

Semantics of Lambeks calculus with additives

Theorem [Brown and Gurr, '95]

- Lambek calculus with additives is complete w.r.t to quantales
- Lambek calculus with additives is complete w.r.t. to relational quantales

See:

- Brown C., Gurr D., (1995)., Relations and non-commutative linear logic, Journal of Pure and Applied Algebra, vol. 105, issue 2., pp 117-136.

Here come the subexponentials.

Medial extraction

The young lady whom Childe Harold met before his pilgrimage

Here come the subexponentials.

Medial extraction

The young lady whom$_i$ Childe Harold met e_i before his pilgrimage

$$\frac{\Gamma, A, \Delta \to B}{\Gamma, !A, \Delta \to B}\ (! \to) \qquad \frac{\Gamma, !A, \Delta, \Theta \to C}{\Gamma, \Delta, !A, \Theta \to C}\ \mathbf{ex}$$

Here come the subexponentials.

Medial extraction

The young lady whom$_i$ Childe Harold met e_i before his pilgrimage

$$\frac{\Gamma, A, \Delta \to B}{\Gamma, !A, \Delta \to B}\,(! \to) \qquad\qquad \frac{\Gamma, !A, \Delta, \Theta \to C}{\Gamma, \Delta, !A, \Theta \to C}\,\textbf{ex}$$

Parasitic extraction

The letter that Young Werther sent to Charlotte without reading.

Here come the subexponentials

Medial extraction

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Parasitic extraction

The letter that$_i$ Young Werther sent e_i to Charlotte without reading e_i.

$$\frac{\Gamma, !A, \Delta, !A, \Theta \to B}{\Gamma, \Delta, !A, \Theta \to B}\,\textbf{ncontr} \qquad \frac{\Gamma, !A, \Delta, !A, \Theta \to B}{\Gamma, !A, \Delta, \Theta \to B}\,\textbf{ncontr}$$

Subexponentials: polymodal case

Exponential modalities were initially introduced by Girard and Lafont within linear logic. We consider the polymodal case of non-commutative (sub)exponentials.

Subexponential signature

A subexponential signature is a quintuple $\Sigma = \langle \mathcal{I}, \preceq, \mathcal{W}, \mathcal{C}, \mathcal{E} \rangle$, where $\mathcal{I} = \{s_1, s_2, \ldots\}$ and $\langle \mathcal{I}, \preceq \rangle$ is a preorder. $\mathcal{W}, \mathcal{C}, \mathcal{E}$ are upwardly closed subsets of $\mathcal{I}$ and $\mathcal{W} \cap \mathcal{C} \subseteq \mathcal{E}$.

$$\dfrac{\dfrac{\Gamma, !A, \Delta, \Theta \to B}{\Gamma, !A, \Delta, !A, \Theta \to B}\ \textbf{weak}}{\Gamma, \Delta, !A, \Theta \to B}\ \textbf{ncontr}$$

Subexponentials: polymodal case

Polymodal subexponential rules for $SMALC_\Sigma$

$$\frac{\Gamma, A, \Delta \to B}{\Gamma, !_s A, \Delta \to B}\ (!_s \to) \qquad \frac{!_{s_1} A_1, \ldots, !_{s_n} A_n \to A}{!_{s_1} A_1, \ldots, !_{s_n} A_n \to !_s A}\ (\to !_s), \forall j, s_j \geq s$$

$$\frac{\Gamma, !_s A, \Delta, \Theta \to C}{\Gamma, \Delta, !_s A, \Theta \to C}\ \mathbf{ex}, s \in \mathcal{E} \qquad \frac{\Gamma, \Delta, !_s A, \Theta \to C}{\Gamma, !_s A, \Delta, \Theta \to C}\ \mathbf{ex}, s \in \mathcal{E}$$

$$\frac{\Gamma, !_s A, \Delta, !_s A, \Theta \to B}{\Gamma, \Delta, !_s A, \Theta \to B}\ \mathbf{ncontr}, s \in \mathcal{C} \qquad \frac{\Gamma, !_s A, \Delta, !_s A, \Theta \to B}{\Gamma, !_s A, \Delta, \Theta \to B}\ \mathbf{ncontr}, s \in \mathcal{C}$$

$$\frac{\Gamma, \Delta \to B}{\Gamma, !_s A, \Delta \to B}\ \mathbf{weak}_!, s \in \mathcal{W}$$

Subexponentials: polymodal case

The current results

1. Cut-rule is admissable
2. SMALC_Σ is undecidable, if $C \neq \varnothing$
3. If $C = \varnothing$, then the decidability problem of SMALC_Σ belongs to PSPACE.

See:

- Kanovich M., Kuznetsov S., Nigam V., Scedrov A., (2016) On the proof theory of non-commutative subexponentials, arXiv:1709.03607.
- Kanovich M., Kuznetsov S., Nigam V., Scedrov A., (2018) Subexponentials in non-commutative linear logic, Math. Structures Comput. Sci., pp 1-33.

Subexponentials algebraically

Quantic conucleus

A quantic conucleus (or open modality) on a quantale $\mathcal{Q}$ is a map $I : \mathcal{Q} \to \mathcal{Q}$ such that

1. $Ix \leqslant x$
2. $Ix = I^2 x$
3. $x \leqslant y \Rightarrow Ix \leqslant Iy$
4. $Ix \cdot Iy = I(Ix \cdot Iy)$

For unital quantale, we require that $Ie = e$, where e is a neutral element.

Quantale semantics

Let $\mathcal{Q}$ be a quantale. Let $\mathcal{S} \subseteq \mathcal{Q}$ be a subquantale.

Proposition

Thus, a map $I_{\mathcal{S}} : \mathcal{Q} \to \mathcal{Q}$ such that $I_{\mathcal{S}}(a) = \bigvee \{ s \in \mathcal{S} \mid s \leqslant a \}$ is a quantic conucleus

Proposition

Therefore, the following maps are quantic conuclei:

1. $I_{\mathcal{S}}(a) = \bigvee \{ s \in \mathcal{S} \mid s \leqslant a \,\&\, s \in \mathcal{Z}(\mathcal{Q}) \}$, where $\mathcal{Z}(\mathcal{Q})$ is the centre of quantale $\mathcal{Q}$
2. $I_{\mathcal{S}}(a) = \bigvee \{ s \in \mathcal{S} \mid s \leqslant a \,\&\, s \leqslant e \}$, where e is a neutral element
3. $I_{\mathcal{S}}(a) = \bigvee \{ s \in \mathcal{S} \mid s \leqslant a, \forall b \in \mathcal{Q}, (b \cdot s) \vee (s \cdot b) \leqslant s \cdot b \cdot s \}$

Interpretation

Let $\mathcal{Q}$ be a quantale. Let $\Sigma = \langle \mathcal{I}, \preceq, \mathcal{W}, \mathcal{C}, \mathcal{E} \rangle$ be a subexponential signature. Suppose, we have $S : \Sigma \to Sub(\mathcal{Q})$, a contravariant map from Σ to the set of subquantales of $\mathcal{Q}$. A subexponential interpretation is a map $\sigma : \Sigma \to \Box_{\mathcal{Q}}$ such that:

A subexponential interpretation

$$\sigma(s_i) = \begin{cases} I_i : \mathcal{Q} \to \mathcal{Q}, \forall a \in Q, I_i(a) = \bigvee \{ s \in S(i) \mid s \leqslant a \}, \text{if } s_i \notin \mathcal{W} \cap \mathcal{C} \cap \mathcal{E} \\ I_i, \forall a \in Q, I_i(a) = \bigvee \{ s \in S(i) \mid s \leqslant a, s \leqslant e \}, \text{if } s_i \in \mathcal{W}, \ e \text{ is a neutral} \\ I_i, \forall a \in Q, I_i(a) = \bigvee \{ s \in S(i) \mid s \leqslant a, s \in \mathcal{Z}(\mathcal{Q}) \}, \text{if } s_i \in \mathcal{E} \\ I_i, \forall a \in Q, I_i(a) = \bigvee \{ s \in S(i) \mid s \leqslant a, \forall b, b \cdot s \vee s \cdot b \leqslant s \cdot b \cdot s \}, \text{if } s_i \in \mathcal{C} \\ \text{if } s_i \text{ belongs to intersection of selected subsets, then we combine the relevant conditions} \end{cases}$$

Interpretation

The notion of an interpretation

Let $\mathcal{Q}$ be a unital quantale. An interpretation is defined inductively:

$$
\begin{aligned}
[\![p_i]\!] &= f(p_i) \\
[\![\mathbb{1}]\!] &= e \\
[\![A \bullet B]\!] &= [\![A]\!] \cdot [\![B]\!] \\
[\![A \backslash B]\!] &= [\![A]\!] \backslash [\![B]\!] \\
[\![A / B]\!] &= [\![A]\!] / [\![B]\!] \\
[\![A \wedge B]\!] &= [\![A]\!] \wedge [\![B]\!] \\
[\![A \vee B]\!] &= [\![A]\!] \vee [\![B]\!] \\
[\![!_{s_i} A]\!] &= \sigma(s_i)[\![A]\!]
\end{aligned}
$$

where $f : PV \to \mathcal{Q}$ is a valuation and $\sigma : \Sigma \to \Box_{\mathcal{Q}}$ is a subexponential interpretation.

Completeness theorem

An entailment relation

$\Gamma \models A$ iff for each valuation f and for each subexponential interpretation σ, $[\![\Gamma]\!] \leqslant [\![A]\!]$, where $[\![\Gamma]\!] = [\![A_1]\!] \cdot \dots \cdot [\![A_n]\!]$, where $\Gamma = \{A_1, \dots, A_n\}$.

Soundness and completeness theorem [Rogozin, 2019]

$\mathrm{SMALC}_\Sigma \vdash \Gamma \to A \Leftrightarrow \Gamma \models A$

Relational completeness

Theorem [Rogozin, 2019]

Any quantale $\mathcal{Q}$ with quantic conucleus I is isomorphic to relational quantale $\hat{\mathcal{Q}}$ with some unary operator $\hat{I}$

where relational quantale is a quantale on subsets of transitive relation initially provided by Brown and Gurr ['93, '95].

Theorem [Rogozin, 2019]

SMALC_{Σ} is sound and complete w.r.t to relational quantales with some family of conuclei

Thank you for attention!