Proofs and surfaces

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Incidence theorems and triangulated surfaces

Incidence theorems in 2-dimensional Euclidean or projective geometry state that some incidences follow from other incidences.

Goal Formalise and extend, within proof theory, a methodology of Richter-Gebert of deriving such theorems from associated triangulated surfaces.

The Menelaus theorem

For three points in the Euclidean plane $\mathbb{R}^2$, let

$$(X, Y; Z) =_{df} \begin{cases} \frac{XZ}{YZ}, & \text{if } Z \text{ is between } X \text{ and } Y, \\ -\frac{XZ}{YZ}, & \text{otherwise.} \end{cases}$$

if X, Y and Z are mutually distinct and colinear. Otherwise, we set $(X, Y; Z)$ to be undefined.

A sextuple (A, B, C, P, Q, R) ($ABCPQR$ for short) of points in $\mathbb{R}^2$ is a *Menelaus configuration* when

$$(B, C; P),\ (C, A; Q) \text{ and } (A, B; R) \text{ are defined}$$

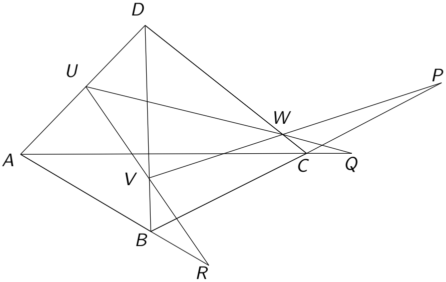
and their product is -1.

Theorem 1.1 (Menelaus)

For a triangle ABC (with A, B, C not colinear), and points P, Q and R (different from the vertices) on the lines BC, CA and AB, it holds that

$$P, Q, R \text{ are } \textit{colinear} \quad \textit{iff} \quad (A, B, C, P, Q, R) \text{ is a } \textit{Menelaus configuration}.$$

(half of) Desargues theorem



$$\frac{\begin{array}{c}(C,D;W)\cdot(D,B;V)\cdot(B,C;P)=-1\\(D,C;W)\cdot(A,D;U)\cdot(C,A;Q)=-1\\(B,D;V)\cdot(D,A;U)\cdot(A,B;R)=-1\end{array}}{(B,C;P)\cdot(C,A;Q)\cdot(A,B;R)=-1}$$

Hence, P, Q, R are colinear.

We could have picked any other triple of triangles forming the faces of *ABCD* **viewed as a tetrahedron**, assuming the corresponding colinearities, and would have derived the colinearity for the missing one.

In logical terms, we have that the following sequent is satisfied:

$$\vdash ABCPQR, ABDVUR, ACDWUQ, BCDWVP$$

where satisfaction means that whenever 3 out of these 4 sextuples is a Menelaus configuration, then so is the fourth.

Homology meets Menelaus

We suppose given a semi-simplical set K, and a function $v\colon K_0 \cup K_1 \to \mathbb{R}^2$.
Consider the operator $\mu\colon K_2 \to (\mathbb{R}^2)^6$

that sends a 2-cell $x \in K_2$ to (vA, vB, vC, vP, vQ, vR) (its realisation),

where A, B, C, P, Q, R are the 0-cells and 1-cells of the boundary of x.

Let c be a cycle on (the chain complex associated with) K. We can write

$$c = \varepsilon_1 x_1 + \varepsilon_2 x_2 + \ldots + \varepsilon_{n-1} x_{n-1} + \varepsilon_n x_n$$

were all x_i's are in K_2 and all ε_i's are 1 or -1. There may be repetitions!

Proposition 2.1

For any $1 \leq i \leq n$, if all μx_j's, for $j \neq i$, are Menelaus configurations, then μx_i is a Menelaus configuration, too.

Proof idea: Extend $y \mapsto (vd_0 y, vd_1 y; vy)$ $(y \in K_1)$ (e.g. $BC \mapsto (B, C; P)$) to a (partial) homomorphism $h\colon (C_1, +, 0) \to (\mathbb{R} \setminus \{0\}, \cdot, 1)$.

A suitable class of semi-simplicial sets

For applications to incidence theorems, it is sufficient to consider $\mathcal{M}$-complexes (for Menelaus), i.e., semi-simplicial sets L such that:

(0) L has a finite number of cells;

(1) for every $m \geq 3$, the set L_m is empty, and every element of $L_0 \cup L_1$ is a face of some element of $L_1 \cup L_2$;

(2) distinct faces map an element of L_{i+1} to distinct elements of L_i $(i \leq 1)$;

(3) every 1-cell of L is a face of exactly two 2-cells of L;

(4) for every $w \in L_0$, the set $L_w = \{u \in L_2 \mid w \text{ is a vertex of } u\}$ is *linked*, i.e., if $u, u' \in L_w$, then there is a sequence of 2-cells starting at u and ending at u', such that every two consecutive 2-cells share an edge having w as a vertex;

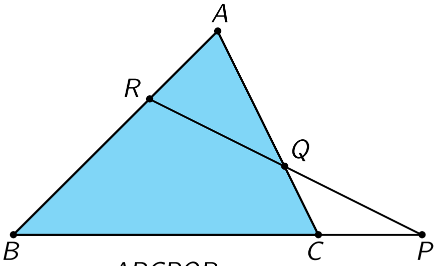
(5) L is *orientable*.

Proposition 2.2

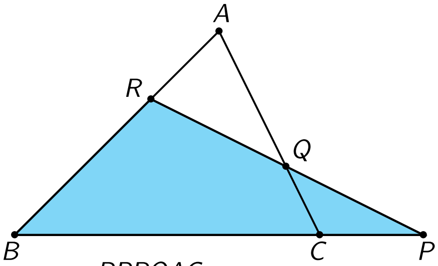
The geometric realisation of an $\mathcal{M}$-complex is a closed orientable 2-manifold.

Permutation and switching of triangles

- If $A_1A_2A_3B_1B_2B_3$ is a Menelaus configuration and π is a permutation of the set $\{1,2,3\}$, then $A_{\pi(1)}A_{\pi(2)}A_{\pi(3)}B_{\pi(1)}B_{\pi(2)}B_{\pi(3)}$ is a Menelaus configuration.
- If $ABCPQR$ is a Menelaus configuration, then $BPRQAC$, $ARQPCB$ and $CPQRAB$ are Menelaus configurations.



$ABCPQR$



$BPRQAC$

The formal system

We fix an arbitrary countable set W. Let

$$F^6(W) = W^6 - \{X_1 \dots X_6 \in W^6 \mid X_i = X_j \text{ for some } i \neq j\}.$$

The (atomic) formulas of our language are the elements of $F^6(W)$.

A *sequent* is a finite multiset Γ of formulas, written $\vdash \Gamma$.

• For every $\mathcal{M}$-complex L such that $L_0 \cup L_1 \subseteq W$, let $\nu\colon L_2 \to F^6(W)$ be defined as $\nu x = (d_1 d_2 x, d_0 d_2 x, d_0 d_0 x, d_0 x, d_1 x, d_2 x)$. Then we set

$$\frac{}{\vdash \{\nu x \mid x \in L_2\}}$$

• Permutations of vertices and switching of triangles:

$$\frac{}{\vdash ABCPQR, BCAQRP} \qquad \frac{}{\vdash ABCPQR, ARQPCB}$$

• In this system, the cut rule looks like this:

$$\frac{\vdash \Gamma, \varphi \qquad \vdash \Delta, \varphi}{\vdash \Gamma, \Delta}$$

Soundness

Proposition 2.3

The Menelaus system is sound.

By this we mean that

for every provable sequent $\vdash \Gamma$,

for any interpretation (i.e., a sufficiently defined partial function v from W to $\mathbb{R}^2$), and

for all $\phi \in \Gamma$,

if the interpretation of each formula in $\Gamma \setminus \{\phi\}$ is a Menelaus configuration, then the interpretation of ϕ is a Menelaus configuration.

[The interpretation $((v \times \ldots \times v) \circ \nu)(x)$

of the "logical" sextuple νx

is the "realised" sextuple μx from above.]

Decidability

For a multiset Γ of formulae, let $\lambda(\Gamma)$ be the set of elements of W occurring in Γ and let $\kappa(\Gamma)$ be the number of elements (possibly with repetition) of Γ.

Lemma 2.4

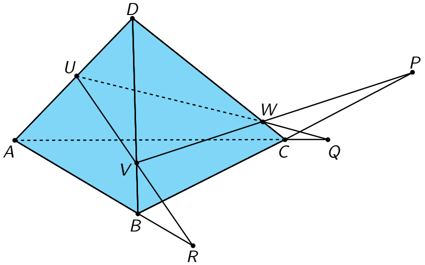
For every sequent $\vdash \Delta$ *that occurs in a derivation of* $\vdash \Gamma$*, we have that* $\lambda(\Delta) \subseteq \lambda(\Gamma)$ *and* $2 \leq \kappa(\Delta) \leq \kappa(\Gamma)$*.*

Proposition 2.5

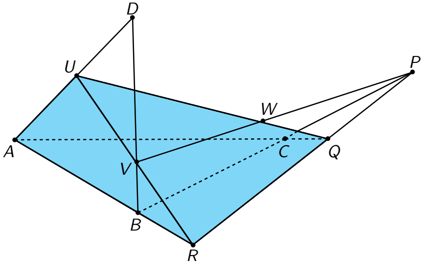
The Menelaus system is decidable.

This follows from the lemma (finiteness of the search space), and from the decidability of the properties defining an $\mathcal{M}$-complex.

Proof of Desargues theorem



$\vdash ABDVUR, BCDWVP, ACDWUQ, ABCPQR$

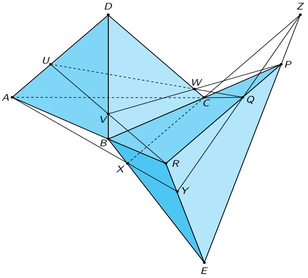


$\vdash ARUVDB, ARQPCB, URQPWV, AQUWDC$

A proof using cuts

Let AU, BV and CW be concurrent lines in $\mathbb{R}^2$, and let X and E be such that B, X and E are colinear. For $\{P\} = BC \cap VW$, $\{Q\} = AC \cap UW$, $\{R\} = AB \cap UV$, $\{Y\} = AX \cap RE$, $\{Z\} = XC \cap EP$, the points Q, Y and Z are colinear. The proof is obtained by cutting three axioms:

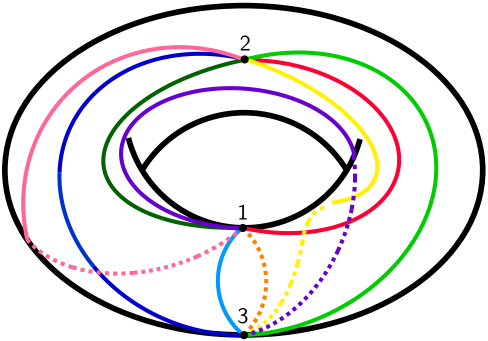
$$\vdash ABDVUR, BCDWVP, ACDWUQ, ABCPQR$$
$$\vdash ABCPQR, BPRQAC$$
$$\vdash BREYXA, BPEZXC, RPEZYQ, BPRQAC$$

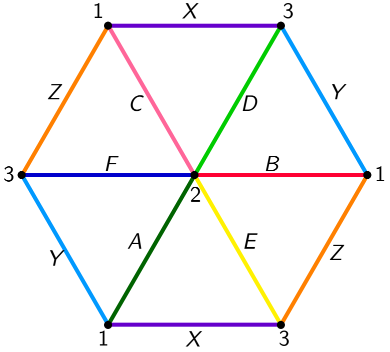


A more sophisticated example

Consider two triples (A, B, C) and (D, E, F) of colinear points, all mutually distinct. Assume that, for $\{X\} = CD \cap AE$ and $\{Z\} = BE \cap CF$, the lines AB, DE and XZ are not concurrent. Let $\{K\} = BE \cap CD$, $\{L\} = AF \cap CD$, $\{M\} = AF \cap BE$, $\{U\} = AE \cap CF$, $\{V\} = AE \cap BD$, $\{W\} = CF \cap BD$. Then the lines KU, LV and MW are concurrent.

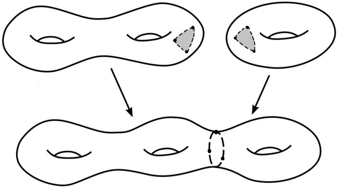
The proof uses the axiom given by the following triangulation of the torus:





The Menelaus cyclic operad $\mathcal{C}$

- For a given set X, we set $\mathcal{C}(X)$ to be the set of isomorphism classes of $\mathcal{M}$-complexes having X as set of 2-cells.
- Compositions are given by **connected sums** (cf. cut rule)!.



- Identities are given by the $\mathcal{M}$-complexes with exactly two 2-cells sharing their 3 faces.

Proposition 3.1

$\mathcal{M}$-complexes are stable under connected sums.

Goal Give a presentation of this cyclic operad by generators and relations.

Reducible $\mathcal{M}$-complexes

Let K be an $\mathcal{M}$-complex and let $T = \{e_0, e_1, e_2\} \subseteq K_1$ be such that $\partial(e_0 - e_1 + e_2) = 0$, i.e. $e_0 - e_1 + e_2$ is a 1-cycle.

Consider the binary relation on K_2 of sharing an edge from $K_1 - T$.

Let τ be the transitive closure of this relation.

We say that T is a *cut-triangle*, when τ is an equivalence relation with exactly two classes. If K contains a cut-triangle, then we say that it is *reducible*, otherwise it is *irreducible*.

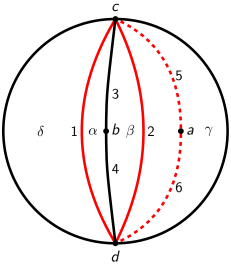
Proposition 3.2

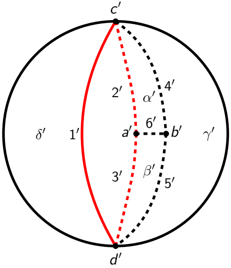
An $\mathcal{M}$-complex K is reducible if and only if it can be obtained as a connected sum of two simpler $\mathcal{M}$-complexes.

Proposition 3.3

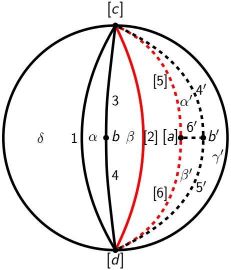
The Menelaus cyclic operad is generated by the irreducible $\mathcal{M}$-complexes.

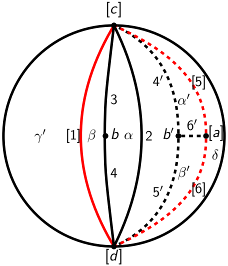
The Menelaus cyclic operad is not free

$K =$ 

$L =$ 

- σ renames the 2-cells α, β, γ, δ of K into β, α, γ', φ
- τ renames the 2-cells α', β', γ', δ' of L into α', β', δ, ψ

$K\ {}_{\gamma}{\circ}_{\delta'}\ L =$ 

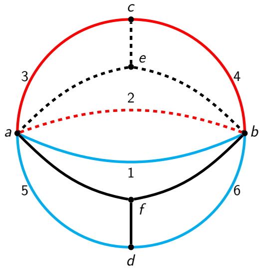
$=$ 

$= K^{\sigma}\ {}_{\varphi}{\circ}_{\psi}\ L^{\tau}$

Disjoint cut-triangles

Let $T_1, T_2 \subseteq K_1$ be cut-triangles of an $\mathcal{M}$-complex K. We say that T_1 is *disjoint* from T_2 if all the edges of T_1 are 1-cells of one of the two $\mathcal{M}$-complexes induced by T_2.

Below, $T_1 = \{2, 3, 4\}$ and $T_2 = \{1, 5, 6\}$ are disjoint:



A presentation of the Menelaus cyclic operad

Theorem 3.4

The Menelaus cyclic operad is the quotient of the free cyclic operad generated by the irreducible $\mathcal{M}$-complexes under the equivalence relation generated by all the equalities of the form

$$\mathcal{T}_1 \, {}_u\circ_v \, \mathcal{T}_2 = \mathcal{T}_1' \, {}_{u'}\circ_{v'} \, \mathcal{T}_2',$$

such that both hand sides evaluate, up to isomorphism, to the same $\mathcal{M}$-complex K, in which the cut-triangles T and T', associated with the pairs (u, v) and (u', v'), are not disjoint.