Completeness properties in abstract algebraic logic

Petr Cintula[1] Carles Noguera[2]

[1]Institute of Computer Science
Czech Academy of Sciences

[2]Institute of Information Theory and Automation
Czech Academy of Sciences

Aim of the talk

Logic: Syntax $\leftrightarrow$ Semantics

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Completeness theorems

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- Study them from the point of view of abstract algebraic logic

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- Consider their different forms for arbitrary classes of matrices

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- Give useful characterizations
- Use them to describe the classes of all models of the logic in question

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Completeness theorems

- Study them from the point of view of abstract algebraic logic
- Consider their different forms for arbitrary classes of matrices
- Give useful characterizations
- Use them to describe the classes of all models of the logic in question
- Explicate the role of equivalence and disjunction connectives

The precedents (from a personal point of view)

1. PC, F. Esteva, J. Gispert, L. Godo, F. Montagna, C. Noguera.
   Distinguished algebraic semantics for t-norm based fuzzy logics.
   Annals of Pure and Applied Logic 160:53–81, 2009.

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2. J. Czelakowski. *Protoalgebraic Logics*. Kluwer, 2001.
3. PC, C. Noguera. Implicational (semilinear) logics III: completeness properties. *Archive for Mathematical Logic* 7:391–420, 2018.

What is a logic (in AAL)

Language $\mathcal{L}$: an at most countable type

$\boldsymbol{Fm}_{\mathcal{L}}$: the absolutely free $\mathcal{L}$-algebra with countably infinite set of generators

we call elements of $\boldsymbol{Fm}_{\mathcal{L}}$ $\mathcal{L}$-formulas

Logic L: a relation between sets of $\mathcal{L}$-formulae and $\mathcal{L}$-formulae s.t.:

we write '$\Gamma \vdash_{\mathrm{L}} \varphi$' instead of '$\langle \Gamma, \varphi \rangle \in \mathrm{L}$' and '$\Gamma \vdash_{\mathrm{L}} \Delta$' instead of '$\Gamma \vdash_{\mathrm{L}} \varphi$ for each $\varphi \in \Delta$'

- $\Gamma, \varphi \vdash_{\mathrm{L}} \varphi$ (Reflexivity)
- If $\Gamma \vdash_{\mathrm{L}} \Delta$ and $\Delta \vdash_{\mathrm{L}} \varphi$, then $\Gamma \vdash_{\mathrm{L}} \varphi$ (Cut)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each substitution σ (Structurality)

A trivial completeness theorem all logics enjoy

Matrix $\mathbf{A}$: a tuple $\langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra and $F \subseteq A$.

$\Gamma \models_{\mathbf{A}} \varphi$ if for each $e\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{A}$ if $e[\Gamma] \subseteq F$, then $e(\varphi) \in F$

$\mathbf{A}$ is an L-model if $\Gamma \vdash_{\mathrm{L}} \varphi$ implies $\Gamma \models_{\mathbf{A}} \varphi$ F is then called an L-filter on $\boldsymbol{A}$

Class of all models: $\mathbf{Mod}(\mathrm{L})$

Fact: theories (deductively closed sets) = L-filters on $\boldsymbol{Fm}_{\mathcal{L}}$

1st completeness theorem: for each Γ and φ, $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbf{Mod}(\mathrm{L})} \varphi$

A less trivial completeness theorem all logics enjoy

Leibniz congruence $\Omega_{\boldsymbol{A}}(F)$: the largest congruence on $\boldsymbol{A}$ s.t.

$\langle x, y\rangle \in \Omega_{\boldsymbol{A}}(F)$ and $x \in F$ implies $y \in F$

Reduced model: $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}(\mathrm{L})$ such that $\Omega_{\boldsymbol{A}}(F) = Id_{\boldsymbol{A}}$

Class of all reduced models: $\mathbf{Mod}^*(\mathrm{L})$

Reduction of a matrix: $\langle \boldsymbol{A}, F\rangle^* = \langle \boldsymbol{A}/\Omega_{\boldsymbol{A}}(F), F/\Omega_{\boldsymbol{A}}(F)\rangle$

Observation: (1) $\mathbf{A}^*$ is reduced (2) $\mathbf{A}^* = (\mathbf{A}^*)^*$ (3) $\models_{\mathbf{A}^*} = \models_{\mathbf{A}}$

2nd completeness theorem: for each Γ and φ, $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbf{Mod}^*(\mathrm{L})} \varphi$

A non-trivial completeness theorem all finitary logics enjoy

Finitary logic: if $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ s.t. $\Gamma' \vdash_{\mathrm{L}} \varphi$

A matrix $\langle \boldsymbol{A}, F\rangle$ is subdirectly irreducible if F is not the intersection of any system of strictly bigger L-filters on $\boldsymbol{A}$

A matrix $\langle \boldsymbol{A}, F\rangle$ is finitely subdirectly irreducible if F is not the intersection of any non-empty finite system of strictly bigger L-filters on $\boldsymbol{A}$

3rd completeness theorem: If L is finitary, then for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$,

$$\Gamma \vdash_{\mathrm{L}} \varphi \text{ iff } \Gamma \models_{\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RSI}}} \varphi \text{ iff } \Gamma \models_{\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RFSI}}} \varphi$$

Three kinds of completeness theorem

Definition

Let L be a logic and $\mathbb{K} \subseteq \mathbf{Mod}(\mathrm{L})$. We say that L has the property of:

- Strong $\mathbb{K}$-completeness, S$\mathbb{K}$C for short, whenever $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$ for every set $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$

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- Finite strong $\mathbb{K}$-completeness, FS$\mathbb{K}$C for short, whenever $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$ for every finite set $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$

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Łukasiewicz logic has $\mathrm{FS}\mathbb{K}\mathrm{C}$ w.r.t. class $\mathbb{K}$ of its finite models but not $\mathrm{S}\mathbb{K}\mathrm{C}$

Any logic has $\mathbb{K}\mathrm{C}$ for $\mathbb{K} = \{\langle \boldsymbol{Fm}_{\mathcal{L}}, \mathrm{Thm}(\mathrm{L})\rangle\}$ but only structurally complete logics have $\mathrm{FS}\mathbb{K}\mathrm{C}$

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Any logic has $\mathbb{K}$C for $\mathbb{K} = \{\langle \boldsymbol{Fm}_{\mathcal{L}}, \mathrm{Thm}(\mathrm{L})\rangle\}$ but only structurally complete logics have FS$\mathbb{K}$C

Recall $\models_{\mathbf{A}^*} = \models_{\mathbf{A}}$, thus from now on we assume $\mathbb{K} \subseteq \mathbf{Mod}^*(\mathrm{L})$

The most general characterizations

Theorem

Let L *be a logic.*

1. L *has the* S$\mathbb{K}$C *iff* $\mathbf{Mod}^*(\mathrm{L}) \subseteq \mathbf{IS}^*\mathbf{P}_{\omega-f}(\mathbb{K})$.
2. L *has the* FS$\mathbb{K}$C *iff* $\mathbf{Mod}^*(\mathrm{L}) \subseteq \mathbf{IS}^*\mathbf{PP}_{\mathrm{U}}(\mathbb{K})$.

$\mathbf{I}$	isomorphic images
$\mathbf{S}^*$	reductions of submatrices
$\mathbf{P}$	products
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What are protoalgebraic and equivalential logics?

Let $\overrightarrow{r}$ be a sequence of atoms (parameters) and $\Leftrightarrow(p, q, \overrightarrow{r}) \subseteq Fm_{\mathcal{L}}$

Convention: given formulae φ and ψ, we set

$$\varphi \Leftrightarrow \psi = \{\chi(\varphi, \psi, \alpha_1, \ldots, \alpha_n) \mid \chi(p, q, r_1, \ldots, r_n) \in \Leftrightarrow \text{ and } \alpha_i \in Fm_{\mathcal{L}}\}$$

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A logic is protoalgebraic if it has a parameterized equivalence set $\Leftrightarrow$, s.t.:

(R) $\vdash_{\mathrm{L}} \varphi \Leftrightarrow \varphi$

(Sym) $\varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} \psi \Leftrightarrow \varphi$

(T) $\varphi \Leftrightarrow \psi, \psi \Leftrightarrow \chi \vdash_{\mathrm{L}} \varphi \Leftrightarrow \chi$

(MP) $\varphi, \varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} \psi$

(CNG) $\varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \varphi, \ldots, \chi_n) \Leftrightarrow c(\chi_1, \ldots, \psi, \ldots, \chi_n)$
for each $\langle c, n\rangle \in \mathcal{L}$ and $i \leq n$

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(CNG)	$\varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \dots, \varphi, \dots, \chi_n) \Leftrightarrow c(\chi_1, \dots, \psi, \dots, \chi_n)$ for each $\langle c, n\rangle \in \mathcal{L}$ and $i \leq n$

A logic is (finitely) equivalential if it has a (finite) equivalence set $\Leftrightarrow(p, q)$

The most general characterizations

Theorem

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1. L *has the* S$\mathbb{K}$C *iff* $\mathbf{Mod}^*(\mathrm{L}) \subseteq \mathbf{IS}^*\mathbf{P}_{\omega-f}(\mathbb{K})$.
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Very little less general characterizations

Theorem

Let L *be a protoalgebraic logic.*

1. L *has the* S$\mathbb{K}$C *iff* $\mathbf{Mod}^*(\mathrm{L}) = \mathbf{I}\mathbf{S}^*\mathbf{P}_{\omega-f}(\mathbb{K})$.
2. L *has the* FS$\mathbb{K}$C *iff* $\mathbf{Mod}^*(\mathrm{L}) \subseteq \mathbf{I}\mathbf{S}^*\mathbf{P}\mathbf{P}_{\mathrm{U}}(\mathbb{K})$.
3. L *has the* $\mathbb{K}$C *iff* $\mathbf{H}(\mathbf{Mod}^*(\mathrm{L})) = \mathbf{H}\mathbf{S}^*\mathbf{P}(\mathbb{K})$.

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Little less general characterizations

Theorem

Let L *be an equivalential logic.*

1. L *has the* S$\mathbb{K}$C *iff* $\mathbf{Mod}^*(\mathrm{L}) = \mathbf{ISP}_{\omega-f}(\mathbb{K})$.
2. L *has the* FS$\mathbb{K}$C *iff* $\mathbf{Mod}^*(\mathrm{L}) \subseteq \mathbf{ISPP}_{\mathrm{U}}(\mathbb{K})$.
3. L *has the* $\mathbb{K}$C *iff* $\mathbf{H}(\mathbf{Mod}^*(\mathrm{L})) = \mathbf{HSP}(\mathbb{K})$.

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Less general characterizations

Theorem

Let L *be a finitary finitely equivalential logic.*

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What are p-disjunctional logics?

Let $\overrightarrow{r}$ be a sequence of atoms (parameters) and $\nabla(p, q, \overrightarrow{r}) \subseteq Fm_{\mathcal{L}}$

Convention: given formulae φ and ψ, we set

$$\varphi \nabla \psi = \{\chi(\varphi, \psi, \alpha_1, \ldots, \alpha_n) \mid \chi(p, q, r_1, \ldots, r_n) \in \nabla \text{ and } \alpha_i \in Fm_{\mathcal{L}}\}$$

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A logic is p-disjunctional if it has a parameterized disjunction set ∇, s.t.:

$$T, \varphi \vdash_{\mathrm{L}} \chi \text{ and } T, \psi \vdash_{\mathrm{L}} \chi \qquad \text{iff} \qquad T, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$$

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In a protoalgebraic logic a disjunction $\{p \vee q\}$ is lattice disjunction if:

(I) $\vdash_{\mathrm{L}} \varphi \vee \varphi \Leftrightarrow \varphi$

(C) $\vdash_{\mathrm{L}} \varphi \vee \psi \Leftrightarrow \psi \vee \varphi$

(A) $\vdash_{\mathrm{L}} \varphi \vee (\psi \vee \chi) \Leftrightarrow (\varphi \vee \psi) \vee \chi$

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Characterization using $\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RSI}}$ and $\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RFSI}}$

Theorem

Let L *be a finitary logic that is protoalgebraic or p-disjunctional.*

1. L *has the* S$\mathbb{K}$C *iff* $\mathbf{Mod}^*(\mathrm{L})^{\omega}_{\mathrm{RSI}} \subseteq \mathbf{IS}^*(\mathbb{K})$.
2. L *has the* FS$\mathbb{K}$C *iff* $\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RFSI}} \subseteq \mathbf{IS}^*\mathbf{P}_{\mathrm{U}}(\mathbb{K})$.

$\mathbb{M}^{\omega}$ stands for the class of at most countable members of $\mathbb{M}$

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3. L *has the* $\mathbb{K}$C *iff* $\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RFSI}} \subseteq \mathbf{HS}^*\mathbf{P}_{\mathrm{U}}(\mathbb{K})$.

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Characterization using $\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RSI}}$ and $\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RFSI}}$

Theorem

Let L *be a finitary protoalgebraic logic with lattice disjunction.*

1. L *has the* S$\mathbb{K}$C *iff* $\mathbf{Mod}^*(\mathrm{L})^{\omega}_{\mathrm{RFSI}} \subseteq \mathbf{IS}^*(\mathbb{K}^+)$.
2. L *has the* FS$\mathbb{K}$C *iff* $\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RFSI}} \subseteq \mathbf{IS}^*\mathbf{P}_{\mathrm{U}}(\mathbb{K})$.
3. L *has the* $\mathbb{K}$C *iff* $\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RFSI}} \subseteq \mathbf{HS}^*\mathbf{P}_{\mathrm{U}}(\mathbb{K})$.

$\mathbb{M}^{\omega}$ stands for the class of at most countable members of $\mathbb{M}$

$\mathbb{K}^+$ stands for the class $\mathbb{K}$ expanded by the trivial reduced matrix

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$\mathbb{M}^{\omega}$ stands for the class of at most countable members of $\mathbb{M}$

$\mathbb{K}^{+}$ stands for the class $\mathbb{K}$ expanded by the trivial reduced matrix

Corollary

Let L *be a finitary protoalgebraic logic with a lattice disjunction. Then*

$$\mathbf{Mod}^*(\mathrm{L})^{\omega}_{\mathrm{RFSI}} \subseteq \mathbf{IS}^*(\mathbf{Mod}^*(\mathrm{L})^{\omega,+}_{\mathrm{RSI}})$$

Characterization using partial embeddability

Partial embedding of a set $X \subseteq \langle \boldsymbol{A}, F\rangle$ into $\langle \boldsymbol{B}, G\rangle$: a one-to-one mapping s.t.

$$x \in F \quad \text{iff} \quad f(x) \in G$$

$$f(\lambda^{\boldsymbol{A}}(x_1, \ldots, x_n)) = \lambda^{\boldsymbol{B}}(f(x_1), \ldots, f(x_n)) \quad \text{whenever} \quad \lambda^{\boldsymbol{A}}(x_1, \ldots, x_n) \in X$$

$\mathbb{K}$ is *partially embeddable* into $\mathbb{M}$ if each finite $X \subseteq \mathbf{A} \in \mathbb{K}$ is partially embeddable into some $\mathbf{B} \in \mathbb{M}$

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$\mathbb{K}$ is *partially embeddable* into $\mathbb{M}$ if each finite $X \subseteq \mathbf{A} \in \mathbb{K}$ is partially embeddable into some $\mathbf{B} \in \mathbb{M}$

Theorem

Let L *be a finitary equivalential logic in a finite language. Then* L *has the* FS$\mathbb{K}$C *iff* $\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RFSI}}$ *is partially embeddable into* $\mathbb{K}^+$.

Consequences for final matrices

Let us by $\mathcal{F}$ denote the class of the finite members of $\mathbf{Mod}^*(\mathrm{L})$

Corollary

Let L *be a protoalgebraic and p-disjunctional logic. TFAE:*

(i) L *is finitary and tabular, i.e., it has* $\mathbb{K}$C *w.r.t. a finite set* $\mathbb{K} \subseteq \mathcal{F}$.

(iii) L *is strongly finite, i.e., it has* S$\mathbb{K}$C *w.r.t. a finite set* $\mathbb{K} \subseteq \mathcal{F}$.

If furthermore L *finitely equivalential with a lattice disjunction we can add:*

(iv) L *is finitary and has the* S$\mathcal{F}$C.

Corollary

Let L *be a protoalgebraic p-disjunctional finitary tabular logic. Then* $\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RFSI}}$ *is finite (up to isomorphism) and*

$$\mathbf{Mod}^*(\mathrm{L})_{\mathrm{RSI}} = \mathbf{Mod}^*(\mathrm{L})_{\mathrm{RFSI}} \subseteq \mathcal{F}.$$

References

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