Time-reversal and homotopical properties of concurrent systems

Cameron Calk
Laboratoire d'Informatique
de l'École Polytechnique (LIX)

Joint work with:
Eric Goubault &
Philippe Malbos

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Concurrent programs and directed topology

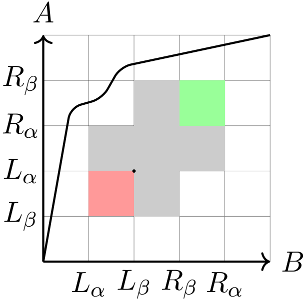
- Directed topology was introduced as a model of concurrent programs in computer science.
  - For n parallel threads, we consider an n-dimensional topological space in which points are states.
  - Paths in this space represent executions of the concurrent program.
  - Since an execution cannot be undone, these paths provide a notion of direction in the space.
  - States (points) are removed when they are unattainable by any execution, creating obstructions (holes).

- We would like to classify executions with respect to obstructions.
  - As in classical topology, algebraic invariants are used as a means of classification.

An example

- For a mutex λ, consider operations
  - L_λ locking the mutex, and R_λ releasing the mutex.
- Given sequential programs
  - $A = (L_\beta; L_\alpha; R_\alpha; R_\beta)$ and $B = (L_\alpha; L_\beta; R_\beta; R_\alpha)$

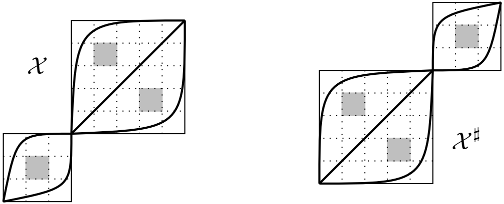
  consider the concurrent program $A||B$.
- We define a geometric realisation of this concurrent program endowed with the structure of a directed space.



Directed homotopy and time-reversal

- We consider directed homotopy $\overrightarrow{\Pi}_n$ (Dubut '17).
- The idea:
  - Consider the topological space consisting of directed paths between points x and y.
  - Apply classical algebraic invariants to these spaces.
  - Allow the end-points x and y to vary.

- Directed homotopy does not capture time-reversal (Hess & Fajstrup '17).



Outline

- Preliminaries : natural systems, directed spaces and directed homotopy.
- Problem : directed homotopy is time-symmetric, that is

$$\overrightarrow{\Pi}_n(\mathcal{X}) \cong \overrightarrow{\Pi}_n(\mathcal{X}^\sharp).$$

  Cause : the order of concatenation of dipaths is not witnessed.
- Sketch of solution :
  - Composition pairings (Porter '16) keep track of the effect of concatenation of dipaths on directed homotopy.
  - Directed homotopy can then be interpreted as a category :

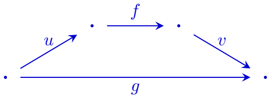
$$\overrightarrow{\Pi}_n(\mathcal{X}) \quad \longleftrightarrow \quad \mathcal{C}^n_{\mathcal{X}}$$

  - Passage to the opposite category captures time-reversal :

$$(\mathcal{C}^n_{\mathcal{X}})^o \cong \mathcal{C}^n_{\mathcal{X}^\sharp}$$

Natural systems on a category

- Given a category $\mathcal{B}$, we consider its factorisation category $\mathcal{F}\mathcal{B}$,
  in which
  - 0-cells are the 1-cells of $\mathcal{B}$.
  - A 1-cell $f \to g$ is a pair (u, v) of 1-cells of $\mathcal{B}$ such that $g = ufv$.



A factorisation $(u, v) : f \longrightarrow g$.

- A natural system on $\mathcal{B}$ with values in $\mathcal{V}$ is a functor

$$D : \mathcal{F}\mathcal{B} \longrightarrow \mathcal{V}.$$

Natural systems

- We define $\mathbf{Nat}(\mathcal{V})$ the category of natural systems with values in $\mathcal{V}$, whose
  - objects are pairs $(\mathcal{B}, D)$, where
$$D : \mathcal{F}\mathcal{B} \to \mathcal{V},$$
  - morphisms $(\mathcal{B}, D) \to (\mathcal{B}', D')$ are pairs (Φ, α), where
$$\Phi : \mathcal{B} \to \mathcal{B}'$$
is a functor and $\alpha : D \Rightarrow \Phi^* D'$ is a natural transformation, with
$$\Phi^* D'(f) = D'(\Phi(f)) \text{ and } \Phi^* D'(u, v) = D'(\Phi(u), \Phi(v)),$$
for f (resp. (u, v)) a 0-cell (resp. 1-cell) of $\mathcal{F}\mathcal{B}$.

From modules to natural systems

- In the abelian (co-)homology theory of a category $\mathcal{C}$ the following are used as coefficients:

left modules	bi-modules	right modules
$\mathcal{C}^{op} \longrightarrow \mathbf{Ab}$	$\mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathbf{Ab}$	$\mathcal{C} \longrightarrow \mathbf{Ab}$

- Natural systems generalize the notions of module and capture the action of composition by morphisms of the category.
- Furthermore, there is an equivalence between the different choices of coefficients in abelian (co-)homology theories:

$$NatSys(\mathcal{C}, \mathbf{Ab}) \quad \longleftrightarrow \quad Ab(\mathbf{Cat}/\mathcal{C})$$

(Quillen, Baues-Wirsching, Jibladze-Pirashvili, . . .)

Directed spaces

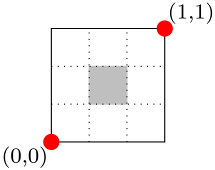
- A directed space $\mathcal{X}$ consists of a pair (X, dX), where
  - X is a topological space,
  - $dX \subseteq X^{[0,1]}$ is the set of directed paths:
    - Every constant path is directed,
    - dX is closed under monotonic reparametrisation,
    - dX is closed under concatenation.
- A dicontinuous map $\mathcal{X} \to \mathcal{Y}$ is a continuous map $\phi : X \to Y$ such that for every path $p : [0,1] \longrightarrow X$ in dX, we have

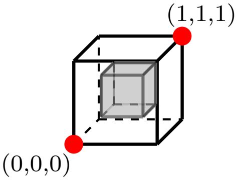
$$(\phi_* p : [0,1] \longrightarrow Y) \in dY.$$

- We denote by **dTop** the category of directed spaces.

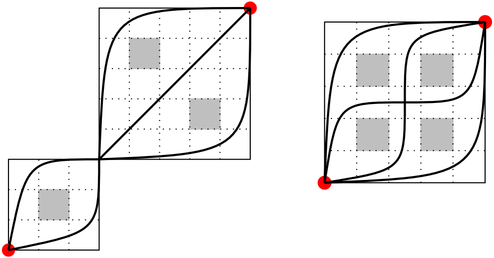
Trace Spaces

- For $\mathcal{X}$ a directed space and x,$y \in X$, $\overrightarrow{Di}(\mathcal{X})(x,y)$ denotes the space of dipaths in dX from x to y equipped with the compact-open topology.
- The trace space of $\mathcal{X}$ from x to y is the quotient of $\overrightarrow{Di}(\mathcal{X})(x,y)$ by monotonic reparametrisation, given the quotient topology. It is denoted by $\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y)$.
- Two partially ordered spaces with non-homeomorphic trace spaces.



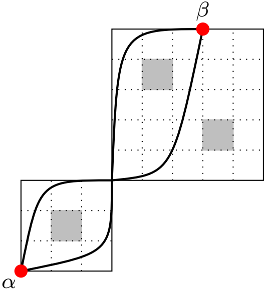


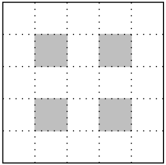
Why natural systems?



- Non-dihomeomorphic directed spaces with homotopy-equivalent trace spaces between extremal points.

Why natural systems?





- Changing the base points distinguishes these directed spaces.
- Specifying a trace chooses a beginning and end point for loops.

The trace diagram

- For a dispace $\mathcal{X} = (X, dX)$, define the category of traces $\overrightarrow{P}(\mathcal{X})$:
  - 0-cells are points of X,
  - 1-cells are given by $\overrightarrow{P}(\mathcal{X})(x, y) = |\overrightarrow{\mathfrak{T}}(\mathcal{X})(x, y)|$,
  - composition is concatenation of dipaths.
- The trace diagram associated to $\mathcal{X}$ is the natural system of pointed topological spaces:

$$
\begin{aligned}
\overrightarrow{T}_*(\mathcal{X}) : \mathcal{F}\overrightarrow{P}(\mathcal{X}) &\longrightarrow \mathbf{Top}_* \\
f &\longmapsto (\overrightarrow{\mathfrak{T}}(\mathcal{X})(x, y), f) \\
(u, v) &\longmapsto (f' \mapsto uf'v)
\end{aligned}
$$

Directed homotopy

- For $n \geq 1$, we define the n^{th} directed homotopy functor of $\mathcal{X}$ as follows :

$$\overrightarrow{\Pi}_n(\mathcal{X}) : \mathcal{F}\overrightarrow{P}(\mathcal{X}) \xrightarrow{\overrightarrow{T}_*(\mathcal{X})} \mathbf{Top}_* \xrightarrow{\pi_{n-1}} \mathcal{V}$$

where $\mathcal{V}$ is $\mathbf{Set}_*$, $\mathbf{Gp}$ or $\mathbf{Ab}$, by composing $\overrightarrow{T}_*(X)$ with the $(n-1)^{th}$ homotopy functor π_{n-1}.

- This induces the n^{th} directed homotopy functor

$$\overrightarrow{\Pi}_n : \mathbf{dTop} \to \mathbf{Nat}(\mathcal{V})$$

associating the pair $(\overrightarrow{P}(\mathcal{X}), \overrightarrow{\Pi}_n(\mathcal{X}))$ to a dispace $\mathcal{X}$.

Time reversal of a directed space

- Given a directed space $\mathcal{X}$, we consider its time-reversal or *opposite dispace*,

$$\mathcal{X}^\sharp := (X, dX^\sharp).$$

- This is the directed space in which the direction has been inverted:

$$dX^\sharp := \{t \mapsto f(1-t) | f \in dX\}.$$

- The passage to the opposite dispace is functorial

$$\mathbf{dTop} \xrightarrow{(-)^\sharp} \mathbf{dTop}.$$

The category of diagrams

- $\overrightarrow{\Pi}_n(\mathcal{X})$ and $\overrightarrow{\Pi}_n(\mathcal{X}^\sharp)$ are in general not comparable in $\mathbf{Nat}(\mathcal{V})$:

$$\overrightarrow{P}(\mathcal{X})^o = \overrightarrow{P}(\mathcal{X}^\sharp).$$

- The category of diagrams in $\mathcal{V}$, denoted by by $\mathbf{Diag}(\mathcal{V})$, has
  - objects pairs $(\mathcal{C}, F)$ where

$$F : \mathcal{C} \to \mathcal{V}.$$

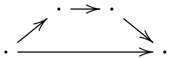
  - morphisms $(\mathcal{C}, F) \to (\mathcal{C}', F')$ are pairs (Φ, α) where

$$\Phi : \mathcal{C} \to \mathcal{C}'$$

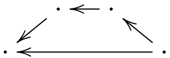
    is a functor and $\alpha : F \Rightarrow F' \circ \Phi$ is a natural transformation.
  - Abusing notation, we have $\overrightarrow{\Pi}_n : \mathbf{dTop} \to \mathbf{Diag}(\mathcal{V})$, sending $\mathcal{X}$ to $(\mathcal{F}\overrightarrow{P}(\mathcal{X}), \overrightarrow{\Pi}_n(\mathcal{X}))$.

Directed homotopy is time-symmetric

- Consider the covariant functor



$$\begin{aligned} \mathcal{F}(^\sharp) : \mathcal{F}\overrightarrow{P}(\mathcal{X}) &\longrightarrow \mathcal{F}(\overrightarrow{P}(\mathcal{X}^\sharp)) \\ f &\longmapsto f^\sharp \\ (u,v) &\longmapsto (v^\sharp, u^\sharp). \end{aligned}$$



- For all $x, y \in X$, there exists a homeomorphism

$$\begin{aligned} \alpha_{x,y} : \overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y) &\longrightarrow \overrightarrow{\mathfrak{T}}(\mathcal{X}^\sharp)(y,x) \\ (t \mapsto f(t)) &\longmapsto (t \mapsto f(1-t)). \end{aligned}$$

- These give the components of a natural isomorphism

$$\begin{aligned} \alpha_f : \overrightarrow{\Pi}_n(\mathcal{X})_f &\longrightarrow (\mathcal{F}(^\sharp)^* \overrightarrow{\Pi}_n(\mathcal{X}^\sharp))_f \\ \sigma = [(s,t) \mapsto \sigma_s(t)] &\longmapsto [(s,t) \mapsto \sigma_s(1-t)] =: \sigma^\sharp \end{aligned}$$

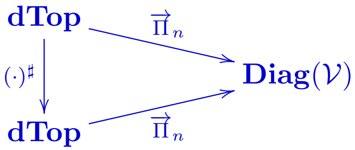
Time reversal and homotopy

- The pair $(\mathcal{F}(^\sharp), \alpha)$ is then an isomorphism

$$(\mathcal{F}\overrightarrow{P}(\mathcal{X}), \overrightarrow{\Pi}_n(\mathcal{X})) \xrightarrow{\cong} (\mathcal{F}\overrightarrow{P}(\mathcal{X}^\sharp), \overrightarrow{\Pi}_n(\mathcal{X}^\sharp))$$

  in $\mathbf{Diag}(\mathcal{V})$.

- The diagram



  thus commutes up to isomorphism (Hess & Fajstrup '17).

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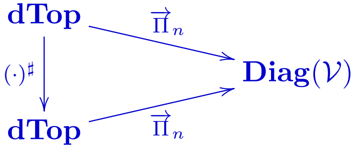


  thus commutes up to isomorphism (Hess & Fajstrup '17).

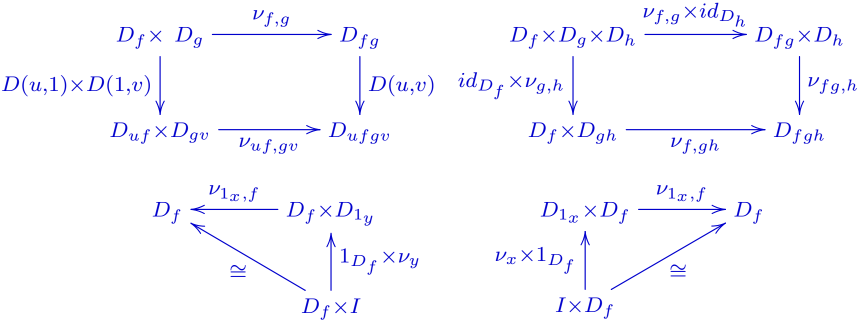
- The functor $\overrightarrow{\Pi}_n$ is time-symmetric with respect to $\mathbf{Diag}(\mathcal{V})$!

Composition pairing

- A composition pairing for a natural system $D : \mathcal{FB} \to (\mathcal{V}, \times)$ is a collection of 1-cells of $\mathcal{V}$:

$$\nu_{f,g} : D_f \times D_g \to D_{fg} \qquad \text{and} \qquad \nu_x : I \to D_{1_x}$$

for all pairs of composable 1-cells and every 0-cell of $\mathcal{B}$, satisfying coherence axioms:



Composition pairing for natural homotopy

Proposition (C., Goubault, Malbos)

For a dispace $\mathcal{X}$ *and all* $n \geq 2$, $\overrightarrow{\Pi}_n(\mathcal{X})$ *admits a composition pairing:*

$$\nu_{f,g}(\sigma, \tau) = \sigma \star \tau.$$

Furthermore, $\overrightarrow{\Pi}_1(\mathcal{X})$ *admits a composition pairing:*

$$\nu_{f,g}([f'], [g']) = [f' \star g']$$

Natural systems as categories

- Now we construct a category $\mathcal{C}^n_{\mathcal{X}}$ from the natural system $\overrightarrow{\Pi}_n(\mathcal{X})$ as follows:
  - 0-cells are points of $\mathcal{X}$
  - 1-cells from x to y are given by

$$\mathcal{C}^n_{\mathcal{X}}(x, y) := \coprod_{f \in \overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y)} \overrightarrow{\Pi}_n(\mathcal{X})_f$$

  - The composition in $\mathcal{C}$ is defined by

$$(\sigma, f) \cdot (\tau, g) := (\nu_{f,g}(\sigma, \tau), f \star g) = (\sigma \star \tau, f \star g)$$

  - (When $n \geq 2$, this defines an internal group in the slice category $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$!)
- The above assignment is functorial

Time reversal of natural homotopy

- We thus have a functorial assignment

$$\overrightarrow{\Pi}_n(\mathcal{X}) \longmapsto \mathcal{C}^n_{\mathcal{X}}$$

- This induces a functor

$$\mathcal{C}^n_{(-)} : \mathbf{dTop} \longrightarrow \mathbf{Cat}$$

Theorem (C., Goubault, Malbos)

For a directed space $\mathcal{X}$, $(\mathcal{C}^n_{\mathcal{X}})^o \cong \mathcal{C}^n_{\mathcal{X}^\sharp}$. *Thus* $\mathcal{C}^n_{(-)}$ *is time-reversal, i.e. the following diagram commutes up to isomorphism*

$$\begin{array}{ccc} \mathbf{dTop} & \xrightarrow{\mathcal{C}^n_{(-)}} & \mathit{Cat} \\ {\scriptstyle (-)^\sharp}\downarrow & & \downarrow{\scriptstyle (-)^o} \\ \mathbf{dTop} & \xrightarrow[\mathcal{C}^n_{(-)}]{} & \mathit{Cat} \end{array}$$

Perspectives

- Explore the effect of time-reversal on rewriting systems via an interpretation of these as directed spaces.

- Explore a notion of relative directed homotopy, and the induced long exact sequence of natural systems

$$\dots \to \overrightarrow{\Pi}_n(\mathcal{A}) \to \overrightarrow{\Pi}_n(\mathcal{X}) \to \overrightarrow{\Pi}_n(\mathcal{X},\mathcal{A}) \xrightarrow{\partial_n} \overrightarrow{\Pi}_{n-1}(\mathcal{A}) \to \dots$$

$$\dots \xrightarrow{v} \overrightarrow{\Pi}_2(\mathcal{X}) \xrightarrow{f} (\overrightarrow{\Pi}_2(\mathcal{X},\mathcal{A}), \overrightarrow{\Pi}_2(\mathcal{X},\mathcal{A})) \xrightarrow{g} \overrightarrow{\Pi}_1(\mathcal{A}) \xrightarrow{h} \overrightarrow{\Pi}_1(\mathcal{X}) \to \overrightarrow{\Pi}_1(\mathcal{X},\mathcal{A}) \to 0,$$

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Thank you