Duality for two-sorted lattices

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  - Priestley duality: distributive lattices $\leftrightarrow$ ordered Stone spaces (Priestley spaces).
  - Jonsson-Tarski duality: modal algebras $\leftrightarrow$ Stone spaces with a relation (general frames).

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- Example: duals of Heyting algebras are precisely those Priestley spaces where the down-set of every clopen set is clopen.
- The extra operations on algebras may require extra structure on spaces: relations, functions, designated subsets.

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- To a logician, it may get hard to view these spaces as semantics for the corresponding logics (points = theories?).
- Mathematically, the spaces we obtain may be harder to understand than the original algebras (!!).
- Certain classes of algebras seem to resist a nice topologization (e.g. non-distributive lattices, MV-algebras).

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- Example: every Nelson algebra can be represented as a (special) product of two (isomorphic) Heyting algebras.

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2. Additional structure may be required, e.g. maps $n\colon \mathbf{A}_+ \rightleftarrows \mathbf{A}_- : p$ between them.
3. The spatial counterpart of $\mathbf{A}$ is thus (e.g.) a pair of Stone spaces (X_+, X_-) equipped with two functions/relations between them.

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- Nelson algebras are structures which have an “intuitionistic” implication and a “classical” negation.
- In one-sorted duality (Odinstov, 2010), the dual of a Nelson algebra is a Priestley space X made of two (overlapping) intuitionistic subspaces X^1, X^2 related by a map g satisfying a bunch of non-trivial properties.

Two-sorted duality

Example: Nelson algebras (Odinstov's one-sorted duality)

DEFINITION 5.1. Let $\mathcal{X} = (X, X^1, \leq, \tau, g)$ be a tuple, where X is a set, $X^1 \subseteq X$, $\leq$ is a partial order on X, $g : X \to X$, and τ is a topology on X. Put

$$X^2 := g(X^1),\ X^+ := \{x \in X \mid x \leq g(x)\},\ X^- := \{x \in X \mid g(x) \leq x\}.$$

The structure $\mathcal{X}$ is said to be an $N4$-*space* if the following conditions are satisfied:

1. $(X, \leq, \tau, g)$ is a De Morgan space, i.e., $(X, \leq, \tau)$ is a Priestley space and g is an order reversing homeomorphism such that $g^2 = id_X$;
2. X^1 is closed in τ, $X = X^1 \cup X^2$, and $X^1 \cap X^2 = X^+ \cap X^-$;
3. $(X^1, \leq\restriction_{X^1}, \tau \restriction_{X^1})$ is a Heyting space;
4. for any $x \in X^1$ and $y \in X^2$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and there exists $z \in X$ such that $x, g(y) \leq z \leq g(x), y$;
5. for any $x \in X^2$ and $y \in X^1$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and $x \leq g(y)$.

This long definition obviously needs some comments.

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Example: Nelson algebras

- Representation (Odintsov, 2004): each Nelson algebra $\mathbf{A}$ can be viewed as a triple $(\mathbf{A}_+, \mathbf{A}_-, F)$ where $\mathbf{A}_+ \cong \mathbf{A}_-$ are Heyting algebras and $F \subseteq A_+$ is a filter.

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- Thus we can take the dual of $\mathbf{A}$ to be just one intuitionistic Priestley space X_+ (or two homeomorphic $X_+ \cong X_-$) together with a closed up-set $\mathcal{C}_F \subseteq X_+$.

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- Thus we can take the dual of $\mathbf{A}$ to be just one intuitionistic Priestley space X_+ (or two homeomorphic $X_+ \cong X_-$) together with a closed up-set $\mathcal{C}_F \subseteq X_+$.
- Morphisms just need to preserve F and $\mathcal{C}_F$, and everything else works: we have a duality.

Two-sorted duality: an abstract framework

- We consider algebraic categories whose objects are tuples $(\mathbf{A}_+, \mathbf{A}_-, n, p)$ where $\mathbf{A}_+, \mathbf{A}_-$ are bounded distributive lattices (with extra operations) and $n\colon \mathbf{A}_+ \to \mathbf{A}_-$, $p\colon \mathbf{A}_- \to \mathbf{A}_+$ are meet-preserving maps ("modal operators").

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- Our objects can thus be viewed as two-sorted algebras (in standard the sense of many-sorted universal algebra), or as diagrams in (e.g.) the category of meet-semilattices.
- Morphisms between two-sorted lattices $(\mathbf{A}_+, \mathbf{A}_-, n, p)$ and $(\mathbf{A}'_+, \mathbf{A}'_-, n', p')$ are pairs (h_+, h_-) of algebraic homomorphisms that make the two diagrams commute, i.e. $h_+ \circ p = p' \circ h_-$ and $n' \circ h_+ = h_- \circ n$.

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- Dual objects are tuples (X_+, X_-, R_n, R_p) where X_+, X_- are (special) Priestley spaces and $R_n \subseteq X_- \times X_+$, $R_p \subseteq X_+ \times X_-$ are closed relations.

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- Spatial morphisms between two-sorted spaces (X_+, X_-, R_n, R_p), (X'_+, X'_-, R'_n, R'_p) are pairs (f_+, f_-) of (special) Priestley-continuous functions that satisfy analogous properties to bounded morphisms of modal logic.

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- By restricting to subcategories we can obtain dualities for various non-classical algebras in a uniform way.

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- Mathematically, our dualities are "easier".
- On the topological side, it is not always easy to recover the one-sorted view from the two-sorted and vice versa.

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Thanks for your attention!