# Resource Reasoning in Duality Theoretic Form

## Stone-Type Dualities for Bunched and Separation Logics[1]

**Simon Docherty**[2] and David Pym

University College London

TACL 2019

[1]SD and Pym. Stone-Type Dualities for Separation Logics. *LMCS* 15(1), 2019.

[2]SD. Bunched Logics: A Uniform Approach. PhD Thesis. 2019

# Resource Reasoning

# Intrinsic vs Descriptive Resource Reasoning

- Abramsky: Two perspectives on logic and structure[3]

[3] Abramsky. Information, Processes and Games. In Philosophy of Information. 2008

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- Linear Logic: control of weakening and contraction via exponentials ⇒ formulae *are* consumable resources (intrinsic).

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     e.g. Kripke semantics of modal logic.
- Linear Logic: control of weakening and contraction via exponentials $\Rightarrow$ formulae *are* consumable resources (intrinsic).
- However! Semantic approaches to LL are *not* plausible *models of* resources (descriptive).

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# The Logic of Bunched Implications[5]

- Control of weakening and contraction at context formation: Two context formers; contexts are tree-shaped "bunches"[4]

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$$!\varphi \multimap \psi \vdash_{\mathrm{LL}} \varphi \multimap \psi \text{ vs } \varphi \to \psi \nvdash_{\mathrm{BI}} \varphi \mathrel{-\!\!*} \psi.$$

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- Intrinsic: sharing interpretation (vs LL's number-of-uses) via $\alpha\lambda$-calculus;
- Descriptive: Kripke semantics of resources (our focus). e.g. money, RAM, chemical substances, quantum phenomena

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# BI's Syntax and Resource Semantics

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$x \vDash \top^*$ iff $x \sqsupseteq e$;
$x \vDash \varphi * \psi$ iff $\exists y, z : x \sqsupseteq y \circ z, y \vDash \varphi$ and $z \vDash \psi$;

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$x \vDash \varphi \mathrel{-\!\!*} \psi$ iff $\forall y : y \vDash \varphi$ implies $x \circ y \vDash \psi$.

# Bunched Logic Duality

# Generalised Resource Semantics of BI

- Real applications: $\circ$ is partial (and possibly) non-deterministic).

## Definition (BI Frame)

$\mathcal{R} = (R, \sqsupseteq, \circ, E)$ where $\sqsupseteq$ a partial order, $\circ : R^2 \to \mathcal{P}(R)$, $E \subseteq X$ and the following conditions are satisfied for all $x, y, z, t, w, e, e'$:

$$(\circ \text{ Closure}) \quad x \sqsupseteq x', y \sqsupseteq y', z \in x \circ y \to \exists z' : z \sqsupseteq z', z' \in x' \circ y'$$

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| (Unit Closure) | $e \in E \wedge e' \sqsupseteq e \to e' \in E$ |
| (Unit Existence) | $\exists e \in E (x \in x \circ e)$ |
| (Coherence) | $e \in E \wedge x \in y \circ e \to x \sqsupseteq y$ |

# BI Algebras

### Definition (BI Algebra)

$(A, \wedge, \vee, \rightarrow, \top, \bot, *, -\!\!*, \top^*)$ with
i) $(A, \wedge, \vee, \rightarrow, \top, \bot)$ a Heyting algebra;
ii) $(A, *, \top^*)$ a commutative monoid;
iii) Residuation: $a * b \leq c$ iff $a \leq b -\!\!* c$.

- Thorough investigations: (Galatos & Jipsen)[6], (Jipsen)[7], (Jipsen & Litak)[8]

[6] Distributive Residuated Frames and Generalized BI Algebras. 2017.
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- Connections to residuated lattices, relation algebra, Kleene algebra, MV algebra....

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# Prime Predicates

- Main technical mechanism in proofs
- A predicate $P$ on tuples of filters and ideals such that
  a) Closed under taking unions of $\subseteq$-chains;
  b) If $P$ holds of tuple with one component $H \cap K$, $P$ holds of tuple with either $H$ or $K$.

### Lemma

*Tuples of filters and ideals satisfying a prime predicate can be extended to a tuple of prime filters and ideals satisfying the prime predicate.*

- Idea: suitable for both squeeze lemma (relevant logic) + correspondence theory (modal logic)

# BI Representation Theorem

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Idea:

- Prime filter frame: $\sqsupseteq$ is $\supseteq$, $E = \{F \mid \top^* \in F\}$, $F \bullet F' = \{G \mid \forall a, b \in A : a \in F, b \in F' \rightarrow a * b \in G\}$

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- Resource semantics clauses generate the algebra of sets.
- Frame axioms guarantee BI algebra.
- Algebraic properties of residuation ensure $h(a) = \{F \mid a \in F\}$ a monomorphism.

# Proof Snippets

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- Unary prime predicate $(-)_1 \in F_y \bullet F_z \wedge F_w \in F_x \bullet (-)_1$.

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Embedding $-\!\!*$: want $h(a) -\!\!* h(b) \subseteq h(a -\!\!* b)$. Suppose $a -\!\!* b \notin F$.

- Need prime filters $G, H$: $H \in F \bullet G$, $a \in G$, $b \notin H$.

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- Thus suitable prime filters $G, H$ exist: inclusion holds.

# BI Duality

- Recipe: Esakia duality for HA + Bimbó-Dunn-Urquhart duality for gaggles + the correspondence theory just established.

## Definition (BI Space)

$\mathcal{X} = (X, \mathcal{O}, \sqsupseteq, \circ, E)$ such that

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4. If $x \notin y \circ z$ then there exist up-closed clopen sets $C_1, C_2$ such that $y \in C_1, z \in C_2$ and $x \notin C_1 * C_2$ (Bimbó-Dunn).

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## Theorem

*Categories of BI algebras and BI spaces are dually equivalent.* □

# More Connectives: Separating Disjunction ("par")

- Extend BI frame with commutative $\triangledown : R^2 \rightarrow \mathcal{P}(R)$
  (Idea: resource intersection)

[9] Hyland, De Paiva. Full Intuitionistic Linear Logic. 1993.

# More Connectives: Separating Disjunction ("par")

- Extend BI frame with commutative $\triangledown : R^2 \to \mathcal{P}(R)$ (Idea: resource intersection)
- $r \vDash \varphi \overset{*}{\vee} \psi$ iff $\forall s, t : r \in s \triangledown t$ implies $s \vDash \varphi$ or $t \vDash \psi$

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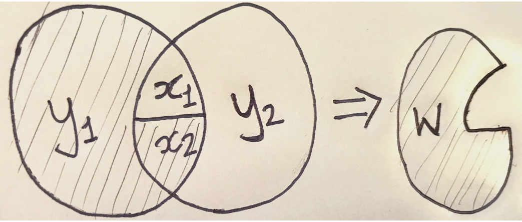
- CLAIM: Corresponds to frame property: $(x_1 \circ x_2) \cap (y_1 \triangledown y_2) \neq \emptyset \to \exists w (y_1 \in x_1 \circ w \wedge x_2 \in w \triangledown y_2)$.




This holds for computer memory models of this logic.

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# Duality Theory for $\mathbin{\overset{*}{\vee}}$: Weak Distribution

- Prime filter operation:
  $F \blacktriangledown G = \{H \mid \forall a, b \in H : a \mathbin{\overset{*}{\vee}} b \in H$ implies $a \in F$ or $b \in G\}$.

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Topological duality:

- Add: If $x \notin y \triangledown z$ then there exists upwards-closed clopen sets $C_1, C_2$ such that $y \notin C_1$, $z \notin C_2$ and $x \in C_1 \mathbin{\overset{*}{\vee}} C_2$.

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We have a systematic treatment of a natural class of bunched logics.

• multiplicative connectives; • modalities; • interesting axioms

# Separation Logic

# A Whirlwind Tour of Separation Logic

- Program logic for programs that manipulate shared data structures[10].

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- FB Infer: Separation Logic static analysis tool deployed at Facebook, Spotify, Amazon, Uber, Instagram, Whatsapp...

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# BI Hyperdoctrines

A *BI hyperdoctrine*[11] is a tuple

$$(\mathbb{P} : \mathrm{C}^{op} \to \mathrm{Poset}, (=_X)_{X \in Ob(\mathrm{C})}, (\exists X_\Gamma, \forall X_\Gamma)_{\Gamma, X \in Ob(\mathrm{C})})$$

such that:

1. $\mathrm{C}$ a category with finite products;
2. $\mathbb{P} : \mathrm{C}^{op} \to \mathrm{Poset}$ a functor: objects to BI algebras, morphisms to homomorphisms;
3. Diagonal morphisms $\Delta_X : X \to X \times X$ has $=_X \in \mathbb{P}(X \times X)$ adjoint at $\top_{\mathbb{P}(X)}$.

$$\top_{\mathbb{P}(X)} \leq \mathbb{P}(\Delta_X)(a) \text{ iff } =_X \leq a\,;$$

4. $\exists X_\Gamma$ and $\forall X_\Gamma$ are left and right adjoint to $\mathbb{P}(\pi_{\Gamma,X})$.

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# Indexed BI Spaces

An *indexed BI Space* is a functor $\mathcal{R} : \mathrm{C} \to \mathrm{BISp}$ such that

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Separation Logic's RAM model!

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## Theorem

*The categories of BI hyperdoctrines and indexed BI spaces are dually equivalent.* □

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# A Goldblatt-Thomason Theorem

- A class of frames $\mathcal{C}$ is definable in bunched logic $\mathcal{L}$ iff exists formulae $\Sigma \subseteq \mathcal{L}$ s.t. $\mathcal{R} \in \mathcal{C}$ iff $\mathcal{R} \vDash \Sigma$.

**Theorem (Bunched Logic Goldblatt Thomason Theorem)**

*A class of $\mathcal{L}$ frames is definable iff it is closed under bounded morphic images*

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- Harder: reflection
  (need: every prime extension the bounded morphic image of an ultrapower frame – extends (Rodenburg 1986)).

# Application of Goldblatt-Thomason to Separation Logic

- Abstract Separation Logic: attempt to capture salient features of memory models as subclasses of BI frames.

## Example

A Separation Algebra is a cancellative, partial commutative monoid.

(Cancellative: $x \circ z = y \circ z$ implies $x = y$.)

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## Theorem

*Separation algebras are not BI definable.*

## Proof.

By failure of Goldblatt-Thomason closure properties. □

- Instead: systematic labelled proof theory to nonetheless capture these classes[17]

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# Further Work & Conclusions

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  Natural duality and many-valued variants of bunched logics.
  Goldblatt-Thomason for first-order bunched logics

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- Full details: my PhD thesis, or SD & Pym. Stone-Type Dualities for Separation Logics. LMCS, 2019.

Thanks!