Topological representations of congruence lattices

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Problem. For a given class $\mathcal{K}$ of algebras describe Con $\mathcal{K}$ =all lattices isomorphic to Con A for some $A \in \mathcal{K}$.

Or, at least,

for given classes $\mathcal{K}$, $\mathcal{L}$ determine if Con $\mathcal{K}$ = Con $\mathcal{L}$
and, if Con $\mathcal{K} \nsubseteq$ Con $\mathcal{L}$, determine

$$\mathrm{Crit}(\mathcal{K}, \mathcal{L}) = \min\{\mathrm{card}(L_c) \mid L \in \mathrm{Con}\,\mathcal{K} \setminus \mathrm{Con}\,\mathcal{L}\}$$

(L_c = compact elements of L)

Two approaches

In the sequel we assume that $\mathcal{K}$ is a finitely generated congruence-distributive variety. Even under such restrictions, the problem of describing of $\operatorname{Con}\mathcal{K}$ is still hard. There are two main approaches: topological representations and lifting of semilattice diagrams. We try to connect these two methods.

Topological approach

$\mathrm{M}(L)$....completely meet-irreducible elements of a lattice L, including the top element
($a = \inf X$ implies $a \in X$)
Fact: if L is algebraic, then every element is a meet of completely meet-irreducible elements.
Topology on $\mathrm{M}(L)$: all sets of the form

$$\mathrm{M}(L) \cap \uparrow x = \{a \in \mathrm{M}(L) \mid a \geq x\}$$

are closed.

Theorem

If L is distributive algebraic, then $L \cong \mathcal{O}(\mathrm{M}(L))$. (The lattice of all proper open subsets of $\mathrm{M}(L)$.

Topological approach

If $L = \mathrm{Con}\, A$ (A in a fin. generated CD variety), then the basis of the topology is given by all sets of the form

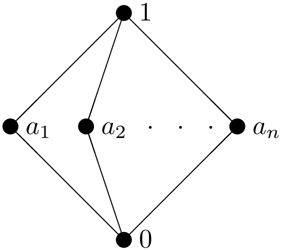
$$U(B, \delta) = \{\alpha \in \mathrm{M}(\mathrm{Con}\, A) \mid \alpha{\restriction}B \leq \delta\},$$

where B is a finite subalgebra of A and $\delta \in \mathrm{Con}\, B$.
Sometimes the properties of $\mathrm{Con}\, A$ are more effectively expressed as topological properties of $\mathrm{M}(\mathrm{Con}\, A)$. A sample:

- If A is a distributive lattice, then $\mathrm{M}(\mathrm{Con}\, A) \setminus \{1\}$ is Hausdorff.
- There exists a countable $B \in \mathbf{M}_3$ (the lattice variety generated by M_3) such that $\mathrm{M}(\mathrm{Con}\, B) \setminus \{1\}$ is not Hausdorff.
- Therefore, $\mathrm{Con}(\mathbf{M}_3) \nsubseteq \mathrm{Con}(\mathbf{D})$.

The topological approach was used to establish e.g. $\mathrm{Crit}(\mathbf{M}_4, \mathbf{M}_3) = \aleph_2$. (But the argument is much more complicated.)

Lattice M_n



Con functor

The Con functor:

For any homomorphism of algebras $f: A \to B$ we define

$$\mathrm{Con}\, f: \mathrm{Con}\, A \to \mathrm{Con}\, B$$

by
$\alpha \mapsto$ congruence generated by $\{(f(x), f(y)) \mid (x, y) \in \alpha\}$.

Fact. $\mathrm{Con}\, f$ preserves $\vee$ and 0, not necessarily $\wedge$.

Lifting of semilattice morphisms

Let

- $\varphi : \ S \to T$ be a $(\vee, 0)$-homomorphisms of lattices;
- $f : \ A \to B$ be a homomorphisms of algebras.

We say that f *lifts* φ, if there are isomorphisms $\psi_1 : \ S \to \operatorname{Con} A$, $\psi_2 : \ T \to \operatorname{Con} B$ such that

$$\begin{array}{ccc} S & \xrightarrow{\ \varphi\ } & T \\ {\scriptstyle\psi_1}\downarrow & & {\scriptstyle\psi_2}\downarrow \\ \operatorname{Con} A & \xrightarrow{\operatorname{Con} f} & \operatorname{Con} B \end{array}$$

commutes.

A generalization: lifting of semilattice diagrams

Result of P. Gillibert

Let $\mathcal{K}$, $\mathcal{L}$ be finitely generated congruence distributive varieties.

Theorem

TFAE

- $\operatorname{Con}\mathcal{K} \nsubseteq \operatorname{Con}\mathcal{L}$;
- *there exists a diagram of finite* $(\vee, 0)$*-semilattices indexed by a finite ordered set liftable in* $\mathcal{K}$ *but not in* $\mathcal{L}$

Looking for a link

So, the list of all finite semilattice diagrams liftable in $\mathcal{K}$ characterizes the class $\mathrm{Con}(\mathcal{K})$. However, it is not clear what the (un)liftability of a particular diagram means for the properties of lattices $\mathrm{Con}\, A$ with $A \in \mathcal{K}$.

We provide a partial answer. We start with diagrams consisting of a single arrow.

Convergence of nets

Let $N = (k_p \mid p \in P)$ be a net in a topological space X, and let $Y \subseteq X$. We say that N *converges precisely* to Y if

- every $y \in Y$ is a limit point of N;
- no $y \in X \setminus Y$ is an accumulation point of N.

Separability

Let $s: S_0 \to S_1$ be a $(\vee, 0)$- homomorphism of finite distributive lattices. Let $s^{\leftarrow}$ be the dual $(\wedge, 1)$-homomorphism defined by

$$s^{\leftarrow}(\beta) = \bigvee \{\alpha \in S_0 \mid s(\alpha) \leq \beta\}.$$

Let X be a topological space, let $\leq$ be its specialization order ($x \leq y$ iff y is in the closure of $\{x\}$).
Let K_i denote the set of all order embeddings $\mathrm{M}(S_i) \to X$ whose range is an upper subset of X.

Definition

We say that X is s-nonseparable, if there exist $k_0 \in K_0$ and a net $(k_p \mid p \in P)$ in K_1 such that for every $\beta \in \mathrm{M}(S_1)$ the net $(k_p(\beta) \mid p \in P)$ converges precisely to the set $\{k_0(\alpha) \mid \alpha \in \mathrm{M}(S_0),\ \alpha \geq s^{\leftarrow}(\beta)\}$.

Result for single arrow

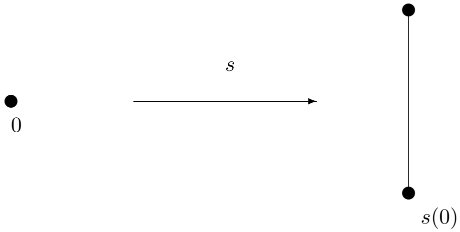
Theorem

TFAE

- $\mathrm{M}(\mathrm{Con}\, A)$ *is s-nonseparable for some $A \in \mathcal{K}$;*
- $\mathrm{M}(\mathrm{Con}\, F(\aleph_0))$ *is s-nonseparable ($F(\aleph_0)$ free in $\mathcal{K}$);*
- *s has a lifting in $\mathcal{K}$.*

Example1

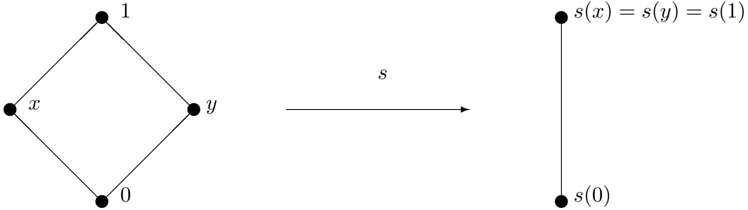
The semilattice homomorphism



has a lifting in $\mathbf{D}$ (distributive lattices), but not in $\mathbf{D}^{01}$ (bounded distributive lattices). Therefore, $\mathrm{Crit}(\mathbf{D}, \mathbf{D}^{01}) \leq \aleph_0$. Intuitively: in $\mathrm{Con}\, D$, where D is a distributive lattice, a sequence of coatoms can converge to the top element. This cannot happen when D is bounded.

Example2

The semilattice homomorphism



has a lifting in $\mathbf{M}_3$ (the embedding of a 3-element chain into M_3 lifts it), but not in $\mathbf{D}$. Therefore, $\mathrm{Crit}(\mathbf{M}_3, \mathbf{D}) \leq \aleph_0$.

Diagrams indexed by finite chains

Let $\mathcal{S}$ be the diagram

$$S_0 \xrightarrow{s_{01}} S_1 \xrightarrow{s_{12}} S_2 \xrightarrow{s_{23}} \dots \xrightarrow{s_{n-1,n}} S_n$$

of finite distributive lattices and $(\vee, 0)$-homomorphisms. Let X be a topological space, let K_i denote the set of all order embeddings $\mathrm{M}(S_i) \to X$ whose range is an upper subset of X.

Definition

We say that X is $\mathcal{S}$-nonseparable, if there exist $k_0 \in K_0$ and nets $(k_p \mid p \in P_1 \times \dots P_i)$ in K_i such that for every $\beta \in \mathrm{M}(S_i)$ and every $r = (p_1, \dots, p_{i-1}) \in P_1 \times \dots P_{i-1}$ the net $(k_{(r,p_i)}(\beta) \mid p_i \in P_i)$ converges precisely to the set $\{k_r(\alpha) \mid \alpha \in \mathrm{M}(S_{i-1}),\ \alpha \geq s^{\leftarrow}_{i-1,1}(\beta)\}$.

Result for finite chains

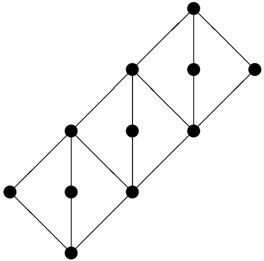
Theorem

TFAE

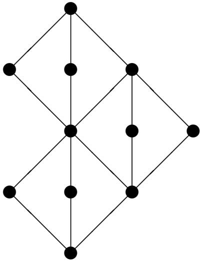
- $\mathrm{M}(\operatorname{Con} A)$ *is* $\mathcal{S}$*-nonseparable for some* $A \in \mathcal{K}$*;*
- $\mathrm{M}(\operatorname{Con} F(\aleph_0))$ *is* $\mathcal{S}$*-nonseparable (*$F(\aleph_0)$ *free in* $\mathcal{K}$*);*
- $\mathcal{S}$ *has a lifting in* $\mathcal{K}$*.*

Example3

Consider the following lattices



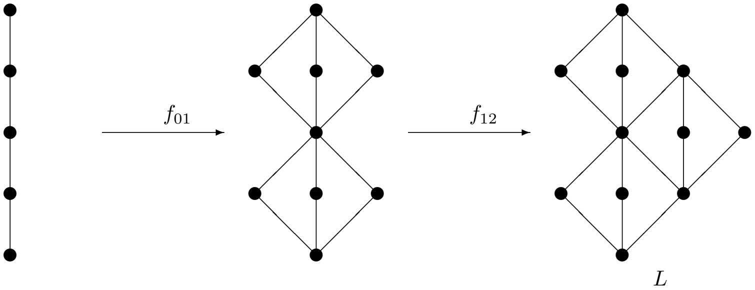
M



L

Example3

Consider the diagram $\mathcal{A}$ in $HSP(L)$:



Then $\mathcal{S} = \mathrm{Con}\,\mathcal{A}$ has a lifting in $HSP(L)$, but not in $HSP(M)$. Therefore, $\mathrm{Crit}(HSP(L), HSP(M)) \leq \aleph_0$.

Congruence intersection

A variety $\mathbf{V}$ has the *Compact Congruence Intersection Property* (CCIP) if the intersection of two compact congruences on any $A \in \mathbf{V}$ is compact.

Examples:

- Boolean algebras;
- distributive lattices;
- Stone algebras;
- $HSP(A)$, where A is a finite algebra generating a CD variety, which has no proper subalgebras.

Congruence intersection

For varieties vith CCIP we have a nicer topological representation of congruence lattices. Since $\mathrm{Con}_c\, A$ is now a distributive lattice, we can consider its Priestley dual space. This space has the same underlying set as before (prime ideals of $\mathrm{Con}_c\, A$ correspond to completely $\wedge$-irreducible elements of $\mathrm{Con}\, A$), but the basis of the topology consists of all sets of the form

$$U(B, \delta) = \{\alpha \in \mathrm{M}(\mathrm{Con}\, A) \mid \alpha{\restriction}B = \delta\},$$

where B is a finite subalgebra of A and $\delta \in \mathrm{Con}\, B$.

Congruence intersection

Using this representation is convenient because

- the spaces $\mathrm{M}(\mathrm{Con}\, A)$ are Hausdorff, so nets can have only one limit points;
- if $f: A \to B$ is a homomorphism of finite algebras, then $\mathrm{Con}\, f$ is a lattice homomorphism (preserves meets);
- $(\mathrm{Con}\, f)^{\leftarrow}(\beta) \in \mathrm{M}(\mathrm{Con}\, A)$ whenever $\beta \in \mathrm{M}(\mathrm{Con}\, B)$.

This enables to simplify the definition of $\mathcal{S}$-nonseparability.

Definition

We say that X is $\mathcal{S}$-nonseparable, if there exist $k_0 \in K_0$ and nets $(k_p \mid p \in P_1 \times \dots P_i)$ in K_i such that for every $\beta \in \mathrm{M}(S_i)$ and every $r = (p_1, \dots, p_{i-1}) \in P_1 \times \dots P_{i-1}$ the net $(k_{(r,p_i)}(\beta) \mid p_i \in P_i)$ converges to $k_r(s^{\leftarrow}_{i-1,1}(\beta))$.

Under this definition, the previous theorems hold exactly as stated.

Finite chains are not enough

There are varieties $\mathcal{K}$ and $\mathcal{L}$ such that $\mathrm{Con}\,\mathcal{K} \neq \mathrm{Con}\,\mathcal{L}$, but exactly the same diagrams indexed by finite chains have lifting in $\mathcal{K}$ as in $\mathcal{L}$. So, we need $\mathcal{S}$-nonseparability for other types of index sets. So far, I am only able to do it in the following special case.

A poset P is a *generalized chain* if it has a smallest element and any two subsets of the form $\downarrow x \setminus \{x\}$ are comparable (with respect to inclusion).

Square diagrams - CCIP version

Let $\mathcal{S}$ be the commutative diagram

$$\begin{array}{ccc} S_0 & \xrightarrow{s_{01}} & S_1 \\ {\scriptstyle s_{02}}\downarrow & & \downarrow{\scriptstyle s_{13}} \\ S_2 & \xrightarrow{s_{23}} & S_3 \end{array}$$

of finite distributive lattices and lattice 0-homomorphisms.

Definition

We say that X is $\mathcal{S}$-nonseparable, if there exist $k_0 \in K_0$ such that for every open set U_{k_0} containing k_0 and every family $(U_k \mid k \in K_1 \cup K_2)$ (U_k containing k) there are $k_1 \in K_1$, $k_2 \in K_2$, $k_3 \in K_3$ such that $k_j(\beta) \in U_{k_i}(s_{ij})$ for every arrow s_{ij} and every $\beta \in \mathrm{M}(S_j)$.

$U_k = (U_k(\alpha) \mid \alpha \in \mathrm{M}(S_i)) \subseteq X^{M(S_i)}$
and $k \in U_k$ means $k(\alpha) \in U_k(\alpha)$ for every α.

Result for square diagrams

$\mathcal{K}$......CCIP
$\mathcal{S}$......square diagram

Theorem

TFAE

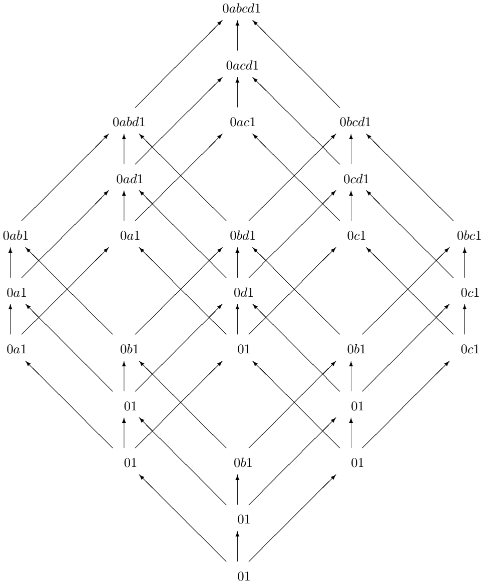
- $\mathrm{M}(\mathrm{Con}\, A)$ *is* $\mathcal{S}$*-nonseparable for some* $A \in \mathcal{K}$*;*
- $\mathrm{M}(\mathrm{Con}\, F(\aleph_1))$ *is* $\mathcal{S}$*-nonseparable (*$F(\aleph_1)$ *free in* $\mathcal{K}$*);*
- $\mathcal{S}$ *has a lifting in* $\mathcal{K}$*.*

A similar theorem holds for diagrams indexed by finite generalized chains.

Finite generalised chains are still not enough

We have $\mathrm{Con}\,\mathbf{M}_4 \neq \mathrm{Con}\,\mathbf{M}_3$, but exactly the same diagrams indexed by finite generalized chains have lifting in these varieties. The diagram distinguishing them:

M3 versus M4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Gillibert's bound

Theorem

Let $\mathcal{K}$ and $\mathcal{L}$ be finitely generated CD varieties such that $\operatorname{Con}\mathcal{K} \nsubseteq \operatorname{Con}\mathcal{L}$. Then $\operatorname{Con} F_{\mathcal{K}}(\aleph_2) \notin \operatorname{Con}\mathcal{L}$.

Conjecture: for CCIP varieties, the cardinality $\aleph_2$ can be replaced by $\aleph_1$.