# Order-enriched solid functors

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## on joint work with Walter Tholen

## TACL 2019

## Nice

Hom-sets are posets with

$$f\left( \begin{array}{c} A \\ \leq \\ B \end{array} \right) g \implies h \cdot f \cdot k \leq h \cdot g \cdot k$$

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Functors preserve the order

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Examples: Pos, SLat, Frm, ..., ordered varieties, Top<sub>0</sub>, ...

Inserter of 
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:  
 $I \xrightarrow{i} A \xrightarrow{g} B$   
(i)  $f \cdot i \leq g \cdot i$   
(ii)  $f \cdot i' \leq g \cdot i' \Rightarrow i'$  factorizes through  $i$   
(iii)  $i$  is order-monic, i.e.  $i \cdot a \leq i \cdot b \Rightarrow a$ 

 $a \leq b$ 

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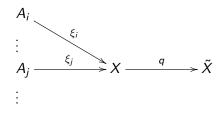
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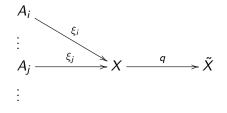
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Dually: coinserters, weighted colimits

Construction of weighted colimits in some categories:

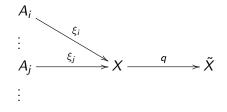


Construction of weighted colimits in some categories:



in SLat w. colim in Pos w. colim in SLat

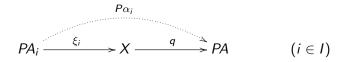
Construction of weighted colimits in some categories:



in Frm w. colim in SLat w. colim in Frm

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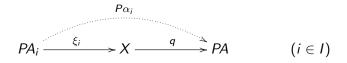
 $P : A \to X$  is a (strongly) order-solid functor if, for every family  $\xi = (\xi_i : PA_i \to X)_{i \in I}$ , there is  $\alpha = (\alpha_i : A_i \to A)_{i \in I}$ , and  $q : X \to PA$ 



with

(i) 
$$P\alpha = q \cdot \xi$$
  
(ii)  $(\alpha, A, q)$  universal with respect to property (i)  
(iii)  $q: X \to PA$  order-*P*-epic:  $Pf \cdot q \leq Pg \cdot q \Longrightarrow f \leq g$   
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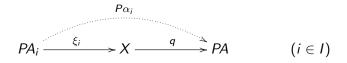


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R.-E. Hoffmann, PhD thesis, 1972

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Walter Tholen, Semi-topological functors I, JPAA, in 1979

Tholen and Wischnewsky, Semi-topological functors II, JPAA, 1979

Street, Tholen, Wischnewsky and Wolff, Semi-topological functors III, JPAA, in 1980

In the book of Adámek, Herrlich and Strecker, and in subsequent papers, they are called <u>solid</u>

Anghel, PhD thesis, 1987

Use of strongly order-solid to distinguish from order-solid in Anghel's sense

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 $\begin{array}{c} P: \mathcal{A} \to \mathcal{X} \text{ strongly order-solid} \\ & \Downarrow \\ \text{If } P \text{ is order-faithful, i.e., for every } A \xrightarrow[f]{g} B \text{, } Pf \leq Pg \implies f \leq g, \\ & \text{then} \end{array}$ 

*P* strongly order-solid  $\Leftrightarrow$  *P* order-solid.

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Use of strongly order-solid to distinguish from order-solid in Anghel's sense

*P* strongly order-solid  $\Leftrightarrow$  *P* order-solid.

Open: Do the two notions agree independently of order-faithfulness ?

#### Theorem

Every strongly order-solid functor  $P:\mathcal{A} \rightarrow \mathcal{X}$ 

(a) is order-faithful;

(b) is an order-right adjoint (i.e., r.a. and units are order-P-epic)

(c) detects weighted colimits.

#### Theorem

Let  $\mathcal{X}$  have inserters. An ordered functor  $P : \mathcal{A} \to \mathcal{X}$  is strongly order-solid if and only if

- (a) P is solid as an ordinary functor;
- (b) A has inserters and P preserves them;
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```
Open: ordinary solid + order-right adjoint + order-faithful \psi strongly order-solid ?
```

#### Theorem

An ordered functor  $P : A \to X$  is strongly order-solid iff P is order-right adjoint, and there exists a class  $\mathcal{E}$  of order-epimorphisms in A such that:

- (a) All adjunction co-units lie in  $\mathcal{E}$ ;
- (b) The pushout of a morphism of E along any morphism exists in A and belongs to E;
- (c) The wide pushout (that is, the cointersection) of any (possibly large) family of morphisms in *E* with common domain exists in *A* and belongs to *E*.

$$\operatorname{Top}_0 \xrightarrow{S} \operatorname{Pos}$$

SX = X with the dual of specialization order

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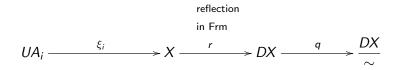
SX = X with the dual of specialization order

$$SA_i \xrightarrow{\xi_i} X \xrightarrow{\operatorname{id}} (X, \tau) \xrightarrow{T_0-\operatorname{reflection}} \overline{X}$$

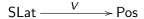
 $\tau =$  down sets U whose pre-image by  $\xi_i$  is open in  $A_i$  for all  $i \in I$ 

$$\mathsf{Frm} \xrightarrow{U} \mathsf{SLat}$$

$$Frm \longrightarrow SLat$$



~ is the least congruence in *DX* with which we obtain frame homomorphisms  $q \cdot r \cdot \xi_i$ , for all *i*.





The composition of two strongly order-solid functors is strongly order-solid.

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In particular,



Examples of strongly order-solid functors

AbMon(Pos) 
$$\xrightarrow{U}$$
 Pos

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# Theorem

For every order-varietal algebraic theory  $\mathcal{T}$ , the functor  $U^{\mathcal{T}}$ : Alg $(\mathcal{T}, \text{Pos}) \rightarrow \text{Pos}$  is strongly order-solid.  $Alg(\mathcal{T}, Pos) = category of ordered algebras for a given algebraic theory <math>\mathcal{T}$  $\mathcal{T}$  order-varietal: the forgetful functor  $U^{\mathcal{T}} : Alg(\mathcal{T}, Pos) \rightarrow Pos$  is an order-right adjoint

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The ordered algebraic functor induced by any morphism of order-varietal algebraic theories

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Example: Frm  $\rightarrow$  SLat

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 $\mathsf{Example:} \ \mathsf{Pos} \to \mathsf{PrOrd}$ 

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# Proposition

In the commutative diagram of preordered functors



with H and J full emdeddings and H a preorder-right adjoint

P' strongly preorder-solid  $\Rightarrow P$  strongly preorder-solid

 $\Rightarrow$  *P* strongly order-solid, if ordered

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Example:  $Pos \rightarrow PrOrd$ 

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Example:

$$\begin{array}{ccc}
\operatorname{Fop}_{0} & & \operatorname{Top} \\
s & & & \downarrow s' \\
\operatorname{Pos} & & & \operatorname{PrOrd}
\end{array}$$

Ordered vector space:  $+: V \times V$  and  $\lambda - : V \rightarrow V$ ,  $\lambda \ge 0$ , are monotone

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Two possible orders:

$$f \bigvee_{i \leq V} g$$

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• if 
$$f(x) \le g(x)$$
, for all  $x \in PV$ 

• if  $f(x) \le g(x)$ , for all  $x \in V$  (equivalently, f = g, since  $f(x) \le g(x) \Rightarrow f(-x) \ge g(-x)$ ) OVec := category of ordered vector spaces with a generating cone, and positive linear maps, ordered by  $f \le g$  iff  $f(v) \le g(v)$  for all  $v \in PV$ 

OVec := category of ordered vector spaces with a generating cone, and positive linear maps, ordered by  $f \le g$  iff  $f(v) \le g(v)$  for all  $v \in PV$ 

The functor

 $P: \mathcal{OV}ec \longrightarrow \mathsf{Pos}$  $V \mapsto PV = \mathsf{positive cone}$ 

is strongly order-solid.

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It fails to preserve inserters: for  $\mathbb{R} \xrightarrow[id]{2-} \mathbb{R}$ , the inserter in Pos is  $\mathbb{R}_0^+$ , but in  $\mathcal{OVec}_=$  it is just  $\{0\}$ .