Extensions of the Stone Duality to the category of zero-dimensional Hausdorff spaces

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Introduction

In 1937, M. Stone [6] proved that there exists a bijective correspondence S_ℓ between the class of all (up to homeomorphism) zero-dimensional locally compact Hausdorff spaces (briefly, *Boolean spaces*) and the class of all (up to isomorphism) generalized Boolean algebras (briefly, GBAs) (or, equivalently, Boolean rings with or without unit). In the class of compact Boolean spaces (briefly, *Stone spaces*) this bijection can be extended to a duality S^t : **Stone** $\longrightarrow$ **Bool** between the category **Stone** of Stone spaces and continuous maps and the category **Bool** of Boolean algebras and Boolean homomorphisms; this is the classical Stone Duality Theorem.

In 1964, H. P. Doctor [4] showed that the Stone bijection S_ℓ can be even extended to a duality between the category **PBoolSp** of all Boolean spaces and all perfect maps between them and the category **GenBoolAlg** of all GBAs and suitable morphisms between them.

Later on, G. Dimov [2, 3] extended the Stone Duality to the category **BoolSp** of Boolean spaces and continuous maps.

In this talk we will extend the Stone Duality to the category **ZHaus** of all zero-dimensional Hausdorff spaces and continuous maps. Moreover, we will define two categories, namely, the categories **DZAlg** and **MZBool**, and will prove that they are dually equivalent to the category **ZHaus**. We will also introduce two other categories, namely, the categories **ZAlg** and **ZBool**, and will show that they are dually equivalent to the category **ZComp** of zero-dimensional Hausdorff compactifications of zero-dimensional Hausdorff spaces. Let us note that the category **ZComp** is a full subcategory of the category **Comp** of all Hausdorff compactifications of Tychonoff spaces, which was defined in the recent paper [1] of G. Bezhanishvili, P. J. Morandi and B. Olberding.

Preliminaries

- We denote by **2** the simplest Boolean algebra containing only 0 and 1, where $0 \neq 1$.
- If A is a Boolean algebra, then $A^+ \stackrel{\mathrm{df}}{=} A \setminus \{0\}$ and $\mathrm{At}(A)$ is the set of all atoms of A.
- If X is a topological space, we denote by $\mathrm{CO}(X)$ the set of all clopen (= closed and open) subsets of X. Obviously, $(\mathrm{CO}(X), \cup, \cap, \setminus, \emptyset, X)$ is a Boolean algebra.
- If $\mathcal{C}$ is a category, we denote by $|\mathcal{C}|$ the class of the objects of $\mathcal{C}$ and by $\mathcal{C}(X, Y)$ the set of all $\mathcal{C}$-morphisms between two $\mathcal{C}$-objects X and Y.
- We denote by **CaBa** the category of complete atomic Boolean algebras and complete Boolean homomorphisms.

We will denote by

$$S^a : \mathbf{Bool} \longrightarrow \mathbf{Stone} \quad \text{and} \quad S^t : \mathbf{Stone} \longrightarrow \mathbf{Bool}$$

the two two contravariant functors which realize the Stone Duality. We set $X_A \stackrel{\text{df}}{=} \mathbf{Bool}(A, \mathbf{2})$ and

$$s_A(a) \stackrel{\text{df}}{=} \{x \in X_A \mid x(a) = 1\}, \tag{1}$$

for every $a \in A$; then

$$S^a(A) \stackrel{\text{df}}{=} (X_A, \mathcal{T}_A),$$

where $\mathcal{T}_A$ is the topology on X_A having as a closed base the family $\{s_A(a) \mid a \in A\}$.

As it is well known, if $\varphi \in \mathbf{Bool}(A, B)$, then
$S^a(\varphi) : S^a(B) \longrightarrow S^a(A), \;\; y \mapsto y \circ \varphi$.

If X is a topological space, then, for every $x \in X$, we denote by

$$t_x : \mathrm{CO}(X) \longrightarrow \mathbf{2}$$

the function defined by $t_x(P) = 1 \leftrightarrow x \in P$, where $P \in \mathrm{CO}(X)$. Then $t_x \in \mathbf{Bool}(\mathrm{CO}(X), \mathbf{2})$. Sometimes we will write t_x^X instead of t_x. We now define a map t_X as follows:

$$t_X : X \longrightarrow \mathbf{Bool}(\mathrm{CO}(X), \mathbf{2}), \;\; x \mapsto t_x.$$

If $X \in |\mathbf{Stone}|$, then $S^t(X) \stackrel{\mathrm{df}}{=} \mathrm{CO}(X)$ and the map $t_X : X \longrightarrow S^a(S^t(X))$ is a homeomorphism.

If $f \in \mathbf{Top}(X, Y)$, then we define the map
$\varphi_f : \mathrm{CO}(Y) \longrightarrow \mathrm{CO}(X)$ by the formula $\varphi_f(P) \stackrel{\mathrm{df}}{=} f^{-1}(P)$ for every $P \in \mathrm{CO}(Y)$. Then φ_f is a Boolean homomorphism.
In particular, when $f \in \mathbf{Stone}(X, Y)$, then $S^t(f) \stackrel{\mathrm{df}}{=} \varphi_f$.

Two extensions of the Stone Duality to the category of zero-dimensional spaces

If X is a zero-dimensional Hausdorff space then, by the Dwinger Theorem, the map $t_X : X \longrightarrow S^a(\mathrm{CO}(X)), \;\; x \mapsto t_x$, is the maximal zero-dimensional Hausdorff compactification of X, i.e., this is the Banaschewski compactification of X. Thus the pair $(\mathrm{CO}(X), t_X(X))$ completely determines the space X. When $X \in |\mathbf{Stone}|$, we have that t_X is a homeomorphism, so that in this case the pair $(\mathrm{CO}(X), t_X(X))$ can be reduced to $\mathrm{CO}(X)$.

The main goal of this talk is to describe abstractly (in an algebraic way) the pairs $(\mathrm{CO}(X), t_X(X))$ and thus to obtain an algebraic description of the zero-dimensional Hausdorff spaces, as well as of the continuous maps between them. As a result, we will arrive to a new Duality Theorem with which we will extend the Stone Duality to the category **ZHaus**.

We will first build a category **DZAlg** dually equivalent to the category **ZHaus** and then we will use the Tarski Duality for presenting in another form the category **DZAlg.** Namely, we will construct a new category **MZBool** and will show that it is equivalent to the category **DZAlg.** Then, clearly, the category **MZBool** will be dually equivalent to the category **ZHaus.**

Definition

A pair (A, X), where A is a Boolean algebra and $X \subseteq \mathbf{Bool}(A, \mathbf{2})$, is called a *Boolean z-algebra* (briefly, *z-algebra*; abbreviated as ZA) if for each $a \in A^+$ there exists $x \in X$ such that $x(a) = 1$.

Notation

If A is a Boolean algebra and $X \subseteq \mathbf{Bool}(A, \mathbf{2})$, we set

$$s_A^X(a) \stackrel{\text{df}}{=} X \cap s_A(a)$$

for each $a \in A$, defining in such a way a map

$$s_A^X : A \longrightarrow P(X), \;\; a \mapsto s_A^X(a).$$

Definition

Let $C \in |\mathbf{CaBa}|$ and A, B be Boolean subalgebras of C. If for every $a \in A$ and any $x \in \mathrm{At}(C)$ such that $x \leq a$ there exists $b \in B$ with $x \leq b \leq a$, then we will say that *A is t-coarser than B in C* or that *B is t-finer than A* in C; in this case we will write $A \preceq_C B$. We will say that the Boolean algebras A and B are *t-equal in C* if $A \preceq_C B$ and $B \preceq_C A$.

Definition

A z-algebra (A, X) is called a *Boolean dz-algebra* (briefly, *dz-algebra*; abbreviated as DZA) if it satisfies the following condition:

(Dw) If B is a Boolean subalgebra of $(P(X), \subseteq)$ and B is t-equal to $s_A^X(A)$ in $(P(X), \subseteq)$, then $B \subseteq s_A^X(A)$.

Example 1.

Let A be a Boolean algebra. Then (A, X_A) is a DZA. (The dz-algebras of this type will be called *compact dz-algebras*.)

Example 2.

Let X be a zero-dimensional Hausdorff space. Then $(\mathrm{CO}(X), t_X(X))$ is a DZA.

Proposition 1.

There is a category **DZAlg** whose objects are all dz-algebras and whose morphisms between any two **DZAlg**-objects (A, X) and (A', X') are all pairs (φ, f) such that $\varphi \in \mathbf{Bool}(A, A')$, $f \in \mathbf{Set}(X', X)$ and $x' \circ \varphi = f(x')$ for every $x' \in X'$.

The composition between two morphisms $(\varphi, f) : (A, X) \longrightarrow (A', X')$ and $(\varphi', f') : (A', X') \longrightarrow (A'', X'')$ is defined to be $(\varphi' \circ \varphi, f \circ f') : (A, X) \longrightarrow (A'', X'')$.

The identity map of an object (A, X) is defined to be (id_A, id_X).

Theorem 1.

The categories **ZHaus** and **DZAlg** are dually equivalent.

Sketch of the proof. For every $X \in |\textbf{ZHaus}|$, set $\hat{X} \stackrel{\mathrm{df}}{=} t_X(X)$ and

$$F(X) \stackrel{\mathrm{df}}{=} (\mathrm{CO}(X), \hat{X}).$$

For $f \in \textbf{ZHaus}(X, Y)$, set

$$F(f) \stackrel{\mathrm{df}}{=} (\varphi_f, \hat{f}),$$

where $\hat{f} : \hat{X} \longrightarrow \hat{Y}$ is defined by $\hat{f}(t_x^X) \stackrel{\mathrm{df}}{=} t_{f(x)}^Y$, for every $x \in X$. Then we show that $F : \textbf{ZHaus} \longrightarrow \textbf{DZAlg}$ is a contravariant functor which is full, faithful and essentially surjective on objects.

Proposition 2.

The category **Bool** is isomorphic to the full subcategory **CDZAlg** of the category **DZAlg** having as objects all compact dz-algebras.

Sketch of the proof. Define a functor

$$E : \mathbf{Bool} \longrightarrow \mathbf{CDZAlg}$$

by setting $E(A) \stackrel{\mathrm{df}}{=} (A, X_A)$, for every $A \in |\mathbf{Bool}|$ and $E(\varphi) \stackrel{\mathrm{df}}{=} (\varphi, S^a(\varphi))$, for every **Bool**-morphism φ.
Define also a functor

$$E' : \mathbf{CDZAlg} \longrightarrow \mathbf{Bool}$$

by setting $E'(A, X_A) \stackrel{\mathrm{df}}{=} A$, for every $(A, X_A) \in |\mathbf{CDZAlg}|$, and $E'(\varphi, f) \stackrel{\mathrm{df}}{=} \varphi$, for every **CDZAlg**-morphism (φ, f).
Then it is easy to see that $E \circ E' = Id_{\mathbf{CDZAlg}}$ and $E' \circ E = Id_{\mathbf{Bool}}$.

Remark 1.

Let $E^s : \mathbf{Stone} \longrightarrow \mathbf{ZHaus}$ and $E^a : \mathbf{CDZAlg} \longrightarrow \mathbf{DZAlg}$ be the embedding functors. Then

$$F \circ E^s = E^a \circ E \circ S^t,$$

where $F : \mathbf{ZHaus} \longrightarrow \mathbf{DZAlg}$ is the dual equivalence from Theorem 1 and $E : \mathbf{Bool} \longrightarrow \mathbf{CDZAlg}$ is the isomorphism from Proposition 2. Therefore, the dual equivalence F is an extension of the dual equivalence $S^t : \mathbf{Stone} \longrightarrow \mathbf{Bool}$.

Now we will define a new category **MZBool** and will show, using Tarski's duality, that it is equivalent to the category **DZAlg**. This will imply immediately that the category **MZBool** is dually equivalent to the category **ZHaus**.

Definition

Let A be a Boolean algebra and $B \in |\mathbf{CaBa}|$. A Boolean monomorphism $\alpha : A \longrightarrow B$ is said to be a *Boolean z-map* (briefly, *z-map*) if every atom of B is a meet of some elements of $\alpha(A)$. A z-map $\alpha : A \longrightarrow B$ is called a *maximal Boolean z-map* (briefly, *mz-map*) if for every Boolean subalgebra C of B which is t-equal to $\alpha(A)$ in B, we have that $C \subseteq \alpha(A)$.

Proposition 3.

There is a category **MZBool** whose objects are all mz-maps and whose morphisms between any two **MZBool**-objects $\alpha : A \longrightarrow B$ and $\alpha' : A' \longrightarrow B'$ are all pairs (φ, σ) such that $\varphi \in$ **Bool**(A, A'), $\sigma \in$ **CaBa**(B, B') and $\alpha' \circ \varphi = \sigma \circ \alpha$.

The composition between two morphisms $(\varphi, \sigma) \in$ **MZBool**(α, α') and $(\varphi', \sigma') \in$ **MZBool**(α', α'') is defined to be $(\varphi' \circ \varphi, \sigma' \circ \sigma) : \alpha \longrightarrow \alpha''$.

The identity map of an **MZBool**-object $\alpha : A \longrightarrow B$ is defined to be (id_A, id_B).

Theorem 2.

The categories **MZBool** and **DZAlg** are equivalent.

Sketch of the proof. We will define a functor

$$G : \mathbf{DZAlg} \longrightarrow \mathbf{MZBool}$$

and will prove that it is full, faithful and essentially surjective on objects.

For every $(A, X) \in |\mathbf{DZAlg}|$, set

$$G(A, X) \stackrel{\text{df}}{=} s_A^X.$$

For every $(\varphi, f) \in \mathbf{DZAlg}((A, X), (A', X'))$, set

$$G(\varphi, f) \stackrel{\text{df}}{=} (\varphi, \sigma_f),$$

where $\sigma_f : (P(X), \subseteq) \longrightarrow (P(X'), \subseteq)$ is defined by the formula $\sigma_f(M) = f^{-1}(M)$ for every $M \in P(X)$.

Clearly, Theorem 1 and Theorem 2 imply the following assertion:

Theorem 3.

The categories **ZHaus** and **MZBool** are dually equivalent.

A duality theorem for the category of zero-dimensional Hausdorff compactifications of zero-dimensional spaces

Proposition [1]

There is a category **Comp** whose objects are Hausdorff compactifications $c : X \longrightarrow Y$ and whose morphisms between any two **Comp** -objects $c : X \longrightarrow Y$ and $c' : X' \longrightarrow Y'$ are all pairs (f, g), where $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are continuous maps such that $g \circ c = c' \circ f$. The composition of two morphisms (f_1, g_1) and (f_2, g_2) is defined to be $(f_2 \circ f_1, g_2 \circ g_1)$. The identity map of a **Comp**-object $c : X \longrightarrow Y$ is defined to be $id_c \stackrel{\text{df}}{=} (id_X, id_Y)$.

Definition

We will denote by **ZComp** the full subcategory of the category **Comp** whose objects are all Hausdorff compactifications $c : X \longrightarrow Y$ for which Y is a zero-dimensional space.

Proposition 4.

There is a category **ZAlg** whose objects are all z-algebras and whose morphisms between any two **ZAlg**-objects (A, X) and (A', X') are all pairs (φ, f) such that $\varphi \in \mathbf{Bool}(A, A')$, $f \in \mathbf{Set}(X', X)$ and $x' \circ \varphi = f(x')$ for every $x' \in X'$. The composition between two morphisms $(\varphi, f) \in \mathbf{DZAlg}((A, X), (A', X'))$ and $(\varphi', f') \in \mathbf{DZAlg}((A', X'), (A'', X''))$ is defined to be $(\varphi' \circ \varphi, f \circ f') : (A, X) \longrightarrow (A'', X'')$; the identity map of an object (A, X) is defined to be (id_A, id_X).

Theorem 4.

The categories **ZComp** and **ZAlg** are dually equivalent.

Proposition 5.

There is a category **ZBool** whose objects are all z-maps and whose morphisms between any two **ZBool**-objects $\alpha : A \longrightarrow B$ and $\alpha' : A' \longrightarrow B'$ are all pairs (φ, σ) such that $\varphi \in$ **Bool**(A, A'), $\sigma \in$ **CaBa**(B, B') and $\alpha' \circ \varphi = \sigma \circ \alpha$. The composition between two morphisms $(\varphi, \sigma) \in$ **ZBool**(α, α') and $(\varphi', \sigma') \in$ **ZBool**(α', α'') is defined to be $(\varphi' \circ \varphi, \sigma' \circ \sigma) : \alpha \longrightarrow \alpha''$; the identity map of an **ZBool**-object $\alpha : A \longrightarrow B$ is defined to be (id_A, id_B).

Theorem 5.

The categories **ZBool** and **ZAlg** are equivalent.

Theorem 6.

The categories **ZComp** and **ZBool** are dually equivalent.

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